NPTEL

NPTEL ONLINE COURSE

Introduction to Abstract

Group Theory

Module 06

Lecture 30– “Odd and even permutations I”

PROF. KRISHNA HANUMANTHU

CHENNAI MATHEMATICAL INSTITUTE

In the last couple of videos we looked at the symmetric group on N letters, we saw how to represent permutations which are elements of SN as cycles and we learned that any permutation can be written as a product of disjoint cycles and we also saw that any permutation can be written as a product of transpositions which may not be disjoint, we saw that a cycle of length K has order K and if we have a product of disjoint cycles the order of the product is simply the lcm of the individual orders. So these are the various things that we have done in the last few videos. I want to now spend more time on writing a permutation as a product of transpositions.

(Refer Slide Time: 01:04)

So if you recall one of the example from, from the previous video we wrote the permutation (12)(13)(14) and we saw that this is same as (23)(25)(12)(45)(15), okay, so this is the same permutation but on the left hand side we have a product description which has which requires 3 transpositions on the right side we have 5 transpositions okay, I, I remarked on this last time, the description product of transpositions is not unique and even the number of transpositions that is required that is also not unique, we have 3 here we have 5 here.

However what is unique is whether we need an odd number of permutations or transpositions are even number of transpositions, so this is one of the examples. We saw another example, we saw that if you take the 3-cycle (132) this can be written has (12)(13), this can also written as (13) (32), okay so here we need 2 transpositions, in this also we need 2 transpositions, both even, of course in this case both 2. In this case these are both odd, even though it is 3 and 5.

So the theorem that I want to prove in this video and it will take me some time to prove this, we will do a various examples to understand the proof and then I will prove. The theorem is informally if you want a theorem it says that if permutation as a description of the product of transpositions in 2 different ways the number of permutations the number of transpositions required in both the things have same parity, so they are both even or they are both odd.

So let me write this more precisely, mathematically, how do we write this? Let me write this as fallows. Let ρ be a permutation, so I am going to use a new Greek letter because I am going to use tau and σ to denote transpositions, so you read this as a rho, rho, okay so let this be any permutation.

Suppose it a 2 representations. Suppose that rho is on the one hand it is σ 1σ 2… σ k, on other hand it is also same as tau 1 , tau 2 , tau t, okay where these are all transpositions, σ1 through σ k and tau 1 through tau t are 2-cycles, remember transpositions are just another name for 2-cycles. So what is the situation? We have a fixed permutation rho, and we have written in fashion as σ 1 to σ k and in another fashion as tau 1 to tau t. Certainly we cannot say k = t but what we can say is that, then both k and tau, sorry t, k is the number of σs and t is the number of tau-s, okay so let me remove the word both here. So then k and t are both even or both odd, so in other words, what I am saying is that they one of them they cannot be one can't be even and the other odd, that is not possible, either both are even or both are odd as illustrated in this example, you have 3 and 5 both are odd 2 and 2 both are odd okay. So you can have both odd or both even.

So the rest of the video is focused on proving this theorem. It will take some set-up, so I am going to slowly introduce to you the techniques involved, so this is not so easy to prove though it would be, as we proceed with the proof, hopefully you will follow the proof and it makes sense to you but the proof is somewhat lengthy.

(Refer Slide Time: 05:50)

So let us systematically prove this. The goal of this video is to prove this theorem, okay, so I will systematically and slowly go through the proof. So in order to prove this let us introduce a polynomial, so let us introduce a polynomial okay, so if you are not very familiar with the notion of polynomial, do not worry about it, we do not need a lot about them, it is just a formal thing and I will do enough examples to make it clear.

So I am go to call this polynomial f x1 up to xn, so think of x1 through xn as variables, are variables or indeterminates. So think of them as just meaningless symbols, okay, they do not have any meaning, they are just symbols, so I will define this to be the following. So I will start with x1, I will subtract from it x2, then I will subtract from it x3, x1-x2, x1-x3, I will do it all the way up to xn, okay so x1 minus the rest, one by one, then I will start with x2 okay and I will x2 to the right of x2 there is x3 so I will start with x3, x2 minus, okay so to basically what we have is x2-x3 , x2-x4 and up to x2- xn, all the way and I finally have x(n-1)–xn , okay so this is the polynomial and formally, if you write compactly if you want to write this this is written like this.

(Refer Slide Time: 07:53)

We take this disjoint, product this symbol stands for product, you are all familiar with the summation sign, this is just the product sign, we take i and j distinct, $i<j$, i can be anything from 1 to $n-1$ and j can be anything from 2 to n right. i can't n be because i in n there will be no j, j must be strictly more than n, similarly j cannot be 2 because j must be strictly more than 1 so this is the product. For any pair of indices where i is less than j, you introduce a term x_i-x_j oaky.

So and remember n is fixed in the theorem. We are working with the symmetric group of n letters, n is any positive integer, we do not want to work with any specific one, the theorem works with any positive integer. But as an example what would be x_1, x_2 so $f(x_1)$ is just, that does not exist, there is no, we don't have two indices like that if you have $n=1$. So $f(x_1, x_2)$ would be simply x_1-x_2. There is just one pair of indices,1 and 2.

What would be $f(x_1,x_2,x_3)$? It would be, you take a x_1-x_2 , x_1-x_3, so starting with x_1 you take every other index, 1 is fixed so 2 and 3 and then starting with x_2 we take every other index and is there is anything else? No, we have you have , here you have $i=1$ $j=2$ but the only possibilities here you have $i=1$ in this case $j=2$ or 3 and $i=2$ you have $j=3$ okay these are the 3 possibilities there will be 3 terms in the product. So this is x_1-x_2 times x_1-x_3 times x_2-x_3.

Similarly I will write one more and hopefully it will become clear to you what is the polynomial. It is a, is an expression, think of it as an expression involving variables x1,x2, …, xn. So here for i=1 you have for i=1 you have j =2, 3 or 4, right so for i=1 you have j=2,3 or 4, so you have x1 –x2, x1-x3 ,x1-x4, okay so what about i=2? So you have i=2, j must be bigger than 2, so j is 3 or 4, so you have x2-x3 x2-x4, and then i=3 then j must be 4, so you have x3 –x4 right. So these are the 6 terms, so this polynomial for x1,x2,x3,x4 will be these 6 terms and so on. Now we can think of what x1,x2 f(x1,x2,x3,x4,x5) would be it will be x1-x2, x1-x3, x1-x4, x1-x5 and so on. So this is what f is.

(Refer Slide Time: 11:47)

So now what I want do is I want to take an element sigma in Sn, okay so σ is just a permutation on n letters and I want to define σ* of f as fallows. So σ * of f will be, so f is remember this, sometimes I do not write the variables, so we have so f is the product xi –xj, 1<i and i strictly < j and j <= n, so σ *f, I will define, so I am going to modify f so σ *f is a modification, I am going to change f a little bit, of f using σ okay, so think of it like this, σ *f is simply a modification using σ. How do we modify? Remember what is σ? Sigma just is a permutation of the letters.

So what I will do this, I will continue to take the product over it pair i, j i strictly < j, but instead of xi-xj, I will have x σ i – x σ j, oaky. This might seem somewhat mysterious but it is not, so I just want to make it very clear to you because it is important to understand what σ * f is before we proceed.

So what is σ *f? I am going to start with f and modify indices based on σ. So as an example, okay so let us suppose we take 2 variables so then so take n=2, so then f is simply x1-x2. So if you take now we are dealing with S2 right so n is 2, S2 is just 2 elements e and (12). This we have seen, the order of S2 is 2 factorial, right and which is 2, it is e and (12) so if you take for any σ we can define, so let us compute this in these two cases.

(Refer Slide Time: 14:34)

e * f is this is a compact way of writing, xσ i –x σ j, but if you spread it out what would it be? e is identity element so nothing changes so you have x (e of 1) –x (e of 2), but e of 1 is 1, e of 2 is 2. So this is just x1-x2, which is just f, right.

On the other hand, what is 12*f? So this is x okay, so now it is easier for me to use the letter σ, so then this is x (σ of 1) – x (σ of 2), so when we right σ of i, remember σ is a function from 1 through n to 1 through n. So σ of i is simply the image of i, okay under σ. So when I write σ of i, I am actually thinking of sigma as a function, so what is what is image of 1 under σ which is (12), it is 2 so it is x2 –x σ 2 will be x1.

So this is not same as f right, this is actually –f. So here we have changed the sign, x1-x2 becomes x2-x1. So this is what happens when n=2.

Let us do n =3 okay. So here what is f? Here f is x1-x2, x1-x3,x2-x3 okay, and here S3 remember has 6 elements, so they are e, (12), (13), (13),(23),(123),(132) okay. Let me do not all of them but illustrate the point, let me do σ = (13) for example.

What is σ *f? So you start with f and you keep the order but you just change the subscripts and see what happens. So it would become x σ 1 – x σ 2, so I am looking it f here x1- x2 that becomes this, x1-x3 becomes x σ 1 – x σ 3, and finally we have x σ 2 –x σ 3. So σ *f is (x σ 1-x σ 2) (x σ 1 - x σ 3) (x σ 2 – x σ 3).

But now let's actually compute these things. x is σ 1 remember is σ is (13) so this is x3. σ 2 what does σ do to 2, nothing so it is 2, 2 is fixed by σ so x3 – x2. σ 1 is 3 so this is x3 – σ 3 is 1 so this is x1, σ 2 is 2 so this is x2 minus σ 3 is 1, so this is x1.

So σ * is (x3 –x2)(x3-x1)(x2-x1) and what is this? This is how do we compare this with f, so every term remember here is switched x1- x2 became x2-x1, x1-x3 became x3-x1 , x2-x3 you have a x3 –x2, so you have -,-, -. So this is –f. Is that clear σ f is – f, because f is a product of 3 terms, σ *f became, those 3 terms are interchanged, the order in which the 2 terms, x1-x2 is interchanged this is – okay so this is –(x2-x3) –(x1-x3) , this is -(x1-x2) right and so this is – f , -1 time -1 times -1 is -1, and then you have just f.

(Refer Slide Time: 19:26)

So σ *f is f. Let us do one more example let us say T = (123), that is another element of S3, so T is 123. So what is T *f same thing so f is, the same process, it will be different but the same procedure we must follow. f is x1-x2, x1-x3,x2-x3 so T*f will be x (T1) –x (T 2) x(T1)- x(T3) x(T2) – x(T3) right so this should be T 1 is 2, T 2 is 3, x2-x3, T1 is 2, T3 is 1 so this is x2-x1, T2 is 3 and T3 is 1, okay so this is x3-x1.

So now how do you compare this with f, f has 3 terms again remember that f has 3 terms x1-x2 ,x1-x3,x2-x3 here we have those three terms but unlike in the previous example one of those terms has not changed x2-x3 remains x2-x3 but other two things we have interchanged. This and this have changed size , right, because it was x1-x2 in f, that has become x2-x1, it was x1-x3 in f that has become x3- x1, x2-x3 remains as it is, so this is actually just f because this is minus of that term times minus of that term, -1 times -1 is 1 so T *f is f okay.

(Refer Slide Time: 21:31)

So what I want to now say is that and I hope you agree based on these examples, so now I am going back to the general situation, if σ is a permutation in SN then σ * of f is either f or –f okay. That is what we found in these examples right it becomes either f or –f. Of course doing examples is no a proof of this, the examples are supposed to give you an idea of why it must be true and how to prove it. And what is a proof?

So I won't write this in detail because it becomes, it is easier if you think about it yourself and try to write down a proof, but I will give you a basic reason why this must be true. Because remember f was product $x_i - x_j$ okay, so it is over all i,j and σ *f is the product $x_{\sigma i} - x_{\sigma j}$, over the same i and j. So if you think about it because σ is a permutation σ is a bijection right remember symmetric group is the set of bijections of this set.

Because it is a bijection, as i varies over 1 to n or 1 to n-1, in this case, σ i also varies, similarly σ j also is an element of 1 to n. So if you take an individual term in f that is translated to some other term only changes will be negative signs as we have seen in this example. For example, x_1-x_3 became x_3-x_2 so we, we must allow that in the definition of f we have x_i-x_j were i is strictly < j always, so 12,13,23 right 12,13,23, in f you never have a bigger index minus a smaller index.

f is defined to be $x_i - x_j$ where i is strictly <j, but once you do σ *f, that will no longer be the case. Sometimes we have a bigger index minus smaller index: x2-x. Because if i<j certainly it is not true that σ i is < σ j, that is not the case. But however that is just a negative of, if you have a bigger index – smallest index, and that is negative of smaller index – bigger index, which appears in f, so f is a product of some terms, σ *f will be product of same number of terms.

But some of these terms, it will be the same terms with possibly negative sign, so if you now accumulate all these negative signs you might either have + or –, as this example shows, you have 3 negative signs giving you –f when σ was (13), when T is (123) you have only 2 negative signs, so that gave you f. So σ *f must be either f or –f, the same terms are preserved up to a change of sign, so σ * is either f or –f. Let me say this is my proof. I won't go into a more formal proof, but I hope this is clear enough for you . So σ *f is f or –f.

(Refer Slide Time: 25:34)

Next I want to do, if σ and T are in SN, what is σ T *f? This is my next question. σ T is another permutation right, so I can apply * of it. How is it related to σ *f and T *f, okay. I want to understand how is it related to this. So again I will explain this by the example that we did. So take n = 3 and σ was 1 let's see σ was 13 and T was 123. So what is σ T, so σ T we are now okay we are all become comfortable hopefully with products of permutations, cycle products.

So here σ T will be 1goes to 2, and 2 is not, 2 is fixed by sigma, so that's 1 goes to 2, 2 goes to 3, 3 goes to 1, so that is 12 so you close, 2 ->3 3->1 so 12 is a cycle 3->1, 1->2 so σ T is (12). And what is σ T *f? If you now do the calculation that we have done earlier, it is x, so f was remember I will write it again here for convenience, f is x1-x2 , x1-x3 ,x2-x3. So σ T *f will be I will just directly write is 1 ->2 in this so x2 and 2->1 so that is x1- x2, 1 -> 2 so x2 and 3 is fixed so that x2 –x3 and 2->1. So and 3 is fixed so x2-x1 right.

I will just look at the subscripts which are the indices of variables and see where they go under σ T. So 1->2, and

2->1, so 1st one changes sign and the 2nd one 1->2 , 3->3 so x2-x3, this does not change sign, no change of sign. Here 2->1 and 3->3, so here also no change in sign, right in the 1st one change in sign okay. So what is this so this? This is –f right, because there is one change of sign. So it is minus of x1-x2 and the other two are fixed, so this is minus f, so σ T * so this is σ T 1st you multiply and take *.

(Refer Slide Time: 28:48)

On the other hand , what is T * f? It was, remember, here it is –f but it is x2-, so it is f, but let me write what we got, okay, it is x2-x3, x2-x1, x3-x1, right x3-x1, okay. So now let us apply, so we have this, let us apply σ * to both sides, so σ * of T *f? What is this see when apply σ *, I simply change subscripts by σ, so σ was (13) right, so x2 –x3, 2 is fixed under σ, so it is x2 and 3 goes to 1, so this is x2-x1, x2-x1, so 2->2 and 1->3, so this is x2-x3, 3->1 and 1 ->3, so you have x1-x3, and if you now compare this with σ T*f okay.

So let us compare them both, we have x2-x1 , x2-x1, x2-x3,x2-x3, so this term is common and this term is common and we have x1-x3, and x1-x3. So we see that, so they are equal, they are same right, so σ, so what we have is, σ T whole star f is applying σ * to

T * f. So you start with f, first apply T * then apply σ * to it. So this is what we have in this example and I will now say this holds in general, okay. So let me quickly say this. So in general we have and why is this?

(Refer Slide Time 31:34)

What is the reason? And the calculations is the following. So what is σ T*f? Which is by definition remember we take x and we apply σ T of i –x σ T of j, remember again, when I for a permutation σ I defined sigma * of f, how did we define this? Let us go back to the definition of σ *, I start with f and each subscript I simply change by σ, right.

So I change a subscript by σ , x σ I – x σ j. Here I am applying it to σ T, so I change i by σ T, j by σ T. But now, this is same as and the product is over this set i < j, i greater than equal to 1, j less than or equal to n, but σ T remember that it is composition right, σ T is the composition of σ and T, the product in, the product in the symmetric group, the binary operation of the symmetric group is the compositions of functions.

So this would be same as x times T of i x sub (σ of T of I) – x sub (σ of T of j). I am not doing anything in this step, I am just observing that σ T of i is σ of T of i, σ T of j is σ of T of j, but what is this? This is applying σ to something, so this is σ * of product 1<=i, i<j , j<= n of x T I – x T j, right, because if I take this and apply σ *, applying σ * is by definition applying σ to each subscript, so I am applying σ to each subscript, so this is σ * of this.

But what is this? So this is σ * of this. But what is this, this is exactly T * of product, the same product xi –x j right.

Because originally you had x i - x j, you apply T * to it and then you get this. But this, what is this? This is just f, so this is σ * of T * of f, remember this is exactly what I have claimed here, σ T * f is σ* of T * f, okay. So, I do not need this interior parenthesis, so just, okay, so in general I have this.

Online Editing and Post Production

Karthick

Ravichandaran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh kumar

Web Studio Team

Anitha

Bharathi

Catherine

Clifford

Deepthi

Dhivya

Divya

Gayathri

Gokulsekar

Halid

Hemavathy

Jagadeeshwaran

Jayanthi

Kamala

Lakshimipriya

Libin

Madhu

Maria Neeta

Mohana

Mohana Sundari

Muralikrishnan

Nivetha

Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Sorna

Subash

Suriyaprakash

Vinothini

Executive Producer

Kannan Krishnamurty

NPTEL Co-ordinates

Prof. Andrew Thangaraj

Prof. Prathap Haridoss

IIT Madras Production

Funded by

Department of Higher Education

Ministry of Human Resource Development

Government of India

HYPERLINK "http://www.mptel.ac.in"www.mptel.ac.in