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Introduction to Abstract

Group Theory

Module 06

Lecture 29 – "Symmetric groups IV"

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So we have seen so far, that any permutation is SN can be written as a disjoint product of cycles, and that is very useful, in particular, for example, you want to calculate the order of the permutation, because the order of permutation, will simply be as LCM of the orders, provided the cycles are disjoint. Next we are going to look at another decomposition for permutations, which is not disjoint, but it is useful in some other situations.

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So let us define a "transposition", a 2-cycle is called a "transposition", so 2-cycles are particularly simple permutations right, what is a 2-cycle? It is something like,(ij), right (ij) is a 2-cycle. What does it do as a permutation, it sends i to j, and j to i, and fixes everything else, and so permutations which are 2-cycles are easy, they just change from i to j, and j to i, and fix everything else, they are called transpositions, because they are just transposing Iiand j, so you are putting i in place of j, and j in place of i, we transpose them, and we do not disturb the other indices.

So the transpositions are easy to understand, and a transposition has order 2, the transposition has order 2 because it is a 2-cycle, so 2-cycles have order 2, in general k cycles have order k.

Now let us see, if we can write any element, let me write the proposition, and then we will look at some examples, and then prove that.

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Every permutation can be written as a product of transpositions or 2-cycles, but, the most important thing to remember is, not necessarily disjoint transpositions, so we can write it as a product of transpositions, but in order to achieve this, we have to give up disjointness, we cannot write disjoint as disjoint transpositions, so before I will proving this, before proving this which is actually not difficult at all, let us look at some examples.

So let us take σ to be (132) in S_5, so actually it can be S_3,S_4,S_5,S_6 and so as on, what does it do, it sends 1 to 3, and 3 to 2, 2 to 1, so it is a 3 cycle.

So remember that it cannot be written as a disjoint, we certainly know, σ cannot be written as a disjoint transpositions, so let me write this cannot it be written as a product of disjoint transpositions. Why can't it be written as product of disjoint transposition, suppose it can be written so if σ is σ_1, σ_2, σ_k, where σ_1 to σ_k are disjoint transpositions. So the reason for why we cannot write σ as a product of disjoint transpositions. If you can then the order of σ is lcm (order (σ_1) …. order(σ_k)), and because it is a product of disjoint cycles, but order of σ_1 is 2, σ_2 is 2 , σ_k is 2, because there all are transpositions.

So the order of σ is 2, the lcm of 2,2,2 any number of times, is 2, but σ is a 3-cycle, so order of σ is 3, by another proposition that we have proved sometime ago, order of σ is 3, because it is a 3-cycle, so if it can be written as a order of product disjoint 2-cycles, its order would be 2, whereas its order is 3, so cannot be written as product of disjoint transpositions. However, we can write it as product of transpositions, if we do not insist on disjointness, how do we do this?

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So (132), for example is it is easy to check, (12)(13), why is this? Because, you can check quickly, this sends 1 to 3, 3 to 2, and 2 to 1, what does this do, 1 to 3, under the 1st one, but then 3 to 3, under the 2nd one, which is good, where does 2 goes 2 , under the 1st one, and 2 goes to 2 , under the 2nd one, which is also good. Finally 3 goes to 1, and 1 goes to 2, so these are equal right, so (132) is written as a product of two transpositions, but they are not disjoint because, 1 is common to both, so they are not disjoint, as we know, you cannot write it as a product of disjoint transpositions.

I did not mention this earlier because we do not need it, but when you write a permutation as a product of disjoint cycles, that product or decomposition is unique, it is not at all difficult to show that, because they are disjoint cycles, you cannot have another product decomposition to disjoint cycles. However because we cannot have disjoint product of transpositions, this product into transpositions is not unique, okay. So (132) happens to be (13)(32), as you can quickly check.

What happens to 1 in this, 1 goes to 1 in the 1st one, 1 goes to 3 under the 2nd one, 2 goes to 3 under the 1st one, 3 goes to 1, under the 2nd one, 3 goes to 2, under the 1st one,2 goes to 2 under the 2nd one, so this is same as (132), right we have proved this. So this is a different decomposition, of the same element, same permutation, namely (132), as a product of transposition, okay. So certainly it is different because, it is (12) and (13) come here, (13) and (32), so they are different, transposition appearing in this, so these are different decomposition, but again it is not disjoint, because 3 is common in both.

So you might wonder, may be the decomposition is not unique, but the number in which, number of transposition is unique, here there are 2 transpositions, here also there are 2 transpositions but as another example, we will show, that even that number is not unique.

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So let us say, I take σ to be (12)(13)(14), so this is a product of 3 transpositions, but you can quickly check that, this is same as, I have checked this, you can do this as an exercise, (23)(25)(12)(45)(15), okay for example, let us randomly pick some element, what happen to let say 1 under both of this. 1 under this decomposition, goes to 4, then 4 goes to 4, and 4 goes to 4, so 1 goes to 4 under the 1st one, what happens to 1 under the 2nd one,1 goes to 5, 5 goes to 4, and 4 is not appearing there, so this is 1 goes to 4, under the same, so similarly you can check that, these two decompositions are the same.

So here you need 5 transpositions, so not only is the decomposition into product of transpositions is not unique, here the number of them is not unique, so we need 3, and we need 5, however, there is some uniqueness in this decomposition into a product of transpositions that I will come to you later, now let us come back to the proof of the proposition.

What do we have to show, we have to show that given any, permutation, in S_n, it can be written as a product of transpositions, first observe that, it suffices to show that a cycle σ can be written as a product of transpositions.

Why is this? So suppose I show that a cycle can be written as a product of transpositions, then why am I able to say that any permutations can be written as a product of transpositions? Because given any permutation, given any permutation σ , we can write σ as $\sigma_1\sigma_2\ldots\sigma_k$, where σ_1 to σ_k are what, disjoint cycles, here disjointness is not that important, but we can write it as a product of cycles, that we already know, we have proved this, any permutation, in S_n, it can be written as a product of cycles, now suppose if I prove that cycles can be written as a product of transpositions.

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So σ1 can be written as a product of, so we have σ_1, σ_2, σ_k, but σ_1 can be written as a product of transpositions, σ_2 can be written as a product of transpositions, so this is σ1, this is σ2 and so on, σ_k can be written as a product of transpositions, then we have written σ right remember we are not claiming anything about disjointness, in fact we cannot claim, there cannot be a decomposition into a product of disjoint transpositions, so we do not care if repeated things happen, σ1 can be written as a product of transposition so can σ 2 and so can σ k. So σ can also be written as a product of transpositions.

So we assume now, σ is a let's say k-cycle, so we only need not consider the case of k-cycles, so say $\sigma = (i_1, i_2 \ldots i_k)$.

So now if you just stare at this for a bit, you can see how to write this as a product of transpositions. So we can write, σ as so I am going to write it like this, so keep in mind that what σ does is sends i1 to i2, so let me put (i1, i2) here. I am going to write from left to right, but when we actually multiply remember, to figure out the image of an index we go from right to left, so if this is the last one, that is there, and if I make sure that i1 does not appear in the remaining ones, then i1 will go to i2 in under the product, so I will do (i2,i3), so now whatever I write here let's ensure that i2 does not appear in any of these, then in order to figure out, what is the image i2 under this product, we have to keep going from right, and we hit i2 here, so i2 will go to i3 under this, but i3 is not present in this, so i2 will go to i3, which is how it should be.

So now I will put (i3, i4), and I will keep doing this, (i(k-3),i(k-2)),(i(k-2),i(k-1)),(i(k-1),ik), okay. So I claim this, we can write like this, in fact we claim that, this is an equality, so let us check this as I said, it is very easy to see this, but let us check one by one, where does i1 go?

For the right hand side product, for the product on the right hand side, i1, remember to figure out where i1 goes we have to go from right, and keep applying i1 every time, we see.

So i1 is not present here, it fixes i1, so we do not care, i1 is not present here, i1 is not present in the 3rd one, from the right, it is not present in any of the 1st k-1, it is present only in 1st k-2, it is present only in the last 1, so we must send i1 to i2, right, σ also sends i1 to i2 remember. Now where does i2 go, we have to go from right and see where i2 first appears, and the 1st time i2 appears remember, the way we are writing this, i2 will not appear, until we hit (i2, i3).

So i2 will go to i3, right, is that clear, because i2 is not present, i2 is already written here in (i2, i3), so subsequent transpositions will only involve higher indices, so i2 will not appear there, i2 will go to i3 here, now possibly, we have to go left and see, but i3 does not appear in the previous one, so i2 will go to i3 under the product. What happens to i3, i3 will not appear, until we hit (i3, i4), and here it will go to i4, but then i4 no longer appears again, i4 does not appear here, so i3 goes to i4, and so on.

What happens to i(k-2)? i(k-2) goes to i(k-1), the 1st one on the right does not involve i(k-2), the 2nd one does, so i(k-2) goes to i(k-1), but i(k-1) does not appear again anywhere else, so i(k-2) goes to i(k-1), the point of this decomposition is, each index appears exactly twice, except i1 and ik, i2 appears here twice, i3 apears twice, and i(k-1) appears twice, but once i(k-2) appears here, only time i(k-1) next it is in the right hand side permutation, so it does not matter what happens previously, i(k-2) goes to i(k-1).

So the 1st one itself involves i(k-1), so we have to check ik, but then ik never appears again anywhere, so i(k-2) goes to i(k-1), so finally check ik. So ik goes to i(k-1) on the 1st thing, now i(k-1) appears again, so it sends i(k-1) to i(k-2), i(k-2) goes to i(k-3) here, and you have to keep tracing back, i(k-3) goes to i(k-4), i4 goes to i3, i3 goes to i2, i2 goes to i1, so in order to determine the image of ik, we start with the right most permutation, and keep tracing it back, all the way back to i1, so ik goes to i1, this is exactly equal to σ, this is exactly equal to σ.

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So we have shown that σ is (i1, i2)(i2,i3)(i3,i4)….(i(k-3),i(k-2))(i(k-2),i(k-1))(i(k-1),ik), and this is a product of transpositions, so we have shown that, give me any k-cycle, (i1, i2….ik), I can write it as a product of transpositions, and we have already said that any permutation is a product of cycles, and every cycle can be written as a product of transpositions, so any permutation can be written as a product of transpositions. This proves the proposition, and we have already looked at some examples.

So the important points to keep in mind is, first, you can write any permutation as a product of transposition. This product is not disjoint, and not unique, even the number of transpositions required is not unique, and in the next video, I am going to look at what kind of uniqueness we can get out of this decomposition into product of transpositions, thank you.

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