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Introduction to Abstract Group Theory

Module 05

Lecture 26- "Symmetric group 1"

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In the course so far we have studied many properties of groups, most recently we have looked at isomorphism theorems, quotient groups and as an application of these we have studied the Cauchey's theorem which says that if a finite group has order (N) and you have a prime number that divides (N) then there is an element of that prime order.

The next important topic in the course is symmetric groups; I am going to spend this and probably the next video studying symmetric groups. We already looked at symmetric groups as an example and we looked at one of these groups in detail. So what are symmetric groups. Lets recall, so let N be a positive integer and we denote the symmetric group on N letters by (SN) so SN denotes the symmetric groups on N letters, what does this mean? So this means the following: so we usually give the letters we call by 1 to N, so (S N) is the group of all bijections of an (N) element set. So you take a set with (N) elements and you consider all bijections of that set, usually we denote the set which has (N) elements, when I say an (N) element set I mean a set having (N) elements. It has (N) elements right, it is very natural to give names to these (N) elements, so let's call them 1, 2, 3 up to (N).

So, these are the (N) elements, do not think of them really numbers. It is sometimes useful to maybe think of them as numbers, but the point is there are (N) elements in that set which we are denoting for convenience as 1,2,3 up to (N). So we already are familiar with S_3, this is one of the groups that we have been looking at in various situations in we saw. How many elements it has and so on, so for example what is the order of S_3? 6 and we have various ways of writing this, so, I introduced in the very beginning one notation where I called the elements so maybe I called them f_1 through f_6 and this we thought is not very useful, it is not convenient because we do not know what f_1, f_2 are and we have to go back and see what they are and because of that I introduced a notation and we will discuss this in more detail today.

This notation that I introduced, so, I am going to spend this video explaining how to understand (S N) and what this notation is and what are its properties and we will consider some basic properties of the symmetric group.

(Refer Slide Time 04:26)

So symmetric groups, remember, just not one group, it is a class of groups, for every positive integer we have one group for example let's quickly settle what S1 and S2 are.

What would be S1? S1 is by definition bijections of a one element set, so and what are they, there is only one bijection right, there is only one bijection from a single element set, remember we call the element 1, there is no surprise here (1 has to go to 1), so it is the identity bijection. Remember E denotes the identity element of the symmetric group and as a function this stands for the identity map which is definitely a bijection. And I should remark here, of course, that when I say group of all bijections I mean under composition. I way back in one of the first videos we saw that bijections under compositions form a group.

So, the binary operation in this group is composition. Every bijection has an inverse, identity bijection serves as the identity element and composition of two bijections is a bijection, so this is a group under composition. So, S1 is the trivial group just containing one element.

What about S2? So we have to look at bijections from (1,2) to (1,2). So there is the identity bijection so (1 goes to 1, and 2 goes to 2), this is there always and we have another bijection. It sends (1 to 2) in order to be different from this 1 must go to some different from 1, so 1 goes to 2 and 2 goes to 1. So let's call this (12) following our notation I am going to recall this again for you later in the video (12) remember is the bijection that sends (1 to 2, 2to 1) and there are no other elements here, no other letters here.

So, it is just E and (1,2), and of course, S3 I have written here, is these six elements so, the first thing we want to understand is and of course S4 will have more elements. It will be bijections of (1,2,3,4) and S5 will be bijections of (1,2,3,4,5) and so on, so first thing is, first question let us settle this first, what is the order of (S N)?

So here we are looking at bijections, so it is a simple set-theoretic count. So (SN) is all bijections from (1, 2 up to N) to itself, so these are bijections, lets think for a minute on how to construct a bijection. Bijection is a function which is 1-1 and onto. So 1 has to go to something, 2 has to go to something, similarly 3 has to go to something, finally (N) has to go to something. First let's see how many options there are for 1, there are (N) options, because 1 can go to 1, 2, and 3 or (N), it can go to any of the (N) letters, so there are (N) options. Now how many option are there for 2, now for 2, remember in order to be a bijection 1and 2 be distinct letters must map to distinct letters, so, we have made a choice for 1 in the beginning, 2 cannot go to that, if 1 goes to 3, for example 2 cannot go to 3, but 2 can go to anything else, so you remove wherever 1 has been sent, and 2 can go to any of the remaining ones, that means there (N-1) options, on where 2 can go, 3 now can go to any of the letters which are different from images (1 and 2) so 3 as (N-2) options, 4 as (N-3) options, (N) finally after you do all this, there is only one option for this, and this is the principle at work if you remember how we worked out S3, S1, S2.

So, you can send 1, so (1,2,3 up to N) is an arbitrary order, we fix them in some way, which order is irrelevant, it is convenient to think of 1 as the first element, 2 as the second element, and so on. So 1 goes to any of the (N) options, 2 goes to any of the remaining (N-1) options, 3 goes to any of the remaining (N-2) options, (N) goes to after having decided the images of 1,2,3 up to (N-1), we have exhausted (N-1) elements of the set 1,2 up to (N), so there is only one option left for (N). Now how many there total? So this is the principal in counting if how many total number of bijection are there.

So the total number of bijections, because 1 can go to (N) of them, it is (N) times 2 can go to (N-1) of them, 3 can go to (N-2) of them, (N-1) can go to 2 of them, so this is the product so, (N-11) times (N times N-1 times N-2 all the way up to 2 times 1) and usually we call this by (N factorial). Read this as (N factorial), so the total number of bijection is (N factorial) because where you send 1 is independent of where you send 2, apart from making sure that 2 has to go to something that is not already the image of 1, once you keep that in mind, 2 can go to anything, so you can fix one of the options for 1 and you can construct a function, so now change the option for 1 and construct same number of functions and so on, so the total number of bijections is (N factorial). Hence the order of (S N) is (N factorial).

So this is the first important property of (S N). (S N) is a group of order (N factorial), and clearly check that this checks out in cases 1, 2, and 3, S1 should be one factorial, which is 1, and that is correct, S2 should be 2 factorial which is 2, which is also correct, S3 order must be 3 factorial which is 6, which is correct, so (S N) is a group of order (N factorial). So the size is increasing, it gets very big very fast, for example S4 as order 24, S5 as order 120, S6 as order 720 and so on. So these orders increase very fast because each time you multiplying by (N) so this is the size of the group.

Now how do we express elements of (S N), so this is the next question, how do we express elements of (S N)?

So there are several ways of expressing elements of (S N), we want to choose one which is most convenient in the sense that it is efficient we do not need to write lot of text and it is easy to work with in terms of multiplication of , in terms of figuring out whether an element is in a subgroup, things liKe that. So how do we express elements? So I want to introduce to you and this is actually introduced to you when we discussed S3 earlier, but I will do this again in more systematically. We use cycle notation. So hopefully this is already familiar to you from the time we discussed S3, so and in any case I will discuss all the details again. So what is the cycle notation?

(Refer Slide Time 14:02)

So it is best explained in terms of, with an example. So let's look at, for example, let's start with this. So before I do the example, just a notation so that I can keep using this, elements of (S N), are called, are usually, permutations, instead of using the word bijection, lets use the word permutation. Permutation is just another word really for bijection, it is about, if you take (N) elements, how do you permute them how do you map those (N) elements to themselves again, so that is really a bijection, but this is more mathematical term so we use the word permutation.

So let's take a permutation and I am going to denote this by sigma, and I am going to use these Greek letters, so sigma is an element of S9, so it is a permutation on (9) letters, so what is this? Lets use one particular way of writing this, so let's use this notation, so let's use this as a matrix, represent this as a matrix.

(Refer Slide Time: 19:52)

Where the first row as all the nine elements, of this set (1,2,3 up to 9), so sigma is a permutation of it, so sigma sends (1 to 3), lets do this (2 to 4). So the top row represents the letters the bottom row represents where they are mapped under sigma. So sigma maps (1 to 3), (2 to 4), (3 to 2), (4 to 1), (5 to 6), (6 to 5), (7 to 8), (8 to 7), (9 to 9), so this sends for example (3 to 2), (6 to 5) and so on, I hope the notation is clear. 1 goes to 3, 2 goes to 4, and so on, so clearly this is nice, clearly it tells you what element sigma does, in the sense that what is the function, remember elements of (S N) which I am calling permutation are really functions.

They are functions from (1,2,3 to N) to (1,2,3 up to N), so in that way sigma is clear, what as a function it does is clear, (1 goes to 3, 2 goes to 4) but certainly you can see that, there is some inefficiency in expressing in sigma, for example the top row is unnecessary. Because we know that it is always 1, 2, 3 up to N. So but if you remove that it is somewhat difficult to keep track of what the function is, so now rewrite this in the following way.

Rewrite sigma in the following way, so what is the way that we want to think of sigma is the following. So let's start with 1, so I can start with anything, let's start with 1, see where it goes under sigma. So I am starting with 1, see where it goes, it goes to 3, and see and repeat till, so let me write that first, so see where it goes under sigma, put it next to, and repeat until we come to 1 again, so that is what we do.

So let's illustrate this, we will start with 1, see where it goes it goes to 3, repeat with 3, so see 3 goes to 2, it is right there, the function sends 3 to 2. We have not come to 1 again, so we will continue. 2 goes to 4 under this, again we have not reached 1, so continue. Where does 4 go? 4 goes to 1, so 1 we have come again.

So we close the bracket. So this is the, we call this, the cycle determined by 1. We call this the cycle determined by 1. Again let me repeat the procedure. We start with 1, see where it goes, put it next to it, so 3, see where 3 goes, it goes to 2, put 2 next to 3, where does 2 go, it goes to 4, so put 4 next to 2, and 4 goes to 1, so you close the bracket. You should think of this as a cycle. So because you start with 1, you go clockwise, go to 3, then you go to 2, then you go to 4, then you come back to 1, okay.

Of course, this whole thing can start with 2 also. If we started with 2, then what happens? See where 2 goes, 2 goes to 4, 4 goes to 1, 1 goes to 3, 3 goes to 2, so we close the bracket. It is the same cycle. On a circle it doesn't matter where you start, right.

So instead of 1 I will start here, so that is (2, 4, 1, 3), if I started with 3 it will be (3, 2, 4, 1), if I started with 4 it will (4, 1, 3, 2), so this is one cycle, sigma is not just this.

(Refer Slide Time: 20:16)

So now in the next step, this is the step 1, step 2, find an index, if, that is not contained in the cycle determined by 1, find an index, so let us call 1, 2, up to N indices, okay so the set is 1, 2, 3,…N, the elements of it are called indices, elements of the symmetric group are called permutations, so permutations permute indices.

So find an index is not contained in the cycle determined by 1, so 1, 3, 2, 4, are contained in the cycle, let's pick 5. You can also chose 6 and also 7, it does not matter, so I am just choosing 5, now find the cycle determined by 5. So in general, we have learned in the step 1, how to find the cycle determined by an index, in this case let us do it for 5.

So you put a bracket and start with 5, under the function where does 5 go, 5 goes to 6, now where does 6 go? 6 goes to 5, so I close the bracket, so I get (5 6), so it is a cycle, so 5 goes to 6 and 6 goes to 5.

So next, you see now what happens, find an index not contained in cycles constructed so far, remember so far we constructed two cycles, (1, 2, 3, 4) and (5, 6), so find an index that is not contained so far, so let us choose 7, and determine the cycle, find the cycle determined by 7. It would be 7 and in the function, what is the image of seven, 7 goes to 8, so 7 goes to 8 and 8 goes to 7, so you take (7, 8) and you close the bracket, this is the third cycle.

(Refer Slide Time: 23:09)

So this is the third cycle, and last cycle will be, find a cycle, so again is the same idea now, find an index not covered so far, which is only 9, 9 is the only index that is not covered so far, and the cycle determined would be in the function, 9 goes to 9, so you just use the index, so you have only 9 in this cycle so, you have the single element so, we say that, we can write sigma equals (1324) was the first cycle, (56) was the next cycle, (78) is the third cycle and, (9) is the last cycle. We usually do not write a cycle, if it has only one index okay, because if it has one index, it is clear that that index will goes to itself, so we usually omit writing 9, so we write as sigma as (1324), (56), (78), so the same function, the same function we have has this cycle decomposition.

(Refer Slide Time: 24:41)

This is called the cycle decomposition of sigma, so what are we doing? We are decomposing sigma as cycles, so a cycle, first of all, is an element, which starts somewhere and ends with that, so 1 goes to 3 in the cycle 3 goes to 2, 2 goes to 4, so cycles are permutations of the form, so now let me introduce a general notation here okay, so this is a cycle, it has K in this cycle, so this is a K-cycle, so we call it as K cycle, what is a K-cycle? It is a cycle with K indices, what does this K-cycle do? What does the K-cycle (i1, i2, i3,…, iK) do?

(Refer Slide Time: 26:16)

It sends i1 to i2, i2 to i3, i3 to i4 right, you understand, i1 the first thing goes to i2, which is right next to i2, i2 goes to i3, which is right next to i3, right next i3 is i4 so, i3 goes to i4 and so on, i(K-1_ goes to iK, because i(k-1) comes just iK, so, i(K-1) goes to iK, and finally where does iK go? Because it is a cycle, you cycle back there is nothing next to iK means, you send iK to i1, so as a cycle you have i1, then i2 always clockwise i4, i5, …, i(K-2), i(K-1), iK, right, so i1 goes to i2, i2 goes to i3, i5 goes to i6, i6 goes to i7, i(K-1) goes to iK, i(K-2) goes to i(K-1), i(K-1) goes to iK, iK goes to i1 so this clearly says that, this same cycle can be written as (i2,i3,i(K-1),iK,i1) right, so start with i2 and end with i1, and you can write them with any of these options, you can start with i3, you can choose any of the k indices as the starting point.

(Refer Slide Time: 28:00)

But now the description of the cycle is not done, I have only told you what happens to these indices, further I need to tell you what happens to indices that are not in this set, if j is an index not contained in { i1, i2, …..iK}, see i1, i2,…., iK are just some indices, which are, remember this is a subset of, so if j is an index not contained in this i1, i2,…,iK, what does the cycle { i1, i2, …..iK} do to j, it sends j to j, it does not do any thing to j, now I have completely described the cycle.

(Refer Slide Time: 28:59)

As an example consider the cycle, let us say (1, 2, 3) in S6 what does it do, as a function (1, 2, 3) is given by, S6 is the symmetric group of 6 letters.

(Refer Slide Time 29:36)

So I need to tell you what is the image of those indices under (1, 2, 3). 1 goes to 2, 2 goes to 3, 3 goes to 1, so all the indices contained in the cycle behave this way, and all the indices that are not contained in the cycle are fixed okay, so we say (1, 2, 3) fixes 4, 5 and 6.

So more generally the cycle (i1, i2, …,iK) fixes any index that is not contained in i1, i2, …., iK okay. So now going back to the example we worked out here, this particular permutation sigma can be written as a product of cycles (1, 3, 2, 4), (5, 6) and (7, 8).

(Refer Slide Time: 30:36)

So in our example above, sigma which was in S9, can be written as a product, okay, of disjoint cycles, okay, let me explain this in a minute, in the following way: (1, 3, 2, 4) (5, 6) (7, 8). These are disjoint cycles because they have no common indices, the indices that (1, 3, 2, 4) contains are 1, 3, 2, 4, (5, 6) contains 5, 6, (7, 8) contains 7 and 8, so there is no common index, so they are disjoint and they are cycles. So sigma can be written as a product of disjoint cycles.

(Refer Slide Time: 32:19)

Okay now if you think about the procedure of how we did this for this particular sigma, hopefully it will be clear to you that we can to this for any sigma, the same procedure that we used for sigma in S9, can be applied to any element in any symmetric group, so we can conclude with a proposition.

(Refer Slide Time: 32:51)

Every element sigma in a symmetric group has, a cycle, has a decomposition as a product of disjoint, right, remember that this clear, because you start with sigma, ask where 1 goes, I am just recapping the procedure that we followed, start with 1, determine the cycle determined by 1, start with 1, see where it goes, you put it next to 1, see where that goes, put it next to that and so on. At some point obviously you will reach 1, because the SN is the symmetric group on N letters, only N indices are there, so you cannot keep forever going without reaching one. So at some point we must reach 1, then you close the bracket, and that is the cycle determined by one, may be you exhausted all the indices, then you have a 9-cycle, all the 9 indices are in the cycle, or maybe you have not exhausted all N indices, N is in general now. So it is possible that you have all N indices in 1 cycle then you are done, element itself is an N-cycle.

Otherwise there is an index that is not contained not in the cycle by determined by 1, then you start with that, and see the cycle determined by that, and you keep doing this, because it is all on N letters, you must have at some point exhausted all the indices, so you are done. So every element of SN has a decomposition as a product of disjoint cycles.

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