NPTEL

NPTEL ONLINE COURSE

Introduction to Abstract

Group Theory

Module 04

Lecture 23-"Third isomorphism theorem"

PROF.KRISHNA HANUMANTHU

CHENNAI MATHEMATICAL INSTITUTE

So far we have looked at first and second isomorphism theorems, in this video let us start with the third isomorphism theorem .

(Refer Slide Time: 00:31)

Recall that isomorphism theorems are statements about isomorphisms among quotient groups, okay, the third isomorphism we will say similar statement to the first and second, so let us see what it says. Let G be a group okay let H, N be two normal subgroups of G, okay, in the second isomorphism theorem we worked with two groups one normal, and the other arbitrary, in this case we take both to be normal H and N are both normal subgroups of G. Then we have two statements, actually I should write let H and let H and N be normal groups of G such that N is contained in H and H is contained G of course. So I am taking two subgroups one containing the other, N is contained in H and both are normal, N and H are both normal in G, then we have two statements.

Then G then H/N, because N is an subgroup of H I can consider H/N, remember N is a normal subgroup of G so it certainly a normal subgroup of H, I want to say H/N is a normal subgroup of G/N. So we will explain this, so H/N first of all I have to think of it as a subgroup of G mod N and then I want to further say it is a normal subgroup of G mod N. And the second statement which is actually what we called third isomorphism theorem is that G/H is isomorphic to G/N mod H/N, okay. So this migHt seem ugly here but the point is H/N is a normal subgroup of G/N by the statement one.

So I can consider the quotient group G mod N modulo H mod N, that is same as G/H. This is a very useful theorem actually, third isomorphism theorem, along with the first isomorphism theorem, they both get used a lot. Here think of this as a sort of canceling, G/N mod H/N, is like almost like when you write ratios of numbers like integers you can cancel, something like that you can do here, that is why this is very useful to write it like this, G/N mod H/N is isomorphic to G/H, that canceling is valid is what the third isomorphism theorem says.

So first of all let us tackle (1). We want to show that H/N is normal subgroup of G/N, in order to do that we first want even think of H/N as a subgroup of G/N, before we even ask if it is a normal subgroup, is it a group subgroup of G/N? Note that we have H is a subgroup of G, so we have an inclusion H in G rigHt.

So this inclusion is simply sending small H to small H. We also have a natural homomorphism G to G/N, so let us put these two things together, we have H to G which is a natural inclusion, so in particular it is an injective map, it is a 1-1 map and then we have the natural surjective map from G to G/N, remember when I talked about the quotient groups I always defined this natural map that exists always g goes to gN, this is a group homomorphism that is the second map G to G/ N.

(Refer Slide Time: 5:25)

So considerer the composition, this composition, what is a composition? H to G/N rigHt but it is actually going via G, so first you send it to G then you send it to the G/N, so this is really nothing but small h going to hN because under this composition here you see that under the first map you take a small h it goes to small h under the first map, in the second map takes small h to small hN so under the composition small h goes to small hN okay.

(Refer Slide Time: 07:34)

What is the kernel of this? Kernel of this map is {h in H| hN=N}. Remember N is the identity element of G/N, identity coset is the identity element, but as before this is {H in H/H in N} in the second isomorphism theorem also this came up, so it is h in H but it is also in N, this of course is H intersection N, but in the new hypothesis remember N is also a subgroup of H, so this is just N, so kernel of this map is N.

So by the first isomorphism theorem, H/N is isomorphic to the image of this map, I don't care what it is, it is a subgroup of G/N is all care about, is isomorphic to a subgroup of G/N and as I have remarked in a previous video after I did the first isomorphism theorem, if you have a situation like this, the technically statement is H/N is isomorphic to a subgroup of G/N, but we can think of as a subgroup of G/N.

So we think of H/N as a subgroup of G/N, we are claiming in the first part of the theorem that H/N is a normal subgroup of G/N. I have justified that it is a subgroup of G/N, now we have to show that it is a normal subgroup. What is normal subgroup? So we have H/N inside G/N, normality means what? You take something from the bigger group something from the smaller group and consider the element by taking inverse and product.

So what is an element of G/N it is of the form gN, what is an element of H/N, it is an element of this form hN, because these are left cosets of N in H, so g is in G, and h is in H. And what do we consider? We will consider this (gN)(hN)(gN) inverse. Lets understand this element, what is this, this is really a operation of cosets, but we have worked with cosets now in several videos, this hopefully it will be clear to you, it is easy to see that this is (gN)(hN), remember in the quotient group G/N the inverse of the coset gN is simply given by g inverse N, this is also something we have shown, while proving that G/N is a group when N is a normal subgroup in G, gN whole inverse is g inverse N okay.

(Refer Slide Time: 13:12)

So now let's write it like this, so we have (gN) (hN) (g inverse N), so we have this because of associativity we can rearrange our brackets, so I am going to write basically I am getting rid of this bracket and write it like this, let us look at this part N, g inverse. In some previous video I commented that if N is normal so note that if N is normal in G then gN =Ng for all g in G, rigHt. So this means that Ng inverse would be equal to g inverse N, so gN, H, g inverse NN, so I am switching Ng inverse writing it as g inverse N but this gN h h inverse and N N is N, this is a property of left cosets, products really of subgroups.

(Refer Slide Time: 12:14)

So this is N, but now I can, using the same property that gN = Ng, pull h and g on this side, so this will be gH N g inverse N, which is same as I am interchanging N and h here, now I will do Ng inverse. So gH, g inverse NN which is of course again gH g inverse N okay, so thougH this seems somewhat magical here some of you migHt worried about whether all this is valid or not, you have to convince yourself that this is valid because all I am doing here is multiplying subsets of group and multiplying subsets of a group follows the same rules as multiplying elements of that group and we are using here the properties of normal subgroups, if N is a subgroup then NN equals N and everything else is correct, so all this is a valid operation okay.

(Refer Slide Time: 13:10)

Let us now take stock, where are we, gN hN g inverse N is equal to gHg inverse N. Now I want to use the normality of H in G, gH g inverse belongs to H rigHt. Because g is in capital G h is in capital H so gH, g inverse is in, g is in G h is in H, so this is in H, so gH, g inverse N is in H mod N, because this is in H this is a left coset of N in H, which is by definition an element of H mod N. So now we have proved that H mod N is a normal subgroup of G mod N, we have started with an arbitrary element of G mod N , and an arbitrary element of H mod N and looked at gN hN gN inverse and concluded that it is in H/N, which is the definition of H mod N being normal in G mod N. So this is (1). (1) is exactly that statement and it is proved now. So let us prove 2 now.

(Refer Slide Time: 14:29)

2 is the statement that G mod H is isomorphic to, we will show G mod N, G mod H is isomorphic to G mod N mod H mod N. So how do we show this? So again whenever you have a statement that you have to prove which involves proving some quotient groups are isomorphic to each other, think of constructing appropriate group homomorphisms, and invoke first isomorphism theorem.

As we have seen in the proof of the second isomorphism theorem, it is really an application of the first isomorphism theorem. And here also it will be an application of the first isomorphism theorem. The map to consider here is the following. Let me consider this map from G mod N to H mod N. What is this map? I will take a coset, element of G mod N which is a coset of capital N in capital G.

So I will take a small g in capital G, so this is not the map to H mod N, this is the map to G mod H. So take a small g in capital G, consider the coset gN and I will map it to gH. So this is the map gN goes to gH, certainly gH is an element of G mod H because it is a coset of capital H in capital G. So φ must be well-defined in order to be a group homomorphism. Before that you have to check well definedness, why do we need to check well definedness here, because we can have gN equal to g prime N and this can happen for two different elements of the group G even if they are different gN could be equal to g prime N.

Suppose this happens, we want to check that φ gN equal φ g prime N. otherwise it is not well defined, you have one element which has two different representations and using the representations if you send it to different elements, it is certainly not a well defined map, so we want to check that gH is equal to g prime H. But if gN is equal to g prime N remember this means that, g inverse times g prime is in N.

This is because you can multiply by g inverse so N will be equal to g inverse g prime N, and that means g inverse g prime will be in capital N. This kind of thing we have seen before, if g inverse g prime is in N, then g inverse g prime is in H, because N is contained in H, by hypothesis N is contained in H. So we have this, but if g inverse g prime is in H, gH equals g prime H.So well definedness is okay.

If you have two representations of the same coset, they will also give you the same coset in G mod H. So if gN equals g prime N, gH equals g prime H. So this is okay, and φ is a group homomorphism, this is easy because what is φ of gN times g prime N. This φ of g g prime N, this is by definition g, g prime H. And this is equal to gH g prime H, and this is φ of gN, but so φ is a group homomorphism. I am checking every detail here, though it is not difficult to check them. What is the image of φ,

(Refer Slide Time: 18:48)

I claim φ is onto. Why? φ is an map from G mod N to G mod H. So let what is an element of G mod H, an arbitrary element of G mod H is a left coset, it is gH. Then φ of gN is gH, so I take the same element small g , gN maps to gH. So φ is onto, this is trivial, you take a coset is is certainly the same element times N and maps to it. So it is onto. What is kernel of φ? This is what we have to check now, what is kernel of φ?

So kernel of φ is by definition all elements of G mod N, it is all elements of G mod N, which are denoted by g capital N and remember these are cosets, all elements of G mod N, such that φ of gN is equal to the identity coset.

Identity element of G mod H is H. So this is equal to that, but this is gN in G mod N, gN in G mod N, such that gH equals to H. Remember φ of gN is by definition gH. So φ of gN is equal to H means, gH equal to H. That means gN is contained in G mod N such that g is in H. So these are cosets of the form gN in G mod N such that the small g is in capital H.

But what is this, if you think for a second about this, these are left cosets of capital N in capital G, which are also left cosets of capital N in capital H, because this g must be in capital H. So this are just in H/N. H mod N remember is hN, h is in H. H mod N is the cosets of capital N in capital H, so that are of the form hN, small h in capital H. But this is really what we are doing here, we are taking all g in H and taking G mod N, gN sorry. All gN instead of calling it small h I am calling it small g here.

So if we are taking small g in H and taking gN, so this is exactly H mod N, so kernel of φ is H mod N.

Now what is the situation, φ is a map from G mod N to G mod H, it is a group homomorphism, onto, kernel equal to H mod N. Now what does the first isomorphism theorem say, again it is the final point is the consequence of first isomorphism theorem, you have a group homomorphism which is onto and kernel is H mod N.

So you have G mod N which is the domain group moduo the kernel which is H mod N, I have computed that, is isomorphic to, remember G to G prime is a group homomorphism means, G mod kernel is isomorphic to image, so I am applying it to G mod N to G mod H, it is a group homomorphism, so G mod N modulo kernel which is H mod N isomorphic to the image which is G mod H, because it is onto and this is exactly what we want to show, G mod N mod H mod N, is G mod H.

So we have finished the proof. So this is exactly the isomorphism that we have claimed and we have proved this. So this completes the proof of the third isomorphism theorem. So together this first, second, third isomorphism theorems are very important and as we have seen in the proofs it is clear that the crucial observation came from the first isomorphism theorem.

Then it is just looking at more and more special cases to get second and third isomorphism theorems and these are used frequently in many theorems and in future classes we will see this, so it is important that you carefully work out the proofs and make sure that you understand all the details. So I will stop here, in the next video I will look at some more applications of these quotient groups. Thank you.

Online Editing and Post Production

Karthick

Ravichandaran

Mohanarangan

Sribalaji

Komathi

VigNesh

Mahesh kumar

Web Studio Team

Anitha

Bharathi

Catherine

Clifford

Deepthi

Dhivya

Divya

Gayathri

Gokulsekar

Halid

Hemavathy

Jagadeeshwaran

Jayanthi

Kamala

Lakshimipriya

Libin

Madhu

Maria Neeta

Mohana

Mohana Sundari

Muralikrishnan

Nivetha

Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Sorna

Subash

Suriyaprakash

Vinothini

Executive Producer

Kannan Krishnamurty

NPTEL Co-ordinates

Prof. Andrew Thangaraj

Prof. Prathap Haridoss

IIT Madras Production

Funded by

Department of HigHer Education

Ministry of Human Resource Development

Government of India

HYPERLINK "http://www.mptel.ac.in"www.mptel.ac.in