NPTEL

ONLINE COURSE

Introduction to Abstract

Group Theory

Module 04

Lecture 22-

Examples and Second isomorphism Theorem

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In the previous video we looked at the first isomorphism theorem and some examples of it, so I am going to before continuing today, I will recall for you what is the first isomorphism theorem.

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First isomorphism theorem, it says that, let phi from G to G prime be a group homomorphism, in this case, then we have an isomorphism from G mod kernel phi to image of phi. So this is the statement of first isomorphism there, it tells that if you go modulo the kernel, you get a group that is isomorphic to the image group, remember kernel is always normal subgroup, so G mod kernel phi has a group structure, it is isomorphic to the image of the group, image of the homomorphism.

So image of phi is a subgroup of G prime okay, and we saw one example on this, we saw that if G is a cyclic group of order N, G is a cyclic group containing N elements, then we saw that G is isomorphic it to Z mod NZ, so any cyclic group of the form Z mod NZ, and this was an immediate corollary of first isomorphism theorem, because we look at the homomorphism from the integers to the group G sending 1 to the generator of the group G, so then kernel will be precisely NZ, it is an onto map, so there will be isomorphism like this.

Let us look at another example of first isomorphism theorem, so recall that C* is the group of nonzero complex numbers with multiplication, these form a group under multiplication, and let's denote R+ to be positive real numbers also with the multiplication.

So we saw in previous videos, these are groups under multiplication. Consider the map, consider the function, let's say phi from C star to R+ which sends a complex number to its absolute value. So if you recall, absolute value is simply this function, A+IB is a complex number, so A and B are real numbers, I is a √ -1, you send it to $A^2 + B^2$, that is this function, absolute value of A+IB is A^2+B^2 , so in order to see what the first isomorphism theorem has to say about this, let us look at the kernel of this map, so what is kernel phi? By definition complex numbers with absolute value 1 okay,

So if you represent complex numbers in the plane, so this the R real axis, this is the imaginary axis right, so this is the kernel phi, it is simply the unit circle, it denoted by S1, this is the set of elements, complex numbers, which have absolute value 1, the unit circle around the origin, that is the kernel phi. What is the image of phi? So image, remember phi is a function from C star to R+ positive real numbers, if R is a positive real number, then absolute value of R is simply R, so phi is actually onto, in other words, the image of phi is all of R+.

So what does the first isomorphism, now say? The first isomorphism theorem gives C* mod S1 is isomorphic to R+ okay, so the first isomorphism says C*/S1 is isomorphic to R+ okay, you should spend some time to think about this example, and this will hopefully help you with understanding quotient groups, so this as groups, C*/S1 is the set of left cosets right, what are left cosets?

These are left cosets of S1 is C*, these are the form {Z S1, Z € C * }. That is simply the set, right, and on this there is a group structure, remember of course that C star is an abelian group, if every subgroup is normal and in any case S1 is a kernel of a group homomorphism, normal, so think of this, the groups C*/S1, as it is the left cosets of S1, but think of them as circles of varying radii okay, so you have radius 1 circle.

That is S1, radius 2 circle, radius 3 circle, radius 1.3 circle, so and if you think about this, if you take all such circles radius will be a positive real number, you take all circles of positive radius, the cosets can be represented by those positive reals and hence we have an isomorphism like this, because if you have all cosets, they are represented by simply the radius, cosets are simply circles of different radius, so they are represented by positive real numbers, so that is a slightly intuitive explanation of this isomorphism, okay.

One more example let us do, this is something that you have seen before.

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Remember GLn (R) is the group of invertible n by n real matrices. Consider the map from GLn (R) to R*, nonzero reals, the determinant map. So a matrix A goes to determinant of (A), since A is invertible, determinant of A is nonzero, it is a real number, so it is in R * it is a certainly group homomorphism, that we have remarked earlier. det is clearly onto also, because if you take any real number, nonzero real number, you can construct a matrix with that determinant, for an example you can just take a diagonal matrices with that number being one of the entries and all the diagonal entries being 1.

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So this is easily seen to be onto. What is the kernel of det? This is the group of matrices in GLn (R) which have determinant 1, we called this SLn (R), in an earlier video. So the first isomorphism theorem gives GLn (R) /SLn(R) os isomorphic to R star, okay, so this is also a consequence of first isomorphism theorem.

So in general if you want to prove statements about quotients groups, or solve problems involving quotient group and isomorphisms involving quotient groups, immediately you should think of first isomorphism theorem, and the other two isomorphism theorems I will do later in this video. One remark I will make before I continue with second isomorphism theorem.

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Let us say that G to G prime is a group homomorphism. Then G/kernel of phi is isomorphic to a subgroup, this is exactly statement of, this is an immediate consequence of the first isomorphism theorem, because first isomorphism theorem says G/ kernel phi is isomorphic to the image of phi, which is certainly a subgroup of G prime, so G/kernel phi is isomorphic to a subgroup of G^1 and hence G/kernel phi can be identified with a subgroup of G^1.

So it is for all practical purposes, we will take it as subgroup of G^1 so we usually think of G/kernel phi as a subgroup of G^1, so sometimes is useful to think of G mod kernel of phi as a subgroup of G prime. In particular if kernel phi ={e}, which is to say phi is 1-1, then G can be thought of as a subgroup of G^1, if G to G^1 is (1-1) group of isomorphism that mean it is injective, G naturally can be thought of as a subgroup, because its image is isomorphic to G, by the first isomorphism theorem. We can replace by G by its image and think of G itself as a subgroup of G^1 okay.

Now that we have looked various examples and a remark about first isomorphism theorem, let us go ahead and do the remaining two isomorphism theorems. So the second isomorphism theorem says the following.

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Let G be a group, okay, I am going to consider two subgroups of G, okay let H and N be two subgroups of G, assume that N is normal in G okay, I am not assuming anything about H, other than that it is a subgroup but I am assuming that N is actually a normal subgroup of G, then the second isomorphism theorem says that H intersection N, so it says several things, let me may be I should write the separate statements.

Then H intersection N is a normal subgroup of H, the product set HN is a subgroup of G and N is a normal subgroup of HN, and finally in (3) we have H modulo H intersection N, by (1) H intersection N is a normal subgroup of H, so this is a group by (2) N is a normal subgroup of HN, so I can consider this okay, So H mod H intersection N is same as HN mod N.

Whenever you see in books statements about second isomorphism theorem, this is what we usually refer to as second isomorphism theorem, the statements (1) and (2) I wrote here are the simply needed to makes sense of the third statements right, because H intersection N needs to be normal subgroup of H, to talk about the left hand side group, similarly HN needs to be group, and N needs to be a normal subgroup of HN, in order to makes an sense of right hand side as a group, so this is again group isomorphism.

I have been using this symbol to always denote group isomorphism, okay. Let us prove this quickly, it is not difficult at all. Okay statement (1) we know that H intersection N, so what is the meaning, so certainly H intersection N is contained in H, it is a subgroup obviously, because H intersection N being an intersections of groups it is a subgroup this is a very exercise because identity is in H and it is in N, so it is in H intersection and N, if the two thing are in intersections their both in H so the product in H they are both in N, so their product is in N. So their product is also in the intersection, so this is a very exercise.

In order to check that it is a normal subgroup what do we have to check? So let us choose h in capital H and n in H intersections N okay, so this we need to check that we need to check that h n h inverse is in H intersection N right, because this is the definition of the normal subgroup.

So recall we say that H is normal in G, definition, if g is in G h is in H then g h g inverse is in H. So you take something from the bigger group and something from the smaller group and consider this element, it must be in the smaller group. Here we are interested in proving that H intersection N is normal in H so we want to take something from H something from the smaller group and consider this element and show that it is in the smaller group okay. First of all note that h n h inverse is in N.

Because h is in G and n is in N and N is normal in G, right . So this is because N is normal in G, that is hypothesis so we are given that N is a normal subgroup of G so you take something in H and it obviously in G because H is a subgroup of G, so small h is in the group G, small n is in group N and the element h n h inverse will be in N. On the other hand, h n h inverse is also in H, because h is in H, n is in H intersection N which is contained in H, so h is in H, n is in H and H is a subgroup. Note that because H is a subgroup and small h is in it and small n is in it, small h, small n small h inverse will be in H, so these two thing together imply that h n h inverse is in H intersection N. So as we needed so (1) is proved because H intersection N is a normal subgroup of capital H.

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Let us prove (2). (2) remember says two things, two statement are made in (2), it says that, it say that, first of all it says that capital H times capital N is a subgroup of G and that capital N is a normal subgroup of capital HN.

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So first of all what is capital HN? This I defined in an earlier video, HN and in general for any two subsets of a group the product is simply this set, you take elements from the first set and elements from the second set and multiply in that order, so you take small h from capital H small n from capital N and you multiply them and you vary small h and small n okay so why is this, so this is certainly a subset of G.

So why is this a subgroup of G? So it is important to note here that if you take two subsets their product is certainly not in general a subgroup. In fact even you if take two subgroups their product is in general not a subgroup. However if one of them happens to be normal it is a subgroup okay let us check one by one the required properties. E belong to H and E belong to N certainly that is because both of them are subgroups.

So E times E which is in HN, remember in order to be subgroup capital HN must contain the identity element, it must be closed under multiplication, it must be closed under taking inverses. We have checked that it closed it contains the identity. Let us take now two elements of the set HN, so H1 N1, H2 N2 let say these are two elements are HN, we want to prove H1 N1 times H2 N2 is in capital HN again.

But here we have to use normality of capital N, so note that N is normal G, this implies (H2 inverse N1 H2) is in N because H2 is an element of the group G and N1 is in capital N, so this element is in capital N, so this elements is in capital N, so say (H2 inverse N1 H 2) is equal to some N3, it is equal to some element of capital N, call it N3. This gives me (N1 H2) is (H2 N3). I am simply multiplying by H2 to the left on both sides. Now what is (H1 N1 H2 N2)? This is equal to, because I do not, remember because of associativity I do not to be careful about putting brackets. N1 H2 is equal to H2 N3, by that calculations so this is H2 N3 N2. This can be written as (H1H2 N3 N2), now H1 H2 are in capital H so this is in capital H product is also in capital H similarly this is in capital N, so this is in HN. So we have said that two things in capital HN means their product is also in HN.

Finally, how do you show inverses? It is the same idea, if you have HN in capital HN, what is HN inverse? This is N inverse H inverse okay, by what we have shown here we have shown that (N1 H2) is (H2 N3), so here what we have really shown is that anything in N times anything in H is you can switch the two you can replace you can bring the H2 to the front but on the right now there will be some element of N, so this by the same calculation as above equal to H inverse times some N1, by the same calculation as above, okay and this is in capital HN. So the point that I want to highlight here and this the point that we used above.

If N is in normal in capital G then for g in G, gN is equal to Ng. See elements of N do not necessarily commute with a fixed element but as a coset they commute, this is something that we have seen before also but this is exactly the calculation that we have done here. So something in N times G, G times something else in N so this is the point that I would like you to remember. So this is what we have used, so HN is a subgroup of G so this certainly proves the a part of the second statement of the theorem and also N is E times N is contained in HN right.

Because HN is product of elements of capital H and elements of capital N, E in an element of H, so you have this. So N is a subgroup okay. To finish the proof of statement 2 of the theorem we want to show that N is actually normal in capital HN but that is obvious because N is certainly normal in HN, because N is normal in even a bigger group, namely G. See what is the picture now we have N sitting inside HN sitting inside G. N is normal in capital G that means you take something in capital G take something in capital N and do that operation GN G inverse, it is in capital N for G in G and N in N, this is certainly going to hold if you take something in capital HN and do this same operation so because HN is already a subgroup of capital G, so if you take something in capital HN it is in G.

So this condition holds, so if you have a subgroup which is normal in the ambient group G then it will be normal in any other subgroup that contains it, so this proves (2), right. (2) is completely proved because (2) is the statement that HN is a subgroup of G which we have showed and we also showed that N is a normal subgroup of HN.

So now let us finally prove the main part of the theorem, which is the isomorphism part of the isomorphism theorem. Now we want to show that, here we will show, so just to recall H mod H intersection N is isomorphic to HN mod N, this is what we want to show. So let consider in order to this, let us consider the map from H to HN mod N, let us call this phi. What is this map? phi of a small h is by definition hN, okay so this is the map. So I want to first of all understand what this really means, remember that, so if you take, what is HN?

So note what is HN mod N? What is HN mod N? These are all elements all cosets, note HN mod N are the left cosets of N in capital HN, so these are elements of the form AN where A is in capital HN, so in general this is the quotient, quotient group, AN where A is in HN, but capital H is contained in HN because of the, I mean this is similar to in previous part when I said N is contained in HN. H Is contained in HN because identity is contained in N.

So we can talk of HN, so hN belongs to HN mod N, if h belongs to H, so this is the explanation for this, I am sending h to hN and keeping in mind that small h is also an element of capital HN. So we want to claim that phi is onto and kernel of phi is H intersection N. Once we prove the claim the theorem will follow immediately from the first isomorphism theorem, so what is the proof of the claim? So why is phi onto? In order to prove phi is onto we want to show that anything in the right hand side is an image of something.

So let AN be element of HN mod N, so A is inside HN, right so A is an element of HN right, it a left coset of capital N in capital HN, so A is an element of HN. So write A as small h times small n, where of course small h is in capital H small n is in capital N.

Now let us look at what is AN, AN is HN N. So this can be written as HNN but what is NN because N is in capital N, NN is just HN so AN is equal to small h times capital N, but this remember is phi H. So we started in arbitrary element of, started with an arbitrary element of HN mod N and showed that it is the image of something in capital H, so this implies phi is onto right, so phi is onto, this much is okay.

What is the kernel of phi? kernel of phi is all elements of group H such that HN phi so the definition is phi of H must be in the identity element. What is the identity element of HN mod N, HN mod N is the quotient group, identity element is simply E times N, so this is the kernel, but this is equal to all elements small h such that hN equals N, EN is equal to N, so I am going to write it as things that map to N are the elements in the kernel. But what does hN equal to N mean? So if hN equal to N, I claim this is true if and only if h is in N, right, this we have seen when we discussed cosets, hN is equal to N if and only if h is in N, certainly so because if h is in N then hN and N have something in common. So they must be identical, any two cosets are either identical or are disjoint, so if h is in N, h is N as well as h is in HN because h is h times identity, so they are identical cosets. On the other hand, if hN is equal to N, then certainly h will be in N because h is in HN, okay so this implies kernel phi, so these are elements I am going to write it simply here, h in H such that h is in N. But what is this set, this set is nothing but things that are in H well as in N, so kernel of phi is simply H intersection N, so the claim is proved.

The claim, claim that phi is onto the kernel phi is H intersection N. Now the first isomorphism theorem says, so the phi is a map from H to HN mod N, phi is onto, kernel phi is H intersection N. So by the first isomorphism theorem, first isomorphism theorem applies to the any group of homomorphism, it says that G mod kernel is isomorphism to image, so in this case image is all of this HN mod HN, and kernel is H mod H intersection N. So H quotiented by H intersection N is isomorphic to H mod the kernel which is H intersection N is isomorphic to image which is all of HN and mod N. Remember this is the theorem that we wanted to prove. The second isomorphism says H mod H intersection N is isomorphic to HN mod N and we have completed the proof of the theorem okay.

This is the, this finishes the proof, okay, let me stop the video here, in this video we have looked at some more examples of the first isomorphism theorem and we have stated and proved the second isomorphism theorem. In the next video I will do the third isomorphism theorem, thank you.

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Funded by

Department of Higher Education

Ministry of Human Resource Development