NPTEL

NPTEL ONLINE COURSE

Introduction to Abstract

Group Theory

Module 04

Lecture 20-"Examples of quotient groups"

PROF. KRISHANA HANUMANTHU

CHENNAI MATHAMATICAL INSTITUTE

(Refer Slide Time: 01:51)

Now let me go ahead and do some examples, to better understand this notion. I am going to take the standard example, the main example that we want, we have learned in beginning namely the group of integers, so this is under addition, actually this is an abelian group right, these G is abelian so, every subgroup is normal, remember normality is a trivial property once you have an abelian subgroup, abelian group, because AB A inverse, what is normal mean? Means if A is in H sorry A is in G, H is an H, you want AH, A inverse is in H, but if you have an abelian group, this is trivial because A, H G is abelian implies AH, A inverse is a A A inverse H, which is H, that is certainly in capital H, so if you have an abelian group, every group is normal, that is trivial, so we can take any subgroup. In the previous slide we had,

(Refer Slide Time: 01:53)

H needed to be a normal subgroup, because now I am working with abelian group, and everything is normal, so I take any subgroup I want. For example, let's take 2Z, remember subgroups of integers are nZ, where N is an integer, so let me start with 2Z. So what is G mod 2Z, G mod H, this is same as Z mod 2Z, so what is this? So again as I said, you have to believe that quotient groups are easy to understand okay, they are easy to understand, you systematically work out what needs to be done.

When I ask you what is Z mod 2Z, you should ask to yourself, what is the set first, what is the underlying set? What is the underlying set? Before you answer what group it is, you want to know what set is it, once you understand the set you can then look at the binary operations. What is the underlying set? It is simply all left cosets of 2Z and in Z, right, what are they? So certainly we have 2Z, remember left cosets are, in this case I am working with the group of integers, so in this case, the operation is addition, so I am going to denote cosets buy M plus 2Z, so these are all the left cosets. As you vary m, are the all the left cosets, right, now though it looks like we get one for each integer as we are seen in the previous videos many of them collapse, many of them are identical, for example you take zero plus 2Z that is same as two plus 2Z, right, that is same as four plus 2Z, that is same as six plus 2Z, that is same as minus four plus 2Z, that is same as 102 plus 2Z, and so on. Okay in fact, if m is even, this is equal two m plus 2Z, for any even integer m okay.

(Refer Slide Time: 04:38)

If, this is of course 2Z, m plus 2Z is equal to 2Z, for every even integer, similarly one plus 2Z is equal to m plus 2Z, for every odd integer okay. This is because, we have done many times, so I do not want to spend a lot time but 2Z remember, is all even integers, if you add one to this, you get minus three, minus one, one, three, five, and so on, if you add two, three to this okay, you get minus one, minus four plus three is minus one, minus two plus minus three, sorry minus four minus one, minus two plus three is one, then you get zero plus three is three, two plus three is five and so on.

So it as a set, these two are equal right, because this is just shifting that zero, and zero plus one is one, minus two plus three is three, so these two are equal, so in fact I mean this is a quick explanation of why, one plus two Z is equal to m plus 2Z, every odd integer would be contained in this set, so this is in fact, so in other words, 2Z is the set of even integer, one plus 2Z is the set of odd integers. What you have to focus on is, when you divide by two, what is the reminder? It can either be zero or one, if it is zero it is an even integer and if it is one, and it is an odd integer,

(Refer Slide Time: 06:46)

So again let's come back to our main focus here, I am trying to understand what is the Z mod 2Z, as a set it is simply 2Z and one plus 2Z, okay as I said again remember that quotient groups are very easy to understand, because first now understood the set and next is to understand the operation, what sometimes confuses is people is the way that we describe the elements here, one is 2Z, and other is one plus 2Z,

But do not worry about it, it is a actually irrelevant what we call this, so let me call them E comma A okay, so E is 2Z and A is one plus 2Z, I am going to rewrite this in our standard notation of groups, 2Z was the identity element and I am going to call it as E, the other element is just called A, so when you write it like this, it becomes fairly clear, right, it is just a two element set which are all even integers and all odd integers, don't think like this if that confuses you, think them as E comma A, okay this is the set and what is the operation?

(Refer Slide Times: 08:10)

There are only two elements here so, it is very easy to define the binary operation, so, what is 2Z plus one plus 2Z, so again I am going to rewrite this as in my simplified notation it is just E times A, so I am going to switch the additive notation to product notation, what is this? This is if you recall the definition of the quotient group here, AH and BH simply was ABH, where AB is the group operation of the group G, in this case this is Z, it is zero plus Z, zero plus one plus 2Z, this is 2Z and this is not surprising because A time E must be A, similarly what is one plus 2Z plus one plus 2Z, all I do is first I add one and one and I have 2Z, that is same as 2Z, in the notation that I want you to think one plus 2Z was A, right, this is A and this is E, this A squared equals E, okay so you have a group that's it you have described the group completely,

(Refer Slide time: 09:39)

So Z mod 2Z is a group with two elements E, A and the multiplication table remember multiplication tables are given by, you have E,A and E,A and what is EE? That is E, this is A it is A and this is E, that's all, so do you agree Z mod 2Z is a very simple group, it is just a group with two elements with the multiplication table given by this. Note that quotient group are really not just groups but you need to always think of the group homomorphism from the group to the quotient group, and what is the natural group homomorphism? Z to Z mod 2Z, so remember that we have a natural homomorphism always, the part of,

(Refer Slide time: 10:58)

this are summary of this very important, not only do we talk about G mod H being a group, we come always, we also discuss the natural group homomorphism from G to G mod H, which is onto and has the kernel equal to the subgroup H, so whatever it is in this case

(Refer Slide time: 11:17)

it must have kernel equal to 2Z, but it is clear now right, what is m phi m is by definition m plus 2Z, so this is, in my notation this is E, if m is even, because remember I am identifying Z mod 2Z as E,A so this is E when m is even, and A if m is odd. So this is the group homomorphism, it is certainly onto because all even integers map to E and all odd integers map to A, so everything in Z mod 2Z is in the image.

And what is kernel of phi, things that map to identity m in Z such that phi m is E, so this is even integers, this is precisely 2Z, okay so we have in this very simple example Z mod 2Z, we have worked out the left cosets, we have worked out the group structure and you have to, remember because it is an abstract group theory course, you have to abstract out the crucial piece of information, I do not care what the form of elements are, 2Z, one plus 2Z that is not very important, so to simplify I am just going to call them E and A, and the multiplication table completely describes the group structure.

(Refer Slide time: 12:58)

And quotient group is given by this group homomorphism, which also we completely understand, so if you understand each piece of this game here, you know what the quotient group is completely okay.

(Refer Slide time: 13:12)

Now I will just, for more exercises for you, to make sure that feel comfortable with group quotients, let's take 6Z, so what would be or rather let me say describe Z mod 6Z. Now I am going to do this quickly because we have done it in full detail for Z mod 2Z, what is the set? First question as I said what is the set? So what are the lest cosets, okay I am not going to describe this in all detail because we have done it for Z mod 2Z, so quickly we will understand, I hope you will agree with me he quotient groups are easy, okay, it is very easy to describe quotient groups, let's first start with, what is the Z mod 6Z, so in other words, what are the left cosets, let me remind you that 6Z is a normal subgroup of Z, so we can talk about the quotient group.

And Z is an abelian group, so any subgroup can be taken, what are the left cosets? There are actually six left cosets, just like there were two left cosets when you were working with the 2Z, there will be 6 left cosets, why is this? Briefly the reason is, you take an integer, you divide by 6, it will have a reminder which is between 0 and 5 and that will determine where it is, which coset will contain it. So this is 6Z, one plus 6Z, two plus 6Z, 3 plus 6Z, and four plus 6Z and five plus 6Z, for example you take 25 okay 25 is an integer. When you divide by six, this is 6 times four plus one, so 25 will belong to this, so 25 is an element of this, what about 56, 56 is when you divide by, this is nine times six plus two right, nine time six it is 54, so this is inside 56, so every integer is in one of them, they cover, they partition the group, the group Z, and any other coset is equal to one of them, for example this means 56 plus 6Z is equal to two plus 6Z, so the quotient group Z mod 6Z has 6 elements and again it might be confusing to keep track of these particular descriptions, I do not need to do that, I am just going to simply take E, A1, A2, A3, A4, A5, E was 6Z, AI is I plus 6Z.

(Refer Slide time: 16:42)

So this one plus 6Z and two plus 6Z and so on, what is the group structure? So what is A1 times A2, I am not going here the multiplication table is going to be six by six, so let me not write the full thing but for example what is the A1 times A2, A1 times A2 is, one plus 6Z plus 2 plus 6Z, that is three plus 6Z, so that is A3, right, so what is A1? So I worked out A1 times A2 let me work out what is A1 times the A1, that is one plus 6Z, plus one plus 6Z, so that is two plus 6Z, that is A2. What is the A1 times, A1 cubed, this I A1 squared and what is A1 cubed?

This is A1 times A1 times A1, so this is one plus 6Z plus one plus 6Z plus one plus 6Z, and this if you think about It is three plus 6Z and that is A3, what is the A1 power four? So this is A1, A1, A1,A1, so this one plus 6Z, one plus 6Z added four times this is 4+6Z that is A4. What is A1 power five? That is A1, A1, A1 A1, A1, one plus 6Z plus 1 plus 6Z six times sorry this is five times you know that I should write A5. And finally what is A1 power 6? A1 six times so this is A6, but what is A6? There is no A6. One plus 6Z plus one plus 6Z, six times. That means you get 6+6Z. What is 6+6Z? Remember we have only 0+6Z up to 5 +6Z. 6+6Z if you think about it is just 6Z. So this E. So A1 power 6 is E.

(Refer Slide Time: 18:59)

So what is the subgroup generated by A1? So this E, A1, $A1^2$, $A2^2$, A32 sorry this is not correct it is A1 2, A1 3, A1 4, A1 5 and then A1 6 versus E. So this is what it is. And this is actually because of the calculation we have made it is E, A1, A2, A3, A4, A5 so this is just Z mod 6Z. Any group which has an element like this has a name for it. What is that name? So, Z mod 6Z is a cyclic group.

This means at this point we can say order of A is 6. So 6 is the smallest positive integer such that A1 power 6 is identity. Okay Z mod 6 is a cyclic group of order 6. Okay so this is just a observation we made but the point is we have understood Z mod 6Z completely. Let me also write down the natural homomorphism from Z to Z mod 6Z. So where does phi of m go? It will go to m plus 6Z.

Okay and it is equal to one of these. It is equal to you write m as 6a, 6 times p plus r. So r is the remainder. So this will be and r remember will be between 0 and 5, when you divide by 6, the remainder is between 0 and less than or it is greater than, it could be 0, it could be 5. So it is between 0 and 5.

So it is r plus 6Z. So this is the, again remember phi is onto and kernel of phi is 6Z. All the things where you, in order for you to be in the kernel, r must be 0, meaning when you divide by 6 the remainder must be 0 that means it is a multiple of 6. Okay, so what we have concluded is Z mod 6Z is a cyclic group and it has a natural map from Z.

(Refer Slide Time: 22:01)

As an exercise I will leave this for you. After we worked out these two examples, for any integer n, Z mod nZ is always a cyclic group, for any integer n, of order n, exactly n elements. So it is the cyclic group of order n, okay so again you have to remember how to list the cosets? So I will let you do this, I will not do the exercise and it is a good exercise to do, to make sure that you understand the notion of a quotient group.

I will only give you a hint which is to consider the left cosets of nZ in Z which again depend on what is the remainder when you divide by n. So you have possible remainders are 0, 1, and 2 up to n-1. So these are the cosets and the product is determined by the standard formula a+nZ+ b+nZ is, these are two left cosets, so it is a+b +n Z.

Okay so this is the product or the addition, binary operation and you can check that this is a group operation and it is a cyclic group because 1+nZ will generate just like here, remember A1 generates it that means A1 is 1+nZ, A1 was 1+nZ, A1 was the generator. So we use that to show that Z mod nZ is a cyclic group.

(Refer Slide Time: 23:54)

So, let me give you one important point here. So recall that we have defined the quotient group as follows. So G is a group H is a normal subgroup. In this setup we look at the set of left cosets and I defined (aH) remember I defined it like this. We have to address the problem of whether this is a well-defined group operation, because even though we have checked all the group properties hold we have not really addressed this problem. Is this is the well-defined group operation?

We need to, why is this important question? Because remember a coset (a H) can be equal to another coset (a prime H) and A and A prime could be very well distinct elements. Similarly, suppose that bH equals b prime H. So, aH equals a prime H, bH equals b prime H. Then in our definition we said aH times bH whatever it is should be equal to a prime H times b prime H. Right, this must happen in order for a valid group operation, a well-defined group operation because this is a single element in that set that is equal to this. Similarly, this is another single element.

That is bH and b prime H are actually same elements of G mod H. So, when you multiply aH and bH it must be the same as a prime H and b prime H and this is very important. I will give you an example let us take G to be S3, that we have discussed in detail and H to be E and 12. Right, we have seen this example in detail. So, in this example let us try to compute, let us say I define, so suppose we just define A H times BH equals AB H. What happens if you just define it like this?

(Refer Slide Time: 26:42)

Let us look at 23H and we know 23H, if you remember the calculation we have done 23H is same as 132H, this we know. So, if you square this element, meaning you multiply it with itself, 23H let us use this description, 23H times 23H right, then in this description what do you get? You get 23 times 23H but 23 is an element of order 2, so 23H is same as eH, H right, but on the other hand we can also use the description 132H. So, 132H is same as 132H, but this is, if our definition we apply, is same as 132 times 132H. What is 132, 132? If you do the product it is simply 123H.

But this is the problem. H is not equal to 123H. Right, H is not equal to 123H. This also we see 123H is a different coset. So, this is not a valid group operation. If you simply declare aH times bH is equal to abH this is not going to give you a valid group operation because 23H is same as 132H but you do it in two different ways, you get two different cosets. So, it is important to note that we need H to be normal. Then why the problem does not happen? Why the problem does not occur?

Because here there is no problem, because we have shown in the above proposition that the product of aH and bH, see this is not the binary operation, this is just a set theoretic product of two subsets of a group, is equal to right, the product of aH and bH is equal to abH. Now if aH is equal to a prime H and bH is equal to b prime H these are the same subsets right, these are the same subsets so this product is also going to equal, at the same time it will equal a prime b prime H, this the proposition but because these two sets are equal these are equal. So, if H is normal then there is no problem because aH and bH as a set, two sets their product is abH. Okay, this is something that we have proved. This is the crucial proposition that we have proved.

So if we have another description, it is just another name for the set but the set is same aH is equal to a prime H because that is only, I mean we are trying to address the problem if aH is equal to aH, a prime H. In this case there is no problem, in earlier case there is a problem because these cosets are same but in our definition we are arbitrarily defining aH and bH to be aH times bH to be abH. Why should it equal a prime b prime H? And sure enough, as our example shows, they are not equal.

However in the proposition we showed that if H is normal the product of the set aH with the product, with the set bH is abH and if the set aH is equal to a prime H and the set bH is equal to b prime H, we have no problem because if you have two sets the product is what it is. Just changing the name of the set does not change anything.

The product is the same. So, in the case that H is normal there is no problem. The group operation is well-defined and we have already checked that all the other properties hold and it becomes a group. Okay, so do these exercises and this will help you understand the notion of a quotient group and in the next video we will look at more examples and look at more properties of quotient groups. Thank you.

Online Editing and Post Production

Karthik

Ravichandran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh Kumar

Web Studio Team

Anitha

Bharathi

Clifford

Deepthi

Dhivya

Divya

Gayathri

Gokulsekar

Halid

Heamvathy

Jagadeeshwaran

Jayanthi

Kamala

Lakshmipriya

Libin

Madhu

Maria Neeta

Mohana

Mohana Sundari

Muralikrishnan

Nivetha

Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Subash

Suriyaprakash

Vinothini

Executive producer

Kannan Krishnamurty

NPTEL Co-ordinators

Prof.Andrew Thangaraj

Prof. Parthap Haridoss

IIT Madras Production

Funded by

Department of Higher Education

Ministry of Human Resource Development

Government of India

HYPERLINK "http://www.nptel.ac.in" www.nptel.ac.in