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Introduction to Abstract Group Theory
Module 03
Lecture 17– "S_3 revisited"
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Okay I want to illustrate this with an example, so I am going to revisit.

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one of the first groups that we studied, S3 revisited, so S3 remember is the symmetric group, the notation the terminology I want to use is symmetric group on 3 letters, in other words, it is the group of bijections from a 3 element set to itself, okay in my earlier notation which was given in my first video, I concretely told what these 6 bijections are and gave them the names f1, f2, f3, f4, f5, f6.

Now I want to give a more logical, more clear, notation for this group, because obviously this is not useful, right, now what is f2, you have to go back to what I called f2. So instead of doing that I am going to give you a different description of S3 and in subsequent lectures I want to use that description. So I want to introduce a new notation to describe elements, I want to use a notation which, which is useful to understand what that element is doing because here, I have no idea what f2 does, no idea what f3 does. So my notation I want to, immediately looking at the notation I need to know what element what it does as a function and I also do not want unnecessary information.

So let me introduce this , this is called the cycle notation, so if you recall what f2 was, I will just do it for 1 element or 2 elements. f2 was the function with sends 1 to 2, 2 to 1 and 3 to 3, right. So this was a function, so let us look at this carefully and how to minimize the data that I have written all these things, a lot of it is unnecessary, for example 3 goes 3, that I already know.

I would like not to mention that at all in my description, so in the new notation, let me write this as simply, (1 2), so all my elements of S3 will denoted by elements 123 within brackets, okay, and I will tell you how to read this. First of all, any number that does not appear here is supposed to go to itself, so 3 is not appearing here, so this is supposed to stand for f2, 3 is not appearing here so that means 3 goes to 3 and where does 1 go? It goes to the next number there, it goes to 2, where does 2 go? It does not have anything to the right, so you cycle back to the beginning of the expression. So 2 goes to 1, so (1 2) sends 1 to 2 , 2 to 1, and 3 to 3. So you can definitely agree with me that this is a much simpler notation than writing all this data here, and the advantage, of course, is also that just by looking at you know what it does as a function: it sends 1 to 2 , 2 to 1 and 3 to 3, so you can replace f2 with (12), in this same way you can replace, use this notation.

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What does f3 do? f3 sends 1 to 3 , 3 to 1 , 2 to 2, so in this notation, in the new notation, I am introducing, I start with 1, okay, where does 1 go under this? It goes to 3, so I put 3 next to 1, and where does 3 go, it goes to 1, so I do not want to write a number again, so I will remove 1 and I start the bracket, I put 1, 1 goes to 3, so I will put 3 there, but 3 goes 1, so I close the bracket because then once you look at the last number on the bracket, before the bracket it goes to the first element in that thing, in other words, it goes to 3, so remember in this new notation if I use (31), it will be the same element, because 3 goes to 1 and 1 goes to 3, and I am not going to introduce, I am not going to write anything about 2 in this notation, because 2 is supposed to go to itself, in the previous term also, (12) also can be written as (21), the point is 2 goes to 1 , 1 goes to 2.

So it will take time for you to get used to this, but this is a very useful notation to represent bijections of sets. 1 goes to 3, 3 goes to 1, 3 goes to 1, 1 goes to 3, that's the same function. What does f4 do? It fixes 1 and it sends 2 to 3 , 3 to 2 and again it is clear, 1 goes to 1, so I do not want to even write 1 in the new notation. So I start with 2, it goes to 3 , 3 goes to 2, so I close the bracket. f5, f5 was, 1 goes to 2 , 2 goes to 3, 3 goes to 1, so this is more interesting.

Now let's put the bracket and start writing the cycle notation. I put 1, 1 goes to 2, so I put 2, 2 goes to 3, now I cannot close the bracket like earlier, because 2 does not go to 1, 2 goes to 3, so I have 3, 3 has not appeared so far, so 3 must be put next to 2, but 3 goes to 1, so 1 is the first element, so I close the bracket. But again just like before I can cycle them and I do not change the function, so for example, I can do this also write this as (231): 2 goes to 3, 3 goes to 1, 1 goes to 2, I can also write this as (312), so there is some ambiguity in which order you, what you start with, but there is no ambiguity in the order in which they appear. So the correct way to think of them is (123), so this is 123 if you go clockwise 123 it does not matter where you start, if you start with 1, you have this description, if you start with 2 you have this description, but 231, it's important to traverse these things in the correct order, if you go clock-wise, 231 or you can started 3, 312.

So that is this cycle, here it is 2 and 3, then you traverse in clock-wise order, so 23 or 32. Finally what is f6? f6 was 1 goes to 3, 3 goes to 2, and 2 goes to 1, so if you write the cycle notation, you put 1 then you put 3 because 1 goes to 3 ,3 goes to 2, and 2 goes to 1 again, so you close it. So in the cycle notation you have 1 and 3 and 2, so (132) is same as you can start with 3, (321) or (213). Remember that in no convention these 2 elements will be same, because 1 goes to 2, whereas 1 goes to 3, here only thing you have to keep mind is in a cycle notation, these are called cycles, the image of an element is determined by what comes after it. In this notation, image of 1 will be 3, image of 3 will be 2 and if the element is at the end of the cycle and the next element next you see the bracket closed, its image is the first element of this cycle, so 2 goes to 1, 1 goes to 3, 3 goes to 2.

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So in this notation now, S3 is, identity element I will now as always denote by e, you have {e, 12,13,23,123,132}, right, and you agree with me hopefully that it is much easier than before, earlier I was calling them f1, f2, f3, f4, f5, f6 and if you just tell you what is f2, you have no way of remembering it right, you have to go and see where we have defined what it is f2 and see what it is. Now it does not matter, I can forget that earlier notation, if I consider now the function 23, it is clear 2 goes to 3, 3 goes to 2, 1 goes to 1, 123 sends 1 to 2, 2 to 3 , 3 to 1 and so on. So let us I haven given this description because this is much more natural to work with this and especially when we are

writing cosets and so on, it is useful to have this notation.
So now let us work out the cosets of a particular subgroup and verify counting formula and the Lagrange's theorem in this case. So this is my ambient group G, consider a subgroup, I can take anything I want, so let me take this sub group, as an example, given by H= {e, 12}. So remember we have already verified that this is a subgroup, because 12 times 12, I should also tell you how to multiply these things. We start with the right hand side, 12 times 12, and you trace what happens, so for example, under the product where does 1 go? 1 goes to 2 under the first element then 2 goes to 1, under this, so 1 goes to 1, so there is no need to write it, if a number is not there it is basically by itself within the bracket. 2 goes to 1 under the first function and 1 goes to 2 so 2 goes to 2 under the product. Similarly 3 goes to 3 this is e, okay, so just as a practice, for example ,we will do this as part of the writing cosets, but what would be 23 times 12 , 23 times 12. As I said, you start on the right hand side element and work from there. Under the product where does 1 go? You do first where does, you ask yourself were does one go under the first function, first element 1 goes to 2 under the first element but 2 goes to 3 under the second element so in the product 1 goes to 3, so you have 13, in the product, now where does 3 go? 3 goes to 3 under the first element, because 3 is not appearing in the first element so 3 goes to 3 and 3 goes to 2 in the second element so in the product 3 goes to 2 okay.
Let me see if I can close the bracket. Where does 2 go under the product? In the first entry, in rightmost entry element 2 goes to 1 right and where we does 1 go under the left entry element 1 goes to itself, so 2 goes to 1 in the product, so the product of 23, 12 is 132.

So now let's come back and calculate the left cosets of H in S3. So this is the problem. Compute the left cosets is H in S3. What are the left cosets? Remember left cosets are elements of G multiplied by h, element is on the left hand side, so this is eH, 12H, 13H, elements are now defined like this, 23H, 123H, 132H, so these are the left cosets. But remember, you will expect that some of them will coincide with each other, so that what we have to figure out.
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So let us compute what is 12H? So 12H remember recall in general what is the left coset at aH, in the general case, it is the set consisting of a times small h, in our example, 12H is and H was {e, 12}. I will fix that here, H is {e,12}. 12H is 12 times e and 12 times 12. I am going to multiply each element of H by 12, so 12 times e and 2 times 12. And what is 12 times e? e is the identity element, so that is 12, and 12 times 12 as I worked out in the previous slide, this is just e, 12 has order 2, so 12H is just {12, e} which is exactly equal to H. So the first two cosets here coincide, so eH is seem as 12H.

Now let us compute 13H. What is 13H? 13 times e and 13 times 12. 13 times e is 13 because e is the identity. What is 13 times 12, and I am going to use the same procedure that I described earlier and you have to do this repeatedly if you are new to this to get used to the product of this cycle notation.
You have to ask yourself put a bracket, put 1, you can take any number right, so let us start with 1 and let us determine the image of 1 under the product, so it is a composition of functions, that is all that is happening here. 1 goes to 2 under the first

function, 2 goes to 2 under the second function so 1 goes to 2 under the composition of the two functions. Where does the 2 go? 2 goes to 1 under the first function and 1 goes to 3 under the second function so 2 goes to 3 under the composition.

So I put 3 there. 3 goes to 3 under the first function because 3 is not appearing in the cycle notation and 3 goes to 1 under second function, so you close. So 13,12 is 123. So this is a coset, so this is what 13H is. Now let us compute 123H. 123H is 123 times e and 123 times 12. Before I compute this, let me just write 123 times e is 123. Before I compute this product 123 times 12, let us look at the previous coset 13 and 123, and this coset has 123.

So the coset 13 times H and the coset 123H have something in common, 123 is common to both of them, and remember our general theorem about equivalence classes of equivalence relation which applies to left cosets of a subgroup in a group: if 2 cosets have anything in common they must be identical, this coset has these 2 cosets have 123 in common, so you expect from that result that these are identical.

So you expect in this place 13. It must be 13, if it is anything else then these cosets have something in common yet there not equal. it has to be 13, but let us verify just for surety that it must be 13. So in other words, what is 123 times 12? Following the same recipe for the product, I put 1 here 1 goes to 2 under the first function 2 goes to 3 under the second function so 1 goes to 3 under the product. 3 goes to 3 here and 3 goes to 1 here so 3 goes to 1 under the product, so we finish the bracket. Now what happen to do 2 goes to 1 here 1 goes to 2 is fixed and no need to write that and sure enough, this is 13. So these 2 cosets are same so this is same.

Similarly we have to compute, let's say 23 times H, what are these? What is this? This 23 times e and 23 times 12, because H is 12, and 23 times e is 23 and 23 times 12, I will let you calculate using the same procedure it will give you 132, okay. Now the only remaining coset we have to compute is, remember there are 6 potential cosets, eH,12H,13H, 23H, 123H, 132H, we have identified the first two, so let I will write this later, we have identified the first two and 13 happen to equal 123, this and this, now let us to 23 and 132, what is 132H? This is 132 times e and 132 times 12, again remember H is {e, 12}. So 132 times e is 132, and exactly as before because they have something in common, they must be identical which you can also verify by explicitly multiplying this out and I strongly urge you to do this you on your own, 132 times 12 it is actually 23, so this 2 cosets are equal, so there are, let us now list all the distinct cosets

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distinct, okay, list all the left cosets, again I am working with this specific example. H is {e, 12}, okay, so this is eH and 12H, this is one coset , this is the first, okay, this is first left coset I wrote. The second one is 13H which also happens to be same as 123H, this is the second left coset and finally we have 23H and that is equal to 132H, this is the third left coset. Remember that S3 has 6 elements, so potentially there are 6 left cosets right because of each element you multiply by H, so eH, 12H,13H, 23H,123H, 132H. So there are 3, 6 potential things but because of identifications because they are not all distinct, you have only there are only, 3 left cosets, there are 3 left cosets of H in G, in other words, the index of H in G is 3, in my notation that I developed earlier, [G :H] is 3, that means G was S3, so let me write S3 colon this particular subgroup is 3. So is the counting

formula verified in this case?
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Recall what is the counting formula? It says order of the group equals the index of H in G times order of H, in our case order was, G was S3, so this is 6, index we just calculated is 3, and what is the order of group? It is 2, there are 2 elements, so 6 = 3 times 2, this is correct, counting formula is verified and Lagrange's theorem is also verified here, of course, for that you don't need to do any coset calculations, right, because order H which is 2 divides order of S3 which is 6.

So as an exercise, I will strongly urge you to do this, to get comfortable with calculations of cosets, compute the left cosets of H, new subgroup I take is, H is {e, 123,132}. So first of all verify that this is a subgroup and compute its cosets in S3. Counting formula tells you how many you expect, counting formula says how many cosets should be there? What is order of H order of H, 3 elements are in H, so order of H is 3, order of S3 is 6. The counting formula says 6 equals index of H in G times cardinality of H, in this case cardinality of, order of H is 3, so the number of cosets should be 2. So verify this, by explicitly computing the left cosets, see that there are 2 left cosets of H in G. So this tells you that, there are 2 left cosets of H, this particular H in S3, so this is an exercise for you I will let you do this I am not going to discuss this, exactly the same calculation, but in this example I also want illustrate the fact that left cosets partition of your group.
And that is clear right, because S3 remember is {e,12,13 23,123,132} and left cosets cover all the 6 elements, so left

cosets are in the previous slide we computed this, eH and 12H are same time so that, and what is eH? eH was, eH was {e, 12}, so 2 elements are in the first left coset, there are two more elements 13 and 12 are in the second left coset, 23 and 132 are in the third left coset, so we also see, so maybe I will just record this here.
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In this example that we worked out, we also see that each left coset, this example is not the exercise, the example of, in the example H was {e, 12}, each left coset has 2 elements, remember this is what you would expect because of a theorem that I proved earlier, because H has 2 elements, each left coset of H has 2 elements. I want to each left coset of H has 2 elements, which is what we noticed in the calculation.
Every left coset, 12H is 12 and e, 13H is 13 and 123, there are 2 elements and there are 3 left cosets, each coset has 2 elements, there are 6/2 = 3 left cosets, again this is counting formula right, number of cosets is the number of elements in the group divided by the number of elements of the subgroup, there are 3 left cosets and left cosets partition, this also we saw, so all 3 properties we proved in general, we have illustrated them in this example and I want you to do the exercise with a different subgroup of S3, namely a subgroup of 3 elements. Here what do you expect? Each left coset should have 3 elements, there should be 2 left cosets and again all the left cosets should partition the group, so do the exercise, compute everything, all the left cosets, and verify that all the properties are satisfied.
So in this video we have done a very important theorem called the Lagrange’s theorem which says that a group, if you have a

finite group G and a subgroup of H of G, the order of H divides the order of G. This is an extremely important theorem and we proved that as a consequence of counting formula, which is also important, which says that order of a finite group is the product of order of the subgroup H and the index of H in G, and we have illustrated all these results in the case of S3 and in order to do that, we have found a new description of elements of S3 using cycle notation, which is very important and that we use repeatedly in future and worked out all the left cosets of a particular subgroup H of S3, thank you.

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