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Introduction to Abstract Group Theory
Module 03
Lecture 16- "Cosets and Lagrange's theorem"

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So, I am going to continue now my discussion about the equivalence classes and how this leads to the notion of a quotient group. So, recall from video on equivalence classes, an equivalence relation on a set is a relation
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where you can say, if S is a set, we can say that two elements are equivalent and in order to be an equivalence relation it must satisfy that any element must be related to itself. If an element is related to A is related to B then B is related to A, if A is related to B, B is related to C then A is related to C. So, these are the properties of an equivalence relation. We discussed some examples of this and the most important example in our situation would be the relation defined by a sub-group of a group.
So, that I will recall in a minute, but an equivalence relation, the main fact that I want you to recall from that video and that we will use today is an equivalence relation, an equivalence relation partitions. So, an equivalence relation on a set S partitions S. So, I discussed this in all those examples of equivalence relations but if S is given like this, you consider equivalence classes. What are equivalence classes?
So, equivalence classes are simply, if you fix for A in S

equivalence class is all elements in S which are equivalent to B, whether I write A is related to B or B is related to A is not important because of the reflexive property of or the symmetric property of the equivalence relation. So, these are the equivalence relations, classes. So, you have A1, A2 and A3 and so on. And the important property is either if you are given to equivalence classes, they are, given two equivalence classes, let's say the class of A and the class of B, then we have two possibilities, then 1) either the class is equal to itself each other the class A is equal to the class B ([A]=[B]).

So they are identical or 2) they are disjoint. They have nothing in common. So, this is a very strong condition, right? You have either disjoint sets or they are identical. So, if you take distinct classes, if A1 class is different from A2 they cannot have anything in common. So, you could cover the entire set like this. So, this is how a set partitions,

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an equivalence class on a set partitions that set. So, the most important example for us and all the examples that we looked at are examples really of this example with different groups and different subgroups. Fix a group, so fix a group G and a subgroup H of G. Fix a group G and fix a sub-group H of G. So, we say that I think I defined if A and B are in G then A is related to B if $A^{-1}B$ is in H.

Right, this is the equivalence relation that we checked, is an equivalence relation. So, what are the equivalence classes? And we discussed this, I think. Equivalence class of A is all B in G such that $A^{-1}B$ is in H. But this is same as AH because if $A^{-1}B$ is in H, $A^{-1}B$ is equal to small h, so B is small a times H. So, B is related to A if and only if B is A times an element of H. So, this is just this set.

This set remember is my notation, the definition is this, this is the definition. So the equivalence class of A is simply (aH). And the most important definition now I want to give you is the following. If H is a sub-group of a group G, the “left cosets of H” are subsets, so, left cosets are simply, you take an arbitrary element of the group and multiply by (aH). So, left cosets are these.

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Similarly, you can define “right cosets” and not surprisingly, these are subsets obtained by multiplying by A on the right, Ha are right cosets. So left cosets are aH right cosets are Ha. So what we have done is for the equivalence relation of this example, for the equivalence relation above by which I mean the previous page, A is related to B if A^{-1} B is in H, for that equivalence relation, the equivalence classes are simply left cosets.

Correct? So equivalence classes of this relation are the left cosets of H. So as I said, the left cosets and right cosets are very important for us and we will constantly keep dealing with them. So you should carefully think about what they are, and they are very simple really, the left cosets are subsets of the form aH and rights cosets are subsets of the form Ha and I will do some examples in a later video to describe these in specific groups, and hopefully that will become clearer to you then.

So, now coming back to the equivalence relation that we are discussing for that equivalence relations the left cossets are the equivalence classes. So, now I want to do an important theorem, this is the first important theorem really of the course and in order to prove that let me observe the following. So, we have noticed two facts. We have noticed already one fact that, since equivalence classes partition the set, in our situation the left cosets of H in G partition G.

So left cosets of H are equivalence classes for an equivalence relation. So, they partition H, which is to say, this means G is the disjoint union of left cosets. The word disjoint means that if you have two distinct left cosets, they disjoint. They have nothing in common. Remember this is automatically true for us because of the property of equivalence classes for an arbitrary equivalence relation on an arbitrary set, two equivalence classes are either equal or they are disjoint. So, in particular in our situation we are looking at a special equivalence class on a group and the equivalence classes are the left cosets so they partition the group. So, the group is the disjoint union of left cosets.

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Now let me prove an important proposition. The number of elements, so fix let H and G, by which I mean G is an arbitrary group and H is a sub-group of G and let a be in G. Then the proposition says the number of elements, then the number of elements in the left coset aH is equal to order of H. Recall order of a group is simply the number of elements. So, order of a group is the number of elements of H.

Remember I am not saying, order of aH because aH is not a group, aH is just a left coset. So, the number of elements of aH is equal to the number of elements of H, that is the proposition.

And the proof is very simple. Consider the map from H to aH given by a small h goes to small a times h. This is a set map. Right, this is a map of sets. Remember aH is actually nothing but a set. It has no further structure; it is just a subset of the group G.

It is a left coset, it is a subset of group G. So, this is a map of sets, give me an element h I will multiply by a and map into this. This of course belongs to aH. I claim that this is a bijective map. Why is this? First of all, is it onto? Yes, because what are the elements of aH? In previous slide I wrote that right, aH or ah as h varies. So, you give me an element of aH that is of the form a times h. So, h maps to that. Right, so it is certainly onto, by definition, everything in aH is a multiple of something in H.

Is it 1-1? Let us prove this. Suppose h1 goes to a h1 and h2 goes to ah2. Let us suppose that they are equal, ah1 equals ah2. But then the cancellation property of groups that we have learned long time ago gives me h1 equals h2, by cancelling A. So, it is 1-1 also, so it is a bijective map. If you have two sets which, there is a bijection between them that means they have the same number of elements.

So, H and aH have the same number of elements. Right, because if there is a bijection that is by definition that means they have the same number of elements. Now let us look at, see one thing that I should mention here, that I should have been careful when I stated the proposition. I am going to assume that, though it is also true in general, G is a finite group, because this is useful only in this situation, G is a finite group.

So, G is has only finitely many elements, in particular H has only finitely many elements because H is a sub-group of G. So, this proves the proposition right. The proposition is proved now.

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Next we have two facts. So, let me draw a picture here. So, this is the group G. You have cosets, remember H itself is a coset because H is nothing but eH. So, that is one coset. You have some other coset a1H, some other coset and so you have several cosets and we have two facts: the first fact is that G is the disjoint union of left cosets. So, these cosets cover G. So, you have this coset union, this coset union this coset union, this coset and so on is all of G.

And the previous proposition says that any two cosets have the same number of elements, because if you aH has the same number of elements as H, a1H has the same number of elements as H but so does a2H, a2H also has same number of elements as H. This proposition works with an arbitrary element a of G. So any two cosets have the same number of elements. So, now let us think about what could be the number of elements of G.

So, we have, G is a disjoint union of H, union a1H, union a2H, anH, because G is a finite group, remember that G is a finite group. So, it has finite many cosets right because they are only finitely many elements. This is my symbol for disjoint union. So, it is a union a1H union a2H union anH. How many elements are over here? Remember n, let me denote this by a2 because this is taken to be a1, n is the number of left cosets of H in G, n because a1H, a2H, a3H, anH, it is a number of left cosets.

So, n is the number of cosets that are distinct among all the coset of H in G. So, now because G is partitioned by these cosets, cardinality of G, number of elements of G, which is order of G is the number of elements of, let me use for simplicity, the same symbol within vertical bars, number of elements of G because G is a disjoint union of these things. Every element of G is in

exactly one of them right.
So, number of elements of G is number of elements of this, plus number of elements of a2H plus number of elements of a3H plus number of elements of a4H and so on. So, number of elements of G is equal to number of elements of e H, number of the elements of a2H, number of elements of a3H and finally number of elements of anH. But by the proposition of the previous slide, number of elements of a2H is also the number of elements of H, a3H is also anumber of elements of H, anH is also number of elements of H.
It does not matter what cosets we are considering they are all H. How many factors here are there? There are n terms. Right, so cardinality of G is the cardinality of H added to itself n times. So, we have that cardinality of G is equal to n times cardinality of H, and this leads us to state our first important theorem.
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Before that let me define the following number, the number of cosets, I am going to stick to left cosets for this calculation, and for this theorem, because everything that we have done is about left cosets. The number of left cosets of a group H in G is called the “index of H in G”. Index of H in G is the number of left cosets of H in G, it is denoted by this symbol, [G: H]. So, the previous slide we ended with this statement, cardinality of G is n times cardinality of H, where n is the number of left cosets.
I have given this a name now. So, we have a “counting formula” so this is an important counting formula in group theory. We are going to encounter other counting formulas which are formulas to count elements of a group in various contexts. The first example of counting formula is this. Order of a group is the

index of the sub-group in H times order of H. So, this is the counting formula.
This is a very important formula that we have to use repeatedly in our future videos. So, please pay close attention to this and try to understand this carefully. I have already proved this. The previous work is a proof of this. Okay, if you are not clear you should go back and listen to this again and convince yourself that we have the counting formula, it says that cardinality of the group or the order of the group is the index of H times the order of H. Here H is a subgroup of G. Okay, so as an immediate corollary of this I am going to write a very important theorem, follows immediately.
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It is called the Lagrange's theorem. So, this is one of the first important theorems in group theory. Again I keep repeating this, at some point in these videos I have started assuming G is a finite group. So, you can talk about cosets and equivalence relations always without assuming that G is a finite group but once I started counting number of elements, I have assumed that G is a finite group.
Everything that follows is about finite groups. For example the statement about counting formula is a statement for finite groups, right because if G is not a finite group there is no sense in the statement, because this is an infinite number and you cannot write something. So, if G is a finite group you can make this. So, similarly Lagrange's theorem is a statement about finite group. So, I will record that here.
Let G be a finite group and let H be a sub-group of G. Then the order of H divides the order of G. Okay, so written differently,

|H| divides, because the symbol for the order is this, order of H divides |G|. This is as I said is a very very important theorem in group theory and it gives you a lot of flexibility in working with groups. And what is the proof? Immediately follows as I said from the counting formula because |G| is something times |H|.
We actually know what this number is but it is immaterial for the Lagrange's theorem. Right, this follows easily from the counting formula. Correct, because |G| is the product of |H| and something else. So, that means |H | divides |G|. So, I am going to work out some simple examples to illustrate this but pause for a minute here and then think carefully about what Lagrange's theorem is saying.
It says that if you have a finite group, it puts a major restriction on possible orders of subgroups. Immediately you can conclude that if, for example G is a group of 6, for example S3, it cannot have a subgroup of order 4. Right, because 4 does not divide 6. So that is what Lagrange's theorem says. Okay, so this is our first application of the study of equivalence relations and left cosets. It gave us a very important result about orders of a finite group and orders of its subgroups.

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