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Introduction to Abstract Group Theory
Module 03
Lecture 14 – "Equivalence relations"
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Okay so far we have studied various properties of groups, subgroups, normal subgroups, group homomorphisms, and isomorphisms of groups and so on. And we next want to study an important operation called quotients of groups and in order to go that I want to talk about cosets of a subgroup in a group. Before that let me recall for you quickly the motion of equiavalence relation on a set which many of you may have seen before.

But it is very important for the course, so let me quickly define it and check its important properties. Then we will apply to the case of a subgroup of a group.

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So review of equivalence relations on a set. So let S be a set okay, for this video we are only going to look at some sets, there is no group here. So let S be a set and equivalence relation on S is a relation, okay, it is relation denoted by the symbol here, it is a relation of among elements of S. So we say that two things are related and we write this symbol A ~ B. This means that A is related to B, so A is related to B is denoted by this symbol A ~ B.

So an equivalence relation is a relation on a set, which is to say a relation among elements of that set, so it is very precisely speaking, it is a subset of a S cross S. But I do not want get into this formal definition because it is confusing, if you have not see it before. So it is simply a relation and when you see this you think of a way of saying that two things are related or it tells you given two elements, if they are related or they are not related. But the relation must satisfy the following conditions.

A must be related to A for every element of S, okay an element must be related to itself. If A is related to B then B is related to A, okay. So this is true for every a,b in S. The third property is called transitive property, if A is related to B, B is

related to C, then A is related to C, so this is true for all a, b, c in S, okay. An element must be related to itself, if an element is related to another that element must be related to A. If an the element related to B, B is related to C, okay. The most important example, a trivial example really, is equality: take for example on Z okay, you say that a is related to a, if a = a. So equality is a basic example of an equivalence relation, every element it is equal to itself, if a=b, then b=a, if a=b, b=c, then a=c. So there is no surprise that equality is an equivalence relation. But there are more interesting equivalence relations and hence you actually do talk about equivalence relations because there are equivalence relations that are not equalities. So another example.
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Take again the integers, but now you say that and you say that a is related to b if a-b is divisible by 4. So this is my definition of a new equivalence relation so this is the definition. For example 5 is related to 5, of course. Because 5-5 is 0. So earlier equality was the only equivalence relation but now we have more, because 5 is also related to 1. And so 5 related to 5 because 4 divides 5-5 which is 0, 5 is related to 1 because 4 devised 5-1 which is 4.
Similarly 5 is related to 21, right, rather, yeah, 21 because 4 divides what is 5-21? That is -16 right and 4 divides -16, so 5 is related to 25, 5 is related to 29, 5 is related to 34, or 33 and so on. So 5 is related to 5, but it is also related to a lot of other integers. But 5 is not related to let me use this symbol to 26, so 5 is not related to because, what is 5-26? 4 does not divide 5-26 which is -21 so 5 does not divide, 4 does not divide it. So 5 is not related to 26 so this is a more interesting example of equivalence relation than equality. So because we are doing group theory, let me define another equivalence relation,
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which is more appropriate for us and more important for us. So let G be a group and let H be a subgroup of G. So we have a group and subgroup of that group, so I define A is related to B, so let me start like this let a, b be two elements of G. We define A is related to B if let's say, A inverse B is in H. So actually I forgot something here, before I continue with the third example, I just wrote something I

never said that it is an equivalence relation.
I should prove it right, so that I will not check, I will quickly say why it should be true. The equivalence relation in the second example is that A is related to B if 4 divides a-b. So certainly, so why is this an equivalence relation. So certainly A is related to A that is okay because A-A is 0 and 4 divides it. If a is related to b then b is related to a because if A is related to B 4 divides A-B but then 4 divides B-A also so B is related to A.
If A divides B and B divides C that means 4 divides A-B and 4 divides B-C but if 4 divides A-B, B-C, 4 divides their sum, which is A-C, so this implies 4 divide A-C, so A is related to C, so we have checked all the three properties. So this is an equivalence relation. Now let us check that this is an equivalence relation, what are we saying, let H be subgroup of G, two things are equivalent if there if this element constructed from them is in the group H.
This is an equivalence relation. Why? First of all, is A related to A? yes, because you have to ask yourself whether A inverse A is in H. It is in H because that is just E, okay so that is okay. If A is related to B, that means A inverse B is in H right, by definition. But because H is a subgroup, A inverse B whole inverse is in H, anything in H contains its inverse, H contains its inverse also, so A inverse B is in H means A inverse B whole inverse is in H, what is A inverse B whole inverse? That is, B inverse times A, right? We have seen this before, A inverse times B whole inverse is B inverse times A inverse inverse, but that is nothing but B inverse A. So if B inverse is in A, this is the meaning of B being related to A. Remember A is related to B if A inverse times B is in H. B inverse times A is in H means that B is related to A. So the second property holds.
Now let's take A related to B, B is related to C, that means $A^{-1}B$ is in H and $B^{-1}C$ is in H, right? $A^{-1}B$ is in H, $B^{-1}C$ is in H. This means because H is a subgroup it is closed under multiplication, so if $A^{-1}B$ is in H, $B^{-1}C$ is in H their product is an H, so $A^{-1}B$ times $B^{-1}C$ is in H, but that means is $A^{-1}C$ is in H because that is the product, that means A is related to C.
So this is an important equivalence relation on a, this is an equivalence relation and this is some thing we will explore in great detail later. So I looked at three equivalence relations on sets, so now some important properties of equivalence relations. Equivalence relations
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give "equivalence classes" so what do I mean? Let us say S is an arbitrary set, I am going back to the setting of general setting of a set and an equivalence relation on that set. So now for A in S, define the equivalence class of A, the equivalence class of A, as we denote by this symbol [A]. So let us see the equivalence class of A is

denoted by A in square brackets, this is all elements in S such that A is related to B okay. This is very simple, the equivalence class of A is the collection of elements that are equal to B, so in our examples, what is equivalence in the first example, in the first example where equality is the equivalence relation equivalence class is very simple.

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Right in the first example equivalence class A is equivalent to B if A = B in the case equivalence class is all elements which are equivalent to B, equivalent to A but this means, so this is B in Z such that B is equivalent to A, but this is same has B in Z, B=A. so this is just nothing but A, right in this case the equivalence class is just the set {A}.

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In the second example we have defined A here S is Z, A is equivalent to B if 4 divides A-B, okay. So I am going to first write down the equivalence class of a specific element, so that you see what is going on, so let us say, 5 that we have seen earlier, what is the equivalence class of 5. So these are all integers such that 5 ~ b, this is same as writing B ~ 5, this equivalence relation has that property right. So this is all integers such that they are equivalent to 5, so this all integers such that 4 divides 5-B.

So if you write these down what would they be? I am just list some of them, so 0 is not an element in this because 5-0 is 5, 4 does not divide it. So 1 is there, 5-1 is divisible by 4, so 1 is an element of equivalence class, the previous element would be -3, right? Because -3 is related to 5 because 5-(-3) is 5+3 = 8, it is divisible by 4, similarly next will be 5, 9, 13, 17, 21, 25, 29, you see the pattern here, right. These are all, the previous element will be -7.

You can see that they all integers, starting with 1 they you keeping increasing by 4, keep decreasing by 4. We have used the notation 4Z remember 4Z, it was 4Z was all integers which are multiples of 4. So this is –8,-4, 0,4,8 and so on right. This is similar to that, it is not quite that, of course it is not that, but it looks like 0 is replaced by 1, 4 is replaced by 5, 8 is replaced by 9, -4 is replaced by -3, so I can write this as 4Z+1.

So you take 4Z and add 1 to each element, so add 1 to each element, so this is 4 Z+1 so the equivalence class of 5 is 4Z+1. We will come back to this and discuss this in more detail, but thing to keep in mind now is 4Z the equivalence class of 5

the equivalence class of 5 is 4 Z+1. I will let you do this as an exercise. what is the equivalence class of 6? is 4Z+2, for 7 it will be 4Z+3, so what about for 8? You will see that it is exactly equal to 4Z. So and for 9 it will be 4Z+1 again, and 10 it will be 4Z+2, so they keep coming back right, it is periodic in that sense, so as I said, I am going to come back to this example and study it more detail later.
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In the third example, the set was a group G, fix a subgroup H and the relation was A is related to B if, so recall that in this example we have declared that A ~ B if A^{-1} B is in H, right, that was the example. You saw that A^{-1} B in H then A is related to B that is an equivalence relation. What are the equivalence classes of? So now let A be in G, what is the equivalence class of this. So this is all elements B in G such that A is related to B, by definition this the equivalence relation, equivalence class, right.
But this is all B in G such that A inverse B is in H right, this means all B in G such that A inverse B is equal to some h, it is an element of capital H, so I am going to give it a name, small h, so this is same as all B in G such that B is Ah, where h is in H, right. So it is all elements in G which can be written as Ah some suitable small h in H, this is nothing but then I can eliminate B from this whole description, it is Ah, where h is in H, so is this clear all the equalities here.
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Originally these are all elements B equivalent to A, that means A^{-1} B is in H, that means A^{-1} B is equal to some small h, which is equivalent to if A^{-1} B= h then b=ah. So I am going to rewrite this as b equals ah, but then I eliminate b from the description completely. So this all ah, where h belongs to H, there is another notation for this, so this is the definition, so define aH to be, this is very similar to the way that we defined subgroups of Z. So I take this subset obtained by multiplying every element of capital H by small a so the equivalence class of A is in this case is just aH, so now I want to define a proposition,
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which is, prove a proposition which is very important for us, let S be a set and let this be an equivalence relation on S. Then the equivalence classes of elements of S partition S. So partition S, that is, S is a disjoint union of equivalence classes. Okay, what happens is, I will prove this in a minute, but what happens is that you start with a set S, you have a equivalence relation on this set, then if you take equivalence classes, they may coincide, but there will not be intersections for two distinct cosets, so these are all equivalence classes, okay, so they partition the set, so they partition the set meaning every element in the set is in one of the equivalence classes. And there is no common element between two distinct equivalence classes, so this is the proposition so let us prove this.

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So first of all give if A belongs to S then A belongs to [A], right, this is because remember one of the first property, the first property of an equivalence relation is that an element is related to itself, so this equivalence class remember, equivalence class of an element is all elements in the set such that B is related to A. So A is related to A, so A is in the equivalence class of A, so certainly S is a union of equivalence classes, correct, if you take all equivalence classes and you take their union you get S. Because every element of S is in an equivalence class, certainly equivalence classes are all subsets of S, so this is fine.

Now on the other hand, I claim that if equivalence class of A and the equivalence class of another element of B are not disjoint, remember disjoint means they have no common elements. If there is not disjoint then they are equal, so what I am saying that if you take the equivalence class of A and equivalence class of B, if they have something in common they must be equal.

So in fact they must be equal, so they can't have something in common and be distinct. What is the proof of this, so let, remember these are not disjoint, meaning they have some common element, so let us say C is in the intersection of, see all of this is happening within S. So this and this are subsets of S and they have something in common, so C is in both of the equivalence classes. That means A is related to C and B is related to C right.

Because C is in [A] , A is related to C, C is in [B], so B is related to C, this implies that, A is related to B, because A is related to C, C is related to B, so by the 3rd property of an equivalence relation, A is related to B, that means, A is in [B], right, so and B is in [A], so A is in the equivalence class of B, and B is in the equivalence class of A, so that means, because if any element, if some other element D is equivalent to A, then it is equivalent to B also, so it is in [B], if it is equivalent to A

and B also, so this is fine.
So this proves that, if two equivalence classes are not disjoint, then they are exactly identical, so this proves the claim.
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Now it follows that, S is a disjoint union of equivalence classes, because, we have either two equivalence classes are disjoint, or they are identical, so because of what we have shown, if they have some elements in common, then they are identical, that is what we shown here, right, if [A] and [B] are not disjoint, they have something in common, they are identical, so if you take distinct, now take distinct, so not equal, equivalence classes, they are all mutually disjoint, because they are distinct, they are not equal.
They are distinct, hence any two of them being distinct are disjoint, and their union is, because every element remember of S, is in an equivalence class, so take that equivalence class, and that is one of the equivalence classes, that we took earlier, so the distinct equivalence classes partition S, this proves the proposition, no need to say really distinct, because we need not repeatedly take it, right, so these are equivalence classes partition S is the statement, and I have proved it, and I will work this out, in the 1st two examples, and leave the 3rd example for the next video.
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So in the 1st example, remember where S was S and a ~ b, if a=b, what are the equivalence classes here, equivalence classes are just singleton sets a, equivalence classes of a is just {a}, so Z is a union of, so certainly it is a union of singleton sets.
In example 2, S was again Z, and a ~b, if 4 divides a-b. Here there are exactly 4 equivalence classes, so namely like the equivalence class of 0 which is 4Z, the equivalence class of 1 which is 4Z+1, equivalence class of 2 which is 4Z+2, equivalence class of 3 which is 4Z+3, okay.
And if you go back and see the examples, the equivalence class of 5 was actually just 4Z+1, which is same as the equivalence class of 1, equivalence class of 6 is 4Z+2, equivalence class of 7 which is 4Z+3, equivalence class of 8 which is 4Z, so you get nothing new, by considering other integers, so Z is the disjoint union of 4Z, 4Z+1 disjoint union 4Z+2, 4Z+3, it is clear that any integer is in one of them.
Because all you need to check is, divide by 4 and see what the remainder is, if the remainder is 0, it is here, if the remainder is 1, it is here, if the remainder is 2, it is here, if the remainder is 3, it is here, and they certainly have nothing in common,

4Z+1 and 4Z+2, cannot have any element in common, so these are the different partitions for the equivalence class. I am going to stop the video here, we have still to discuss the equivalence classes of the relation, given by a subgroup of a group, that we will do in the next video, but to recall, what we have done in this video, we have talked about equivalence classes on arbitrary sets, we looked at some examples, and I proved a proposition saying that equivalence classes of an equivalence relation of a set, partition that set, and which is very important for us in the context of group theory, because it will allow us to define cosets of a subgroup and talk about quotient groups later on. So I will stop the video here, and continue with cosets next video, thank you.

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