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Introduction to Abstract
Group Theory
Module 03
Lecture 13-"Normal subgroups"
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Okay, so with that let me define an important concept of normal subgroups,

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which is related to both the notion of subgroups and group homomorphisms. So the definition is, let G be a group, a normal subgroup H of G is a subgroup, so I am putting an adjective before subgroup, it is normal subgroup, so first it must be a subgroup, and it must satisfy, more conditions. A normal subgroup H of G is a subgroup satisfying the following property, so it is a subgroup that satisfies the given g in G, and h in H, ghg-1 is in G, sorry it is in H, so given an element of the group and an element of the subgroup, I must have that, $g\ h\ g^{-1}$, so the group element times the subgroup element times the inverse of the group element, must again be an element of H, okay so this is what we call a normal subgroup. So an immediate example. If G is abelian, then every subgroup is normal, okay this is easy to check because, a normal subgroup is a subgroup, which has an additional property that ,if g is in G, and h is in H, $g\ h\ g^{-1}$ is in H, the second condition is automatically true in an abelian group, because if $g\ h\ g^{-1}$ in an abelian group, you can multiply in any order.
This is same as $g\ g^{-1}h$,so I can interchange, so now $g\ g^{-1}h$ is nothing but h. Remember that I have to start with h in H, and g in G, because h is in H, ghg invers is in H, so there is no problem in an abelian group, so normal subgroups, really are interesting if you do not have an abelian group, if you have an abelian group, every subgroup is normal, it is not an interesting new concept, and on the other hand,

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if you have a non-abelian group, you can have subgroups , which are not normal, okay for example, take G to be $S_{3,}$again remember is the set of bijections from a 3 element set to itself, and in my notation these are {f1,f2,f3,f4,f5,f6}. If you take the subgroup H to be {f1,f2}, and I will let you check this, H is a subgroup of S3 , because f1 is identity element, and $f2^2$ is f1, so I will remind you what is f2, f2 is the function which sends 1 to

2, 2 to 1, and 3 to 3, and so $f2^2$ is f1, which is the identity function, so it is a subgroup, but H is not normal in $S_{3.}$
So H is a subgroup consisting of f1and f2, I want to show H is not normal in $S_{3,}$so let us take f3, f3 if you recall, is the element , which sends 1 to 3, 2 to2 , and 3 to 1, so let us do, so remember, a normal subgroup is a subgroup satisfying: take something in the group, something in the subgroup, and do this, g h g^{-1} it is in H, so I am taking something in the subgroup namely f2, and something in the group namely f_3, and I want to calculate $f_3\, f_2 f_3^{-1}$, but f_3^{-1} if you think about it, is just $f_{3,}$ because the f_3 has order 2, and f_3^2 is also f1, so f_3 is f_3^{-1} .
So what is f3 composed with f2 composed with f3, so let us do this systematically. And under f3, 1 goes to 3, and under f2, 3 goes to 3, and under f3, 1 goes to 1, right and where does 2 go ,so this is the f3, this is the f1 and then f3. 2 goes to 2 under f 3 1 under f2, 2 goes 1 under f2 and 1 goes to 3 under f3. Similarly 3 goes to 1 under f3, 1 goes to 2 under f2 and 2 goes to 2 under f3. So f3f2f3 is the function which sends 1 to 2, 2 to 3 and 3 to 2 right, and this is not equal to f2, right it is not equal to 2 and it is certainly not equal to f1. So f3f2f3 inverse is not in H which recall, so H is not normal. So it is not a normal subgroup.
Abelian groups all subgroups of abelian groups are normal but non-abelian groups have subgroups which are not normal. I want to now introduce two important examples of normal subgroups.
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Okay, so let us take any group homomorphism: let us say this is a group homomorphism. Then kernel of phi which we already seen is a subgroup of G is actually normal subgroup of G, so we already saw kernel is a subgroup. To check normality, what do we need to check, so let g be in G and let h be in kernel phi. We want to check that g h h inverse is in kernel phi. What is the meaning of being in kernel if you apply phi to it must send it to the identity, so let us check that. What is phi of ghg inverse.
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It is because phi is a group homomophisim this is phi g phi h phi g inverse, okay. This is phi of g what is phi of h because h is in the kernel phi of h is eg' and phi (g) whole inverse so eg prime is the identity this is just phi of g inverse which is eg' so ghg inverse okay. So this implies kernel phi is a normal subgroup, so kernels are always normal subgroups, so this is an important example of normal subgroup. Whenever we encounter a group of homomorphism and take its kernel it is automatically a normal subgroup.
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A second important example of a normal subgroup is the "centre of a group". Centre of a group is always a normal subgroup, so recall what is the centre of a group g it is denoted by Z (G) this is all elements g in G such that ag =ga for all, again already saw that Z(G) is a subgroup. That is not surprising, that is not new to us. To check the normality just like before we take a group element and a subgroup element and let us check if, what do we want to check, we want to check that ghg inverse is in Z(G), what is meaning of being Z(G)? It means that it commutes with everything in the group so let a in G be arbitrary we want to check that ghg inverse commutes with it so let us write ghg inverse times a, so I want to rewrite this as gh g inverse a. Now let us use property that h is an element of the centre, if a h is element of a centre, h commutes with everything so since h is in the centre let me write here.

h is in Z(g), h commutes with everything, so in particular it commutes with g inverse a, so h times g inverse a is g inverse a times h. I am not disturbing this g so I am writing this like this. But now actually I should have been, there is no need to do all this sorry, so it is much easier than this, because you want to check that gh g inverse Z(G) but because h commutes with everything.

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So this is g inverse time h, right? h is in the centre, so h g inverse is g inverse h, so I want to use that, this just h, which is that taken to be in centre. So if g is in the group h is in centre, ghg inverse is in the centre, okay. Centre is a normal subgroup and kernels are normal subgroups. These are two very important examples of normal subgroups okay, so I will stop this video here, in this video we have looked at normal subgroups, and we have looked at group isomorhipisms okay, and this is, these are both very important properties and especially normal subgroups will keep coming up a lot in our course and next time I will start talking about cosets of a group and talk about quotients of groups, quotient groups, thank you.

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