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Introduction to Abstract

Group Theory

Module 02

Lecture 10- "Group homomorphisms and examples"

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Okay, so in this video we are going to learn about homomorphism of groups and talk about when two groups are same. Ok, so let's start with this definition of a group homomorphism. So,

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let G and G prime be two groups. So, I am going to write the definition and then we will discuss some examples. So, let G and G prime be two groups, so this is the definition. A homomorphism of groups, so it is a homomorphism of groups, is a function, so usually we denote this by phi (φ) ok, this is read as phi. Φ is a function from G to G prime it's a set-theoretic function first of all.

It's a function from G to G prime such that it satisfies the following condition. Φ of (ab) is equal to φ of a times φ of b, $\varphi(ab) = \varphi(a)\,\varphi(b)$. That's all. So, we want φ of a, b equals φ of a times φ of b. Okay, so let us just recall our convention here when I write ab remember this is for all what are a & b you have to ask is true for all a & b in G, so a & b are elements of the group G, the domain group and in the bracket here I have ab which means a times b in the group G. I apply the binary operation of the group G to a & b.

Φ of a and φ of b are elements of the group G prime and here I write φ of a times φ of b and that really means that I am applying the group operation of G prime. So, the left hand side group operation of G is being used and right hand side group operation of G prime is used. Okay, so let's keep this is mind. So, it is not any set function but a set function which satisfies this. A good way to remember this is to say this is a set function which satisfies or which respects the group operations of G and G prime.

Whether you multiply first, so this is the way you should think of this whether you multiply first in the group G and then apply φ or you apply φ first and then multiply in the group G prime, you get the same answer. This is what a group homomorphism is. Okay, so whether you multiply first and then apply G or apply G multiply and then apply φ it is same as apply φ and then multiply.

So, multiply and apply φ. Remember when I use the word multiply I really use it to stand for the binary operation of the group G, it's not multiplication of ordinary integers or rational numbers. Right, because G is an arbitrary abstract group. So, multiply is just a convenient word to indicate the binary operation. So, multiply and apply φ that is what we have done in the left hand side is same as apply φ and multiply, which is what we have done on the right hand side. We applied φ first and then multiplied, so we started with a & b we first multiplied them and then applied φ to get φ of ab. On the right hand side we have apply φ first to a and b separately and then multiply. So, a group homomorphism is one were these two operations are going to give you the same answer. Okay let us start with some after the definition let us look at some basic examples and before we continue study the properties of

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group homomorphisms. Let us look at some examples. If you look at, okay let us look at this example. Let us look at our group that we have studied in detail, the group of integers again when I write Z and do not say what the operation is, it is understood that the operation is addition. So, for example let's say I take an integer a and multiply by so you fix an integer n and I define φ of a to be n times a.

So, this is nothing but a + a + a…, n times. Is this a group homomorphism? Is this a group homomorphism? Let's check. So, remember what is that we have to check? We have to check that if you apply the binary operation first and then if you perform the binary operation and then apply φ it should be same applying the binary, apply the function and then perform the binary operation.

So in this example it amounts to checking, we want to check a and b are two integers you want to check that after adding them and then applying φ or first applying φ and then add. This should be true for all integers a & b. Right, but in order to check this let's individually calculate them. What is φ of a + b by definition? Φ of a is n times a. So, φ of a + b simply n times a + b. φ of a is na and φ of, sorry this is actually b okay, φ of a + b should be φ of a + φ of b and φ of b is nb. so, n times a + b is equal to na + nb, yes. So, is this a group homomorphism? Yes.

Okay, so this is a group homomorphism. So, this is a good example of a group homomorphism, multiplying by a fixed integer n. Okay, let us look at another example. So, let us say I consider φ so let maybe I keep using the word letter φ for all these homomorphisms. Let us say I have {1, -1}. I take this to be G prime. We have not seen this explicitly before but is this a group?

Is G prime a group? It is, under our usual multiplication. Right because 1 is the identity element, 1 is the identity element right and -1 is the second element -1 squared is -1 squared is 1, -1 times 1 is -1 and so on. So, this is really G prime is actually a subgroup if you think about it, of let's say rational numbers under multiplication. So, I will make that note here and it is for you to check.

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Note that G prime is a subgroup, subgroup of Q*. If you recall, Q* is the group of non-zero rational numbers under multiplication. So, this is G prime is just a subgroup of it. So, I need to identify G prime as a group right? In order to talk of group homomorphisms so we need two groups. We know that Z is a group and now I am considering the group {1,-1}. That is the group. So, what is the map φ?

So, let us define the map φ, so define φ of A to be so we have two cases. So, let's define it to be 1 if A is even, so if A is even I define φ of A to be 1 and let us say if A is odd, I define it to be -1. So, φ of A is simply going to keep track of whether the number A is odd or even. If it is even we send it to 1, if it is odd we send it to -1. So, it is a function from the integers to the set {1,-1}. Is this a group homomorphism? Is φ a group homomorphism?

So, again remember one has to check that φ of a+b should be equal to φ of a + .Okay so now this is a good example because it will make you deal with the notation that we have been using so far. If you go back and see the definition of a group homomorphism I should say φ of AB equals φ of a times φ of b but remember I have clarified at that point that the multiplication here is whatever is a binary operation in the group that we are dealing with. So, in this example we have Z on the left hand side which is a group under addition and G prime which is {1, -1} which is a group under multiplication.

So, in the definition we have to check that φ of ab is φ of a times φ of b. Right, this is what we have to check, but in this specific example, the left hand side is actually φ of a+b because the binary operation here is a+b the addition should be equal to and what is the binary operation on the right hand side? It is just the multiplication, so it is φ of A, times φ of B. is this true for all A, B in Z. Let's check this.

Note that the image of φ is either 1 or -1 depending on whether A is even or odd. So, let us take two even numbers. So, let us say are both even. If they are both even, φ of A equal φ of B equals 1 because if both are even they both go to 1 and note that then a+b is also even. So, φ of a+b is also 1. If two even numbers are added to each other may also have an even number. We again get even number.

So, φ of a+b is 1 and remember then this condition is satisfied. So, φ of a+b is φ of a times φ of b because this is 1 and this is 1 times 1. So this is okay. On the other hand if A & B are both odd, then what do we have then φ of A and φ of B are both -1 but if A & B are both odd, what is A+B? Sum of two odd numbers is even so A+B is even then what is φ of A+B? That is 1. Again this condition is satisfied right because φ of A+B is 1, φ of A, times φ of B is -1 times -1. So, this is again okay. So, if both are odd also φ of A+B equals φ of A times φ of B. right. Now let us check the other case.

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So, if A is even and B is odd let us say in this case what happens? So, we have φ of A is 1, φ of B is -1 because A is even and B is odd, A+B is odd. Right, A is multiple of 2, B is not a multiple of 2 and when you add you will not get multiple of 2. So, A+B is odd, so φ of A+B is -1 again remember the function is odd numbers go to -1 and even numbers go to1. So, in this case again we have φ of A+B is φ of A times φ of B because this is -1 and this is 1 and this is -1. So, this is okay. So, similarly if A is odd B is even you can check that the same condition holds. So, φ is a group homomorphism.

Okay, so the function which sends an integer to 1 or -1 depending on whether it is even or odd is a group homomorphism. Okay, so the next obvious example would be to consider, so this is probably 3, is the function, the previous function modified slightly. So, let us keep the same groups Z and {1,-1} but now send A to -1 if A is even, 1 if A is odd. Okay, so I have interchanged the previous function.

So, here I sent even numbers to 1, odd numbers to -1. Now I have sent I am sending even numbers to -1, odd numbers to 1. Is φ a group homomorphism? Is φ a group homomorphism? I will leave this as an exercise for you. It is a very easy exercise you can check that it is not a group homomorphism. You can produce two numbers which violate the definition of a group homomorphism. Okay, so this is an example of a function which is not a group homomorphism. So, let me not do anything about this solution because I would like you to try to do this on your own.

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One more example. Let's recall $Gl_n(R)$ so, I do not know if I used the same notation before. Let me use G this notation here Gl_n (R).

Maybe I used some other notation in the past but the set Gl_n (R) remember stands for all invertible n by n real matrices, under multiplication. Okay, so we take all invertible matrices which have size n by n and which have real entries and I make it a group by multiplication because now every matrix in this set has an inverse because it is invertible. Product of two invertible matrices is invertible and inverses exist by definition and so on… multiplication of matrices is associative. Identity is there, so it is a group. This is the group under multiplication.

Now we all know what determinants of functions are, so I consider the determinant function. So, let us take φ from Gl_n (R) to non zero real numbers. So, I take a matrix A and I define φ of A to be determinant of A. So, just simply written det (A). So, det (A) is the function which sends A to determinant of A.

This is the function. So, is φ a group homomorphism? First of all before you even attempt to answer this question, you have to ask yourself what are the two groups involved. I just recall for you what is Gl_n(R). Gl_n(R) is the group of all invertible n by n real matrices under multiplication but what is R*? R* is all nonzero real numbers, also under multiplication right. So, this is also a group that we have seen before.

If you take nonzero real numbers and consider the binary operation given by multiplication it's a group. So, now I can ask: is this is a group of homomorphism? It is the function from one group to another group and the answer is yes and it follows from the properties if you have seen matrices before properties of the determinant. What do we have to check? We have to check again remember the definition of a group homomorphism is φ of ab should be same as φ of a times φ of a.

I am going to use letters capital A, capital B because those that is more standard to denote matrices by capital A, capital B. Again I recall for you the definition of group homomorphism is φ of ab must equal φ of a times φ of b. So, is this true? Yes because what is φ of AB this is determinant of AB and what is φ of A times φ of B? This is determinant of A times determinant of B and if you have studied matrix theory before and when you studied determinant, one of the first properties that you will see is that det (A B) is equal to determinant of A times determinant of B. This is a property of determinant so this is a well-known property of determinants. So, because this is a well-known property of determinants, φ is a group homomorphism. So, this is an important example of a group homomorphism. So, let us look at one more example. So 5 in my counting.

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So, let's say G is an arbitrary group. So, I am going to take any group and fix an element A in G, fix an element A in G. Consider the following function from the group of integers to G, which sends φ of A to sorry φ of now because I have used A for the group element, let me use n for the integer φ of n to be, I define it to be, A power n. Remember again A power n stands for a*a*a…n times.

So, I am applying the binary operation of our group G to A, n times so, this is φ of n equals A power n. Now is this a group homomorphism? So, again remember the definition is φ of m+n because the group Z is under addition we have to check φ of m+n should be equal to φ of m times φ of n because that is the group operation on G. My notation is just φ m times φ n. So, I do not write * unless I want emphasize the point here. So, is this true? But what is φ of m+n? This is A power m+n by definition of phi, m+n goes to A power m+n. This is true and what is φ of m times φ of n?

This is A power m times A power n. Okay, so now the question is this true? Is A power m+n equal to A power n dot A power m and the answer is yes, this is true and this came up in some of the work that we have done in the past when we worked out some examples or looked at some properties of multiplication. So, I am not going to spend a lot of time doing this but quickly I will do this. What is this?

This is a*a*a…. m+n times and this is first I will do a*a …m times and then I do * note that I am using dot and star interchangeably here. Second we have n times. Okay, so we have a*a*a, m times and then *n times but this is, if you use associatively of group operation and expand this out you have to do first A here and then you take this and do this A power n so this is remember A power n (A^n).

So, this is A power n + or rather I should use I am using the distributive property the associative property. So, then you have another a * power n like that and this is n times. So, this is a n+1*a n+1*n+1. So if you keep doing this one by one you get a power m+n. Okay, so this is an exercise for you to check this. So, check this as an exercise. Okay, so I remarked this on this earlier when I use the exponential notation I can use the usual properties of exponents that you have studied about real numbers and integers and rational numbers. So, this will go through. Okay, so this is a group homomorphism. So, let us do one more example and then we will study some properties.

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Now fix again a group but assume now that it is abelian. So, what is an abelian group? Recall G is abelian if ab equals ba for all a, b in G. Okay, so any two elements in G commute with each other.

So, this is an abelian group. Okay, so now I consider the map φ from G to G which sends A to A squared. Okay, so what I am doing? I am taking an element A and I am simply squaring it. Is this is a group homomorphism?

So, we have to check remember check φ of ab should equal φ of a times φ of b. Let's just write both sides of this equation. What is φ of ab? This is ab squared right? That is the definition send any element to its square and what is this? This is a squared times b squared, φ of ab is ab whole squared. Φ of a times phi of b is a squared times b squared. Now are these two equal? And here is where we have to use the abelian hypothesis.

So, let us just write it down so ab whole squared is ab ab, right, this is the definition of ab whole squared ab then ab because the group is associative so I am going to just remove the brackets and write this as ab ab. What is a squared b squared? This is a, a, because that is what a squared is times bb. So, again I will remove brackets so this is a a b b. See there is no reason in general that these two are equal. In general these are not equal.

But note that we are in an abelian group. I am starting with an abelian group. So, aabb I can interchange in any way I want. So, I can interchange the middle two elements that is abab. So, if I interchange this because ab equals ba I have noted that ab equals ba for all the ab, ab equals ba so this is ab times ab. So, this is true. So, the map which stands a to a squared is a group homomorphism if the group is abelian. So, now the question naturally arises:

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Is φ a group homomorphism when G is not abelian? Okay is this is a group homomorphism when G is not abelian. Note that our proof here suggests that it need not be because in general we may not be able to interchange A & B here. But unless we get an example we can't be sure. Just because the proof that we are trying to give here does not work in the non abelian case. The statement itself is wrong for non abelian case, you cannot say that maybe there is some other proof. Okay, so unless you give an example where this is not a group homomorphism when G is not abelian you cannot conclusively settle this question.

So, in order to understand this think of the groups that you know that are not abelian. The simplest such group is S3, so consider remember that in an exercise in an earlier video we have shown that any group of order up to 5 is abelian. So, 1, 2, 3, 4, 5 is abelian. And hence G3 S3 which has 6 elements is the smallest non abelian group and in the notation that I introduced way back in one of the first videos, I used these letters to denote the elements here. These are all bijections of the set {1, 2, 3} to itself.

S3 is always the set of bijection from {1, 2, 3} to {1, 2, 3} so, now consider the map from S3 to S3, I map FI φ sends FI to FI squared, for every I from 1 to 6. Right, so 1 goes, F1 goes to F1 squared, F2 goes to F2 squared and so on. So, now I want you to check as an exercise here and this is a straight forward exercise it is a good way to again remind yourselves about the group operation of S3, φ is not a group homomorphism.

Okay, so you can check that φ is not a group homomorphism. For example maybe I will just quickly tell you how to check this. If you recall F2, so I will recall for you from the first video. F2 sends 1 to 2, 2 to 1, 3 to 3 and F3 sends 1 to 3, 2 to 2, and 3 to 1. So, where does F2, F3 go? So let us do this. If you multiply F2 and F3 1 goes to first you send 1 to 3 then 3 to 3, 1 goes to 3, 2 goes to 2 and then 2 goes to 1, 3 goes to 1 and 1 goes to 1. So, is this clear? So, F2 was this, F3 was this and F2 F3 is this and this if you recall again the notation that I developed in the video where I discussed this in detail this is same as 1 goes to 3, 2 goes 1, 3 goes to 2. So, in this notation is F6 because F2, F3 is F6. So, now if this is a group homomorphism we want

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Where does F2 go under φ now this is F2 squared. We checked that F2 is actually in order two element in, when we discussed orders of elements of a group. So, F2 squared is F1 similarly F3 is also in order two element. So, φ of F3 is F3 squared and that is 1 namely F1, the identity element. Okay, in other words φ of F2 times F3 is φ of F2 F3 is F6 φ of F6 is F6 squared and F6 is an order 3 element of S3. So, this is not F1. On the other hand, φ of F2 times φ of F3 is F1 times F1 which is F1. Okay, so φ of F2 F3 is not equal to φ of F2 φ of F3.

Okay, so I have actually checked all the details here. You have a nonabelian group where the multiplication and squaring is not a group homomorphism. So, this exercise is completely checked. However if the group is abelian you have that is a group homomorphism. Okay, so I will stop the video now, this video. In the next video we are going to study some basic properties of group homomorphisms and learn more about subgroups attached to group homomorphisms. Thank you.