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Transform Techniques for Engineers  
More Examples on Fourier Series of a Periodic Signal  
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# Transform Techniques for Engineers

## *More Examples on Fourier Series of a Periodic Signal*

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Welcome back, in the last video we have seen how a Fourier series converges when  $F$  is a piecewise differentiable function, that is also called piecewise smooth functions, okay, so this we have already proved that when  $F$  is a piecewise differentiable function, that means the derivative exists and that is piecewise continuous function.

Then the Fourier series converges point-wise to the function or the signal  $F(x)$ , when  $F$  is continuous, only continuous function on the interval  $-L/2$  to  $L/2$  then the Fourier series, when if the Fourier series converges, if it converges then it has to converge to the signal function  $F(x)$  at each point  $F(x)$  when at each point of  $X$  where  $F$  is continuous, otherwise it converges to the value of the jump, average value of the jump across the discontinuous point, okay, it is continuous point  $X$ . So if  $F$  is discontinuous at  $X$ , if the Fourier series converges then it has to converge to the average value  $F(x+) + F(x-)$  divided by 2, so half of that so there's basically average value of those left side and right side value of the function at that point, discontinuous point  $X$ .

So we'll see today, so just an example, we will just give you, start with an example and we will see how most of the functions what you see, whatever function you take elementary function with period  $L$  which is obviously it's a piecewise smooth functions that's why you may not see a function that is only continuous but not differentiable function, it's very difficult at every point if you want, so whatever function you take that's mostly piecewise differentiable functions, so

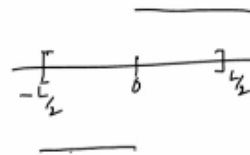
that's why your Fourier series always converges to the function or average value of the function at the discontinuous points, okay.

So once you know how to compute the Fourier series, so in Mathematica or Matlab you can actually calculate and you can plot the function in terms of this Fourier series, you consider the Fourier series and you take few terms, end terms and the Fourier series and then you plot it, sum of all these functions if you plot it together it is actually converges to the function  $F$ , the signal between  $-L$  to  $L$ ,  $-L/2$  to  $L/2$  so that is the period on which it is, interval on which  $F$  is defined.

But the thing is, but the problem with this is, there is something called Gibbs phenomena, when the Fourier series doesn't converge to, I mean if the Fourier series converged to the average value of that function at a discontinuous point there is something called Gibbs phenomena occurs, that means when you compute whatever may be the number of terms in your Fourier series you will always see that there is an error when you actually plot that sum it's always the jumps, jumps will be, suppose the upper jump if you look at the jump left hand side so there may be always some overestimate or underestimate, and if it is overestimate there so you will see the underestimate, underestimate in the right side jump that is the function value at the right side when you plot it with  $N$  terms in your Fourier series, if it is overestimate on the left side you will see the underestimate in the right side, or vice versa if it is underestimation in the left side you will see the overestimate in the right side, that means on an average, finally you are taking the average value of it, sum of the average, sum of these functions,  $F$  of, sum of the function value at both sides that is  $F(x+) + F(x-)/2$ , so one of them, so one is overestimated and the other is underestimated, so eventually so the error is, error won't be there, so finally it converges to the average value, but you cannot fix a number of terms so that you cannot fix however large the number is, there will always be an error, error on the left side of the curve at some in the neighborhood of a point that is where  $F$  is discontinuous, jump discontinuity okay. So we will see with an example that is called Gibbs phenomena, so before we do that we'll just do this example, we'll just take the signal function or  $-L/2$  to  $L/2$ , the simple function will consider, and we get the Fourier series for that, so what is this example is, consider, find the Fourier series, find the Fourier series, find the Fourier series for the function  $F(x)$ , what I consider is a simple function I consider so that will establish for this function, what do we mean by Gibbs phenomena, through this example we will establish that phenomena, that's called Gibbs phenomena, this is  $1/2$  is actually  $-L/2$  to  $L/2$ ,  $-L/2$  to  $0$ , this is  $0$  and this is between  $0$  to  $L/2$ , this is  $L$ , so if you consider this function  $-L/2$  to  $L/2$ , so what you have is initially you have a  $-1/2$  and again you have  $+1/2$ , so what is so from the theorem, from the theory that you learned is clearly see that this function is smooth here is a differentiable function, what is the derivative here? Within this derivative is simply  $0$  here, this is a constant function here, here it is  $0$ , okay.

Example: Find the Fourier Series for the function

$$f(x) = \begin{cases} -\frac{1}{2}, & -\frac{L}{2} < x < 0 \\ \frac{1}{2}, & 0 < x < \frac{L}{2} \end{cases}$$



So derivative is actually continuous even this side is derivative is 0, derivative is 0, so derivative is actually 0 everywhere inside this  $-L/2$  to  $0$ , and  $0$  to  $L/2$ , at  $0$  if you look at the right limit and left limit of the derivative they are also  $0$  so it is a continuous function, but otherwise so because of that the given function is piecewise differentiable function, piecewise in fact is a continuous function, okay, so we'll write piecewise differentiable function, so implies by the result that we have Fourier series, Fourier series converges to the function  $F(x)$  given function, okay.

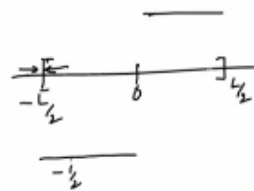
So what is that Fourier series? That is  $\sum_{N=-\infty}^{\infty} C_N \cos N \omega x$ , sorry  $\sum_{N=-\infty}^{\infty} C_N e^{iN \omega x}$ , this converges to  $F(x)$ , so what does it mean? That



$$= f(x) \quad \forall -\frac{L}{2} < x < \frac{L}{2}$$

Example: Find the Fourier Series for the function

$$f(x) = \begin{cases} -\frac{1}{2}, & -\frac{L}{2} < x < 0 \\ \frac{1}{2}, & 0 < x < \frac{L}{2} \end{cases}$$



Sol: given function is piecewise differentiable function.

$\Rightarrow$  the Fourier series converges to  $f(x)$ .

$$\text{i.e., } \sum_{n=-\infty}^{\infty} C_n e^{in\omega x} = f(x); \quad x \in -\frac{L}{2},$$

means this one, this is equal to X for every X, for every X wherever X is continuous, so where is this one, so where is X is continuous?  $-L/2$  to 0 it is continuous, so an X belongs to  $-L/2$ , so including the point 2, so here it is not, at this point the right side this value is  $-1/2$ , and the left side means, so these values you have to consider here so that is right side, so that is right side so there is a left side of, so this limit is actually the value of the function is  $-1/2$  so this limit is  $-1/2$  at this point so here if you look at this side these values you can always view the values here so you have to go here, so in this, this is a simply function value  $1/2$ , so one side is function value is  $-1/2$ , other side is function value is  $+1/2$ , so here function is not continuous, the function value here and here, the endpoints should be same so they are different, so implies a function value is not continuous at the endpoints, that is how you can recognize because you look at the limit of this function, limit  $F(x)$  as X goes to  $-L/2$  positive side is actually  $-1/2$  half clearly, is a constant function here.

Now what happens to this limit  $F(x)$ ? As X goes to  $-L/2$  - negative side that means this side, these values are same because it's periodic they're here, why is it periodic? You can always extend as a periodic function, you can go on like this, so again you can go down and up like that, so values here or exactly values here, so if you view here they are like this, so these values are same as values here, okay, so that is the reason, so values on the left side are same as values here, so if you view this one this is same as a limit  $F(x)$ , as X goes to  $+L/2$ ,  $L/2$  negative, so these is same as  $1/2$ , so you see that value one side is  $+1/2$ , other side is  $-1/2$  so implies it's a discontinuous function at the endpoint, so whether wherever at the points it is continuous, it's only in the open intervals this union 0 open interval  $L/2$ , okay from the result, so we will calculate what this is.

Handwritten work on a digital whiteboard:

Graph:  $f(x) = \begin{cases} -\frac{1}{2}, & -\frac{L}{2} < x < 0 \\ \frac{1}{2}, & 0 < x < \frac{L}{2} \end{cases}$

Sol: given function is piecewise differentiable function.

$\Rightarrow$  The Fourier series converges to  $f(x)$ .

i.e.,  $\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x} = f(x); x \in (-\frac{L}{2}, 0) \cup (0, \frac{L}{2})$

$\omega_0 = \frac{2\pi}{L}$

$C_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-in\omega_0 x} dx$

Limit calculations:

$$\lim_{x \rightarrow -\frac{L}{2}^+} f(x) = -\frac{1}{2} \checkmark$$

$$\lim_{x \rightarrow -\frac{L}{2}^-} f(x) = \lim_{x \rightarrow \frac{L}{2}^-} f(x) = \frac{1}{2} \checkmark$$

So what are your CN's? CN's are simply  $1/L$  integral  $-L/2$  to  $L/2$   $F(x) E$  power  $-IN W$  naught X, what is your W naught? W naught is  $2 \pi/L DX$  so these are your coefficients, Fourier coefficients so if we calculate  $1/L$  so you can write this  $-L/2$  to 0,  $F(x)$  is  $-L/2$  to 0 it is  $-1/2$ , so this is  $-1/2 E$  power  $-IN \omega$  naught X DX + 0 to  $1/2$ , 0 to  $L/2$  it is  $+1/2$ ,  $1/2 E$  power  $-IN \omega$  naught X DX, so I just split this integral into two parts, so what you get is  $-1/L$ , and this becomes  $E$  power  $-IN \omega$  naught X divided by  $IN \omega$  naught, that is the anti-derivative

-L/2 to 0, and this is minus minus plus, so again you have a -I, and you bring this I, I comes up it becomes -I, because I square = 1 so this implies I times I = -1 so this is -1, so I = -1/I or -I is 1/I so this is from complex variables, I is square root of -1, so I means imaginary number that is square root of -1, so this 1 plus again because of here it's minus, for the other integral 1/2L, so there is a 2 missing here, so half that is half here, 1/2 L and you have here IN omega naught into E power -IN omega naught X, and if you bring this up is going to be + I, again for this 0 to L/2, that is the limits you have to apply, so what you get is 0, 0 is -I/2L if you take it out what you have is 1/N MW naught - E power - IN omega naught L/2 divided by NM omega naught + E power, now here this is for the first term.

Fourier coefficient  $C_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x} dx$

$$= \frac{1}{L} \left[ \int_{-L/2}^0 \frac{1}{2} e^{-in\omega_0 x} dx + \int_0^{L/2} \frac{1}{2} e^{-in\omega_0 x} dx \right]$$

$$= \frac{-i}{2L} \frac{e^{-in\omega_0 x}}{n\omega_0} \Big|_{-L/2}^0 + \frac{i}{2L} \frac{e^{-in\omega_0 x}}{n\omega_0} \Big|_0^{L/2}$$

$$= \frac{-i}{2L} \left[ \frac{1}{n\omega_0} - \frac{e^{in\omega_0 L/2}}{n\omega_0} \right] + \frac{i}{2L} \left[ \frac{e^{-in\omega_0 L/2}}{-n\omega_0} \right]$$

$i = \sqrt{-1}$  ·  $i^2 = -1 \Rightarrow i \cdot i = -1$   
 $-i = \frac{1}{i}$  ✓

The second term is I/2L, what you get is, E power - IN omega naught L/2, and this is a thing this is going to be minus minus plus, when you put X equal to, this is going to be plus, and this term have L/2 by N omega naught -1/N omega naught, so you get -I/2L N omega naught and here also -2I/2L N omega naught + I/2L, right, I/2L E power I/2L N omega naught, and this is, this E power IN omega naught L/2 into + E power - IN omega naught L/2, so that will give me 2 cos, so 2 2 goes here this is going to be cos N omega naught L/2, that's what is the result, so this is equal to -2, so 2I, so have I/LN omega naught + I/LN omega naught cos N omega naught L/2, so where is this? Your CN's, N is running from, if I put N = 0 there is a trouble here, so N = 0 I cannot because it's 1/N, this is undefined term, so it's an CN infinity, so when N = 0 you have to integrate separately, and by putting N = 0 this integration, this is valid only for N is + or - 1, + or - 2 and so on, okay, so finally this is actually equal to I by, so omega naught is 2 pi/L, so 2 pi/N, L omega naught is 2 pi N, and you have - 1 + cos 2 pi N, cos 2 N pi is -1 power N okay, so I/LNW, so is I divided by 2 pi N, so cos N omega naught L/2 is cos 2 N pi, cos what is that one? 2 pi/L, L L goes, 2 2 goes, so cos N pi, cos N pi is -1 power N okay.

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$$= -\frac{i}{2L} \left[ \frac{1}{n\omega_0} - \frac{e^{-in\omega_0 L}}{n\omega_0} \right] + \frac{i}{2L} \left[ \frac{e^{in\omega_0 L}}{n\omega_0} - \frac{1}{n\omega_0} \right]$$

$$= -\frac{i}{2Ln\omega_0} - \frac{i}{2Ln\omega_0} + \frac{i}{Ln\omega_0} \cos n\omega_0 \frac{L}{2}$$

$$C_n = -\frac{i}{Ln\omega_0} + \frac{i}{Ln\omega_0} \cos n\omega_0 \frac{L}{2}, \quad n = \pm 1, \pm 2, \dots$$

$$= \frac{i}{2\pi n} [-1 + (-1)^n]$$

So this is what you get for  $N$  is from  $+1$  or  $-1$ ,  $+2$  or  $-2$  and so on, so if you see this this is actually equal to  $0$ ,  $0$  if  $N$  is even then  $-i/\pi N$  if  $N$  is odd, is it true if  $N$  is odd  $N = 1$  or  $-1$  and what you see is it's going to be  $-2$ ,  $-2$  goes  $-i/\pi N$ , okay, this is what I have so, and this is only, and  $N$  is nonzero, this is  $2\pi/N$ , right, yeah that's fine, so this is what you get, this is your  $C_N$ , and which is  $N$  is not equal to  $0$ , but we need  $C_0$  also,  $C_0$  is actually  $1/L$  integral  $-L/2$  to  $L/2$   $F(x) e^{-iN\omega_0 x}$ , when  $N = 0$  that is  $1$ , you have  $DX$ , this is your  $C$  here, this is your  $C_0$ , so  $1/L$  and what you get is  $-1/2$   $-L/2$  to  $0$   $F(x)$  that is  $-1/2$   $DX$   $+1/2$   $0$  to  $L/2$  and  $DX$ , what you get is  $1/L$  so this is going to be  $-1/2$  minus  $1/2L$ , and this is going to be  $0$   $-L/2$ , so that is  $L/2 + 1/2L$   $L/2$ , so this is going to be, this is actually  $0$ , so  $C_0$  is  $0$ , and  $C_N$  is this,  $C_N$  is actually  $0$  if  $N$  is even, if  $N$  is odd it is actually value is this, so this implies your Fourier series is, Fourier series for the given function  $\sum C_N e^{-iN\omega_0 x}$ , right so  $iN\omega_0 x$ ,  $N$  is running from minus infinity to infinity this is equal to, you have  $\sum$ ,  $N$  is from minus infinity to infinity, and  $N$  is not equal to  $0$  because  $N = 0$  it's anyway  $0$ , and what you have is odd.

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$$= \frac{i}{2\pi n} [-1 + (-1)^n], \quad n = \pm 1, \pm 2, \dots$$

$$= \begin{cases} 0, & \text{if } n \text{ is even } \checkmark \\ -\frac{i}{\pi n}, & \text{if } n \text{ is odd, } n \neq 0. \end{cases}$$

$$c_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) dx = \frac{1}{L} \left( -\frac{1}{2} \int_{-L/2}^0 dx + \frac{1}{2} \int_0^{L/2} dx \right)$$

$$= -\frac{1}{2L} \left( \frac{L}{2} \right) + \frac{1}{2L} \left( \frac{L}{2} \right) = 0 \checkmark$$

$\Rightarrow$  Fourier Series is  $\sum_{n=-\infty}^{\infty} c_n e^{in\pi x}$

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When N is odd, odd means you can, when N is odd, only for odd you have  $-i/\pi N$  into  $E$  power  $IN$  omega naught  $X$ , so you have this,  $N$  is not,  $N$  is from minus infinity to infinity,  $N$  is not equal to 0, and  $N$  is odd, so what is this one? This is actually equal to, so you can write  $-i$  divided by  $\pi$  times, if  $N$  is odd so you can say  $2N-1$ ,  $N$  is running from 1, 2, 3 onwards, okay,  $N$  is odd means that,  $N$  is odd means is equivalent to,  $N = 2K-1$ ,  $K$  is running from, and  $N$  is not equal to 0, means they are equivalent to this so you have  $K$  is running from 1 to infinity, okay in fact minus infinity infinity, + or - 1 right, + or - 1, + or - 2 and so on, this is what they are still odd, both sides so you have  $2K-1$   $E$  power  $I$ ,  $N$  is  $2K-1$   $W$  naught  $X$ ,  $W$  naught is  $2\pi/L$ , so  $W$  naught  $X$ , this is equal to  $K$  is from, so  $K$  I just remove  $K$ , so here  $N$  is minus infinity to infinity, use the index  $N$  and this is a cos, that is  $C$ , given function  $F$  is, a Fourier series is this and this has to be equal to, this has to be equal to  $F(x)$  wherever it is continuous, so I write this here, so  $F(x)$  is this, so this implies  $F(x)$  is this,  $F(x)$  is so where is this valid?  $X$  belongs to  $-L/2$  to 0, union 0 to  $L/2$ , so in this open interval it is true, so what you get is  $F(x)$ .

Now  $F(x)$  if you look at this left-hand side,  $F(x)$  is a real value function, so right hand side you should take only a real part, imaginary point has to be 0, okay, so all together so that also you

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$$\Rightarrow f(x) = \sum_{\substack{n=-\infty \\ n \neq 0, n \text{ is odd}}}^{\infty} \left(-\frac{\epsilon}{\pi n}\right) e^{in\omega x}.$$

$n \neq 0$  is odd  $\Leftrightarrow n = (2k-1), k = \pm 1, \pm 2, \dots$

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$$\Rightarrow f(x) = \sum_{k=-\infty}^{\infty} \frac{-\epsilon}{\pi(2k-1)} e^{i(2k-1)\omega x}, \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right)$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \dots$$

will have a extra result so what you have is, if you just separately this complex number, if you write real part and imaginary part separately what is the, imaginary part is - I and what you get is  $\cos 2N$ , I'm writing in terms of  $N$ ,  $2N-1$  omega naught  $X$  divided by pi times, so pi anyway constant goes out  $2N-1$  so this is the imaginary part plus a real part is  $I$  square,  $I$  square - 1, minus minus plus what you are left with is  $N$  is from minus infinity to infinity,  $1$  over pi times  $2N-1$  you have sine, sine  $2N-1$  omega naught  $X$ , that's what it is, okay, so this has to be equal to the real part which is this, so this implies given function  $F(x)$  that is, given function  $F(x)$  which is  $-1/2$  and  $+1/2$  in those intervals is actually equal to,  $N$  is from minus infinity to infinity,  $1$  over pi, pi is constant, sine function  $2N-1$  W naught  $X$  that is  $2 \pi/L X$  divided by this is all sine thing, you have to, this is sine of all thing so into divided by  $2N-1$  is this, and what is left, what is left is this one, so that has to be 0.





$$\Rightarrow f(x) = \sum_{k=-\infty}^{\infty} \frac{-e^{-i(2k-1)\frac{\pi}{2}x}}{\pi(2k-1)}, \quad x \in \left(-\frac{1}{2}, 0\right) \cup \left(0, \frac{1}{2}\right)$$

$$\Rightarrow \underline{f(x)} = \frac{-1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\cos(2n-1)\frac{\pi}{2}x}{2n-1} + \sum_{n=-\infty}^{\infty} \frac{\sin(2n-1)\frac{\pi}{2}x}{\pi(2n-1)}$$

$$\Rightarrow f(x) = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin((2n-1)\frac{\pi}{2}x)}{2n-1}, \quad \boxed{\sum_{n=-\infty}^{\infty} \frac{\cos(2n-1)\frac{\pi}{2}x}{2n-1} = 0}$$

So this is what you get a new identity but it may be the fact divided by it has to be true, this is equal to 0, this we don't care what it is but that value is actually 0, you get an identity the fact that the sum of all these things are 0, and what you get is this, this is your Fourier series that converges to the point or we can do something else here, so what we do is instead of this, so what you get is I can rewrite, because it's a Fourier series is converges in this, we can always rewrite, this is a minus  $1/\pi$  sigma,  $K$  is from, and of course  $K$  is not equal to 0, so that is what is the thing, and if I don't use  $K$ ,  $N$  is from 1 to infinity and what you get is  $1/2N-1$  E power  $1/2N-1$  W naught  $X$  plus so if I add here so this is going to be  $N$  is from - infinity to -1, again  $1/2N-1$  E power  $1/2N-1$ , I just split it into two parts.

$n = -\infty$   
 $n \neq 0, n \text{ is odd}$       $n = (2k-1), k = \pm 1, \pm 2, \dots$

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$$\Rightarrow f(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} -\frac{i}{\pi(2k-1)} e^{i(2k-1)\omega_0 x}, \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right)$$

$$\Rightarrow f(x) = -\frac{i}{\pi} \left[ \sum_{n=1}^{\infty} \frac{1}{(2n-1)} e^{i(2n-1)\omega_0 x} + \sum_{n=-\infty}^{-1} \frac{e^{i(2n-1)\omega_0 x}}{(2n-1)} \right]$$

$$= -\frac{i}{\pi}$$

So now from  $-i/\pi$  so let's not rewrite, we'll do the same thing, so this implies  $F(x)$  equal to, this is running from,  $K$  is from minus infinity to infinity, and  $K$  is not equal to 0. and you get  $-i/\pi$  by  $2K-1$  this is going to be  $E$  power, I'll just leave it like this, so  $E$  power, so if you rewrite put this instead of  $K$  you write it as  $N$  as an index,  $N$  is not equal to 0 and  $N$  is running from minus infinity to infinity integers,  $-I$  over this and  $E$  power  $I 2N-1$   $W$  naught  $X$  this is valid in this interval, so this will give you your because  $F$ , left hand side the given function is a real valid function, a real value has to be equal to right side the real value, whatever remaining a imaginary part that has to be 0, so you have to equate, you can equate both sides real and imaginary parts, so here left hand side real part is itself  $F$ , and if you take the real part here that is going to be  $\sum N$  is from minus infinity to infinity,  $N$  is not equal to 0 and what you get is  $1/\pi \sin$ ,  $I \sin 2N-1$   $W$  naught  $X$  divided by  $2N-1$ , so that  $-I$  becomes plus, so this is what you get the Fourier series for  $F$ , and that is happen to be sine series which is valid here, union the open interval this one.

So what happens, there is another part that is a imaginary part, that imaginary part if you equate, real part you equated, now you equate the imaginary part here, in this equality what you get is 0 equal to  $\sum N$  is from minus infinity to infinity, and  $N$  is not equal to 0, and what is the imaginary part? Is  $-1/\pi 1/2N-1$  and this is  $\cos 2N-1$   $W$  naught  $X$ , the  $W$  naught is  $2\pi/L$ , so for every  $X$ ,  $X$  in this interval  $-L/2$  to  $0$ , union  $0$  to  $L/2$  this quantity is actually 0, so  $-PI$  is constant, so this overall this constant has to be 0, this is the equality you get because of, as a

$$\Rightarrow \checkmark f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} -\frac{i}{\pi(2n-1)} e^{i(2n-1)\omega_0 x}, \quad x \in (-\frac{L}{2}, 0) \cup (0, \frac{L}{2}).$$

$$\Rightarrow f(x) = \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}, \quad x \in (-\frac{L}{2}, 0) \cup (0, \frac{L}{2}).$$

$$0 = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\cos(2n-1)\omega_0 x}{2n-1}, \quad \omega_0 = \frac{2\pi}{L}.$$

$\nexists x \in (-\frac{L}{2}, 0) \cup (0, \frac{L}{2}).$

byproduct from the Fourier series, okay, this is anyway we don't use this material, so what we require is this is your Fourier series for the given function that is piecewise continuous function, piecewise smooth function, piecewise differentiable function which is like this  $-1/2$  and  $+1/2$  here, this is valid only because this value here and value here they are not same is also discontinuous point, this endpoints are also discontinuous points, another point is  $0$  is the discontinuous point, so you have three discontinuous points at which what happens. So what happens at  $0$ ? First consider at  $X = 0$ , at  $X = 0$  what we see  $F(0)$ ,  $F(0+)$  is  $1/2$ ,  $F(0-)$  is  $-1/2$  so the average value is simply  $1/2$ ,  $F(0+) + F(0-)$  divided by  $2$  is  $1/2 + 1/2$  divided by  $2$  that is  $1/2$ , and what we have seen already that this Fourier series which is  $1/\pi$  times sigma  $N$  is from minus infinity to infinity,  $N$  is not equal to  $0$  sine  $2N-1$   $\omega_0 x$  divided by  $2N-1$  this if you take  $X = 0$ , this is equal to  $1/2$  when  $X = 0$ ,  $X = 0$  so average value at  $0$ , so what is this one average value of this is  $1/2$  and this is  $-1/2$  that is what we made a mistake, so here it is  $-1/2$

$$\Rightarrow f(x) = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{-i}{\pi(2k-1)} e^{i(2k-1)\omega_0 x}, \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right)$$

$$\Rightarrow \checkmark f(x) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{-i}{\pi(2n-1)} e^{i(2n-1)\omega_0 x}, \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right)$$

$$\Rightarrow f(x) = \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}, \quad x \in \left(-\frac{L}{2}, 0\right) \cup \left(0, \frac{L}{2}\right) \checkmark$$

$$\Rightarrow \left. \begin{aligned} 0 &= \sum_{n=1}^{\infty} \frac{\cos(2n-1)\omega_0 x}{2n-1} \\ &- \sum_{n=1}^{\infty} \frac{\cos(2n-1)\omega_0 x}{2n-1} \end{aligned} \right\} \omega_0 = \frac{2\pi}{L}$$

when you add this is going to be  $1/2 - 1/2$  divided by 2 which is 0, so clearly the average value has to be 0, so when you put  $X = 0$ , so obviously this Fourier series which is also 0 which is true.

Now what happens at  $X = -L/2$  or  $+L/2$ ? Both places what happens, because their function is discontinuous function you calculate  $F(-L/2)$  and  $+$ , this  $+$  side is same as it is actually equal to  $-1/2$ , and  $F(-L/2)$  actually you need  $+L/2 -$  side, at the endpoints you need, right, at every endpoint what happens the Fourier series converges to  $F(L/2) + F(-L/2)$  so you just calculate  $-L/2$  and  $-L/2$  so these are values which is  $1/2$ , so again  $F(-L/2) + F(L/2)$  divided by 2 is again 0, because they are  $-1/2 + 1/2$ , this value and this value again this is  $1/2$  and this is  $-1/2$ , average value is 0, so there also it is 0 which you can easily see here, so  $1/\pi$  if you put here  $X = -$  or  $+ 1/2$  what you end up is  $1/\pi N$  is from minus infinity to infinity,  $N$  is not equal to 0, sine  $2N-1$  W naught  $+$  or  $- 1/2$  you put  $+$  or  $- 1/2$  as  $X$ , what happens to this? An  $X = +$  or  $- 1/2$  any one of them if you put  $L/2, +$  or  $- L/2$  at both the end points value has to be 0 from the theory, is it really true? So if you look at term  $\sin 2N-1, 2\pi/L X = +$  or  $- L/2$ , so that comes  $+$  or  $-$  because

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$x \neq 0$                        $x = 0$

At  $x = \pm \frac{L}{2}$ :  $f(-\frac{L}{2}) = -\frac{1}{2}$ ,  $f(\frac{L}{2}) = \frac{1}{2}$ ,  $\frac{f(-\frac{L}{2}) + f(\frac{L}{2})}{2} = 0$

$$\frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1} \Bigg|_{x=\pm \frac{L}{2}} = 0 \checkmark$$


---

$\pm \sum_n \frac{\sin(2n-1)\pi \frac{L}{2}}{2n-1} = 0 \checkmark$

of sine, it comes out of sine, so 2 2 goes and pi times this which is 0, we know that this is true, and N be 0 so that is why each term for every N nonzero or whatever, when N is nonzero, N is running from - infinity, denominator never 0, a numerator is 0, so because W not = -2 pi/L, so this is 0, so it's verified, so this is how you see that the Fourier series converges point to the function at a continuous point when the function is continuous at that point, otherwise its average value it converges to the average value, that is what we can see from this example.

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$\pm \sum_n \frac{\sin(2n-1)\pi \frac{L}{2}}{2n-1} = 0 \checkmark$

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So what is that I am trying to say from this function? Will use it, will demonstrate what is a Gibbs phenomena, I'll just try to explain and I will remark on, we will remark convergence of

some other thing, so Gibbs phenomena is we will try to see this with some other example in the next video.

Yeah, Gibbs phenomena what you are trying to see is you have this, this is your function which is, this is your X axis, this is your L/2, this is your 0, and this is your -L/2, okay, this is your -1/2, and this is your +1/2 which is constant function, okay.

Now I'll try to take in this Fourier series, what is your Fourier series?  $\frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}$  is from minus infinity infinity, N is not equal to 0, so this is your Fourier series in this, I fix my N, W naught X /2N-1, I fix my N, M I take some N terms let us say some 10 terms, 5 terms, 10 terms,

Handwritten notes and graph illustrating Gibbs' phenomenon:

Equation 1: 
$$0 = \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1} \Big|_{x=\pm \frac{L}{2}} = 0 \checkmark$$

Equation 2: 
$$\pm \lim_{x \rightarrow \frac{L}{2}} \frac{\sin(2n-1)\frac{2\pi x}{L}}{2n-1} = 0 \checkmark$$

Equation 3: 
$$\frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1)\omega_0 x}{2n-1}$$

Equation 4: 
$$n = 5, 10, \dots, 1000, \dots$$

Graph: A plot of the function  $f(x) = \begin{cases} 1 & 0 < x < L/2 \\ -1 & -L/2 < x < 0 \end{cases}$ . The x-axis is labeled with  $-L/2$ ,  $0$ , and  $L/2$ . The y-axis is labeled with  $1$  and  $-1/2$ . The graph shows the function and the partial sums of the Fourier series, which exhibit overshoot near the discontinuity at  $x=0$ .

any number, okay, thousand terms and so on, you can take anything, when you do what you see is that even surely as N becomes bigger and bigger this Fourier series is actually converges, see when you to do this one this is converging but the problem here is you will see that there is an error here, see that it's actually it's underestimated, this is less than more value, less value, is actually underestimation of 1/2, what is underestimation 1/2 is? Over estimation means more value, underestimation means it comes down, so here you underestimated, overestimated okay, so it's overestimated here, so what happens, in this here it will underestimate, this side it'll underestimate so you will see that this kind of plot, so the value the limit value is here and the limit value is here lower.

So the jump is same okay, actual value is this value and this value + or -, -1/2 and +1/2 difference is, average will is 0 but here the average value so what happens here, so when it underestimates  $1/2 + 1/2 - \text{some Epsilon}$ , here  $-1/2 - \text{Epsilon}$ , so this amount if it is underestimated that means it's added, it's added - epsilon, if it is overestimated like this and

$$0 = \frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1) \frac{\pi x}{L}}{(2n-1)} \Big|_{x=\pm \frac{L}{2}} = 0 \checkmark$$

---

$$\pm \sum_n \frac{\sin(2n-1) \frac{\pi x}{L}}{(2n-1)} = 0 \checkmark$$

$$\frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1) \frac{\pi x}{L}}{(2n-1)}$$

$n = 5, 10, \dots, 1000, \dots$

Gibbs's Phenomenon:  $\frac{1}{2}(\pm \epsilon)$

here it is underestimated, okay, so it is something like this I can show you by plotting it, everything is converging here, but here it starts there is errors, error will be more here is less and then it goes here and here it is more, sorry, this should go and here is fine and there is small errors and then it's more here, so if it's less here and this is more here, so if it's more, if it is less here this will be more here at this point, the limit, limiting values, so whatever may be this is + or - will be added or subtracted.

$$\pm \sum_n \frac{\sin(2n-1) \frac{\pi x}{L}}{(2n-1)} = 0 \checkmark$$

$$\frac{1}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{\sin(2n-1) \frac{\pi x}{L}}{(2n-1)}$$

$n = 5, 10, \dots, 1000, \dots$

Gibbs's Phenomenon:  $\frac{1}{2}(\pm \epsilon)$

Finally a jump that is this value would be same, so this value finally the convergence value is same, but whatever may be N, big any number of terms you take in this Fourier series you will not be able to capture the actual image of the function that is actual value, so that is the step

function you will never able to cover it with your these functions by choosing any number of terms, okay, so even if you choose big number there will always be some error here, so more error here I mean if it is lesser than this value you get more value here, so or an average a Fourier series actually converges to the average value of it which is actually same, if it comes down here and it will go down here, if it goes up here this will go up here the error, so that is called a Gibbs phenomena, we will explain this with an example maybe in the next video, and explain what is the Gibbs phenomena through that example. Thank you very much.

[Music]

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