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Transform Techniques for Engineers
Conditions for the Convergence of Fourier
Series (continued)
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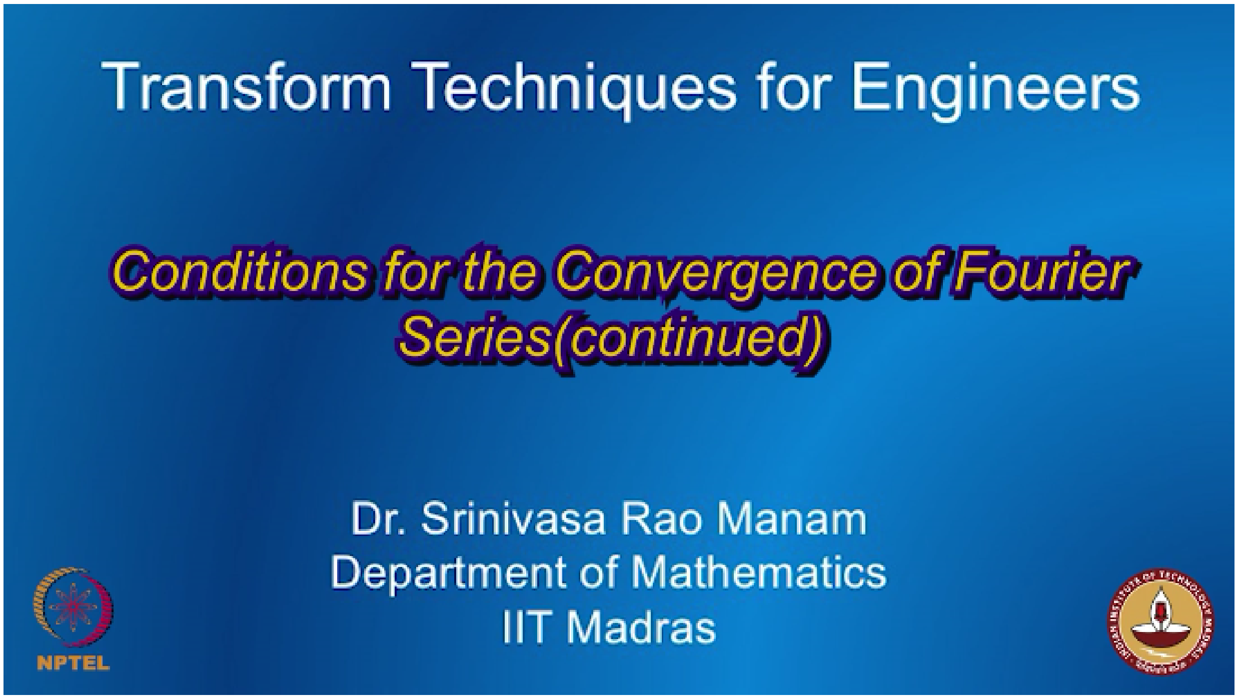

Hello, in continuation of the discussion that we had on the last video, on the convergence of Fourier series to the function F, when F is a periodic differentiable function that is piecewise, so basically that means function, when the function F is piecewise differentiable periodic function that means it's derivative function that is F dash(x) is a piecewise periodic function.
So with period L, if such a function F, what is its Fourier series and it's convergence, it converges to the function F itself whenever X belongs to the period interval –L/2 to L/2, so as of now we'll just, we're only talking about the open interval –L/2 to L/2 and eventually we discuss at the end, once we prove this we will discuss the endpoints also, or the points of discontinuity, so these things we will see, so anyway so there's no point of discontinuity because we are considering differentiable function that means it's continuous function, okay.

Theorem: If $f(x)$ is a piecewise differentiable periodic function with period L;

then $$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+)+f(x^-)\right);\quad x\in\left(-\frac{L}{2},\frac{L}{2}\right).$$
$$= f(x),\quad x\in\left(-\frac{L}{2},\frac{L}{2}\right).$$

$f(x) = f(x^+) = \lim_{t\to x^+} f(t)$

$f(x) = f(x^-) = \lim_{t\to x^-} f(t)$

Proof: Let $$S_n(x) = \sum_{k=-n}^{n} c_k e^{ik\omega_0 x} \checkmark$$
$$= \sum_{k=-n}^{n} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-ik\omega_0 t}\,dt\; e^{ik\omega_0 x}$$
$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\cdot \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}\,dt.$$
$$= \frac{1}{L}\int_{0}^{L/2} f(t)\sum_{k=-n}^{n} e^{-ik\omega_0(t-x)}\,dt + \frac{1}{L}\int_{-L/2}^{0} f(t)\sum_{k=-n}^{n} e^{-ik\omega_0(t-x)}.$$

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So we were discussing about the proof of this, we started with this, we consider the series, a partial sums of that series and we just simplified and in between we define what is DN, and

$$= \frac{1}{L}\int_{0}^{L/2} f(t)\sum_{k=-n}^{n} e^{-ik\omega_0(t-x)}\,dt + \frac{1}{L}\int_{-L/2}^{0} f(t)\sum_{k=-n}^{n} e^{-ik\omega_0(t-x)}.$$

Let $$D_n(x) = \sum_{k=-n}^{n} e^{-ik\omega_0 x} = e^{in\omega_0 x} + e^{i(n-1)\omega_0 x} + \dots + e^{-in\omega_0 x} \checkmark$$
$$= e^{in\omega_0 x}\left(1 + e^{-i\omega_0 x} + e^{-i2\omega_0 x} + \dots + e^{-i2n\omega_0 x}\right)$$
$$= e^{in\omega_0 x}\cdot\left(\frac{1 - e^{-i\omega_0 x(2n+1)}}{1 - e^{-i\omega_0 x}}\right)$$
$$= \frac{e^{in\omega_0 x} - e^{-i(n+1)\omega_0 x}}{1 - e^{-i\omega_0 x}}$$
$$= \frac{e^{i\frac{\omega_0}{2}x}\left(e^{in\omega_0 x} - e^{-i(n+1)\omega_0 x}\right)}{\left(e^{i\frac{\omega_0}{2}x} - e^{-i\frac{\omega_0}{2}x}\right)}$$
$$e^{i(n+\frac{1}{2})\omega_0 x} - e^{i(n+\frac{1}{2})\omega_0 x}$$

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then when you substitute and you observe that DN is this, DN of this function is actually this,

$$= \frac{e^{i(n+\frac{1}{2})\omega_0 x} - e^{-i(n+\frac{1}{2})\omega_0 x}}{e^{i\frac{\omega_0}{2}x} - e^{-i\frac{\omega_0}{2}x}}.$$

$$D_n(x) = \frac{\sin(n+\frac{1}{2})\omega_0 x}{\sin \frac{\omega_0}{2} x}.$$

$D_n(-x) = D_n(x)$. even function.

$$S_n(x) = \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, D_n(t-x)\, dt$$

$t - x = x'$

$dt = dx'$

$$= \frac{1}{L}\int^{L/2} f(x+x')\, D_n(x')\, dx'$$

simple sine function involving simple sine functions and also we observe that it is an even function and then it becomes this, partial sum becomes this once you change of variable in this integrand T-X = X dash will give you, so this type of function, so this partial sum will, only so

$$S_n(x) = \frac{1}{L}\int_0^{L/2} f(x-t)\, D_n(t)\, dt + \frac{1}{L}\int_0^{L/2} f(x+t)\, D_n(t)\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) + f(x-t)\right] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin \frac{\omega_0}{2} t}\, dt$$

far we consider only partial sum of that series, Fourier series and we end up having this expression.

So what we do now, we will just rewrite this, so we'll rewrite and see that as N goes to infinity this converges to F(x), so that's what we are going to show, so we will show as N goes to infinity this partial sums of the series, Fourier series it converges to F(x), let's see how we will do this, so I'm just going to do some manipulation here, so we'll write 1/L so it's from 0 to L/2, what I do is I just subtract something from this quantity so that I have to add, once you subtract and you also have to add, so we are fixing X, X we are fixing, and T is the integrating variable, so we write F(x+t) - F(x+), X+ and X- we have defined that means limit of F(t), as T goes to X+, as T goes to X- that means coming from right side or left side of X, that limit is this F at X+ and X-, okay.

$$= \frac{1}{L}\int_{-L/2} f(x+x')\, D_n(x')\, dx' + \frac{1}{L}\int_{0} f(x+x')\, D_n(x')\, dx'$$

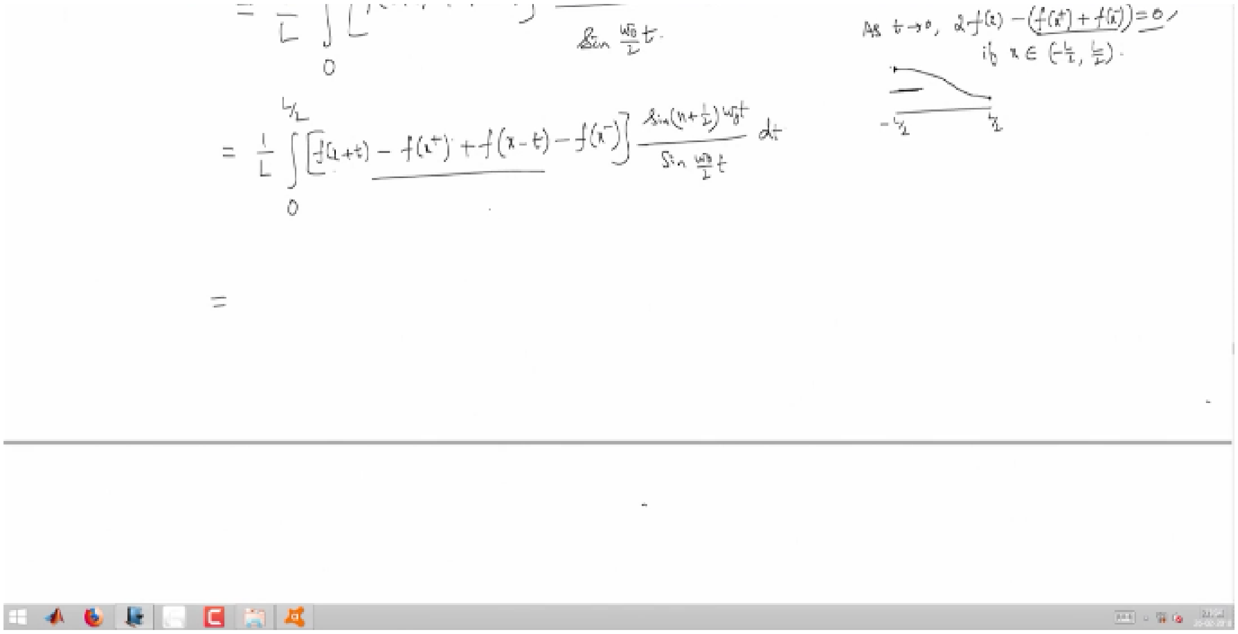
$x' = -t$

$$S_n(x) = \frac{1}{L}\int_0^{L/2} f(x-t)\, D_n(t)\, dt + \frac{1}{L}\int_0^{L/2} f(x+t)\, D_n(t)\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) + f(x-t)\right] \frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0}{2} t}\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) - f(x^+)\right.$$

And this is what I subtracted plus, from this also you can do the same, so X-T – F(x-) okay, and then this divided by, and you have this sine N+2, N+1/2 W naught T DT divided by sine W naught/2T DT that's what it is, so see as T, so T is between 0 to L/2, so as T goes to 0 and I have the quantity and this is 0 and you have the top quantity that is the numerator, this part is 0, so as T goes to 0, FX- FX+ +FX- so what happens as T goes to 0, if you look at it as T goes to 0 this quantity becomes F(x), basically two times F(x) – F(x+) + F(x-), because this is continuous function, because anyway is actually is 0, if X is in between, if X belongs to – L/2 to L/2, if it is in the open interval, if it is the endpoints we don't know really, so if it is like this your function is like this, if it's repeated from – L/2 to L/2, so you see this endpoint here is that this is the different values, okay, so two times of, this is really not you don't know exactly what this means, okay, so if X is this, if X equal to, if you take –L/2 and this this difference average value is here, so you cannot say that, so in that case we don't know, so it's not 0, okay, so you actually show that is actually 0, it has to be 0 otherwise this integral doesn't make sense, because denominator we have 0, so numerator has to be 0.



So anyway so that observation, so just to avoid that and we also know that, just remove this here, we also know that these limit F(x+t) – F(x+) divided by X+T -X + that is T, so this limit T goes to 0 is nothing but, this is anyway X side, X positive side minus that limiting value, so there's something like this here, so you take some value here this is X+T, this is X+ so that will come from a value of the function from this limit, so this is actually F derivative at X+ at this value so this is a derivative at X+, if it's piecewise this may be different from and the other side X-, that is need not be same because what we construct is a piecewise a differentiable function, okay, so this need not be same as this, so this is what it is, so you take this, so because we assume that is piecewise differentiable function so this is finite.
Similarly we have T goes to 0, F(x-t) – F(x-) divided by T again, X-T -X okay, so this is also F dash at X-, this is also finite because that's what we assume, so to get this quantity we only divide, we need to divide with T, so if we divide T we have to multiply T, so you rewrite, there's a most elementary way of doing, showing the convergence of this Fourier series, so I write this F(x+t) – F(x+) + F(x-t) – F(x-) this I divide with T for both the parts, and also I write T divided by sine omega naught by 2 T, okay, into that whatever remaining part N+1/2 omega naught DT, so I just multiplied, anyway it’s T is from 0 to L/2, so T is nonzero so you can always divide.

$$= \frac{1}{L}\int_0 \Big[\cdots \Big] \frac{\cdots}{\sin\frac{\omega_0}{2}t}$$

$$= \frac{1}{L}\int_0^{L/2} \Big[f(x+t) - f(x^+) + f(x-t) - f(x^-)\Big] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \frac{\big[f(x+t) - f(x^+) + f(x-t) - f(x^-)\big]}{t} \frac{t}{\sin\frac{\omega_0 t}{2}} \sin(n+\tfrac{1}{2})\omega_0 t\, dt$$

$$\lim_{t\to 0} \frac{f(x+t) - f(x^+)}{t} = f'(x^+) < \infty$$

$$\lim_{t\to 0} \frac{f(x-t) - f(x^-)}{t} = f'(x^-) < \infty$$

As T goes to 0 it doesn't mean that T is 0, so T goes to 0 means T is nonzero and it's approaching 0, so I can always divide L, divide with T both sides so this quantity I call something, the quantity here call it something else so, so once I add and subtract I have to, so here I subtracted so I have to add so that quantity is whatever is left with is -L 0 to L/2 I subtracted so I have to add F(x+) + F(x-) into this quantity sine N+1/2 omega naught T/sine omega naught, omega naught/2T DT this integral is addition so I have to add here also, 1/L 0 to L/2 this quantity says anywhere nothing to do with that integrant variable so I write this outside F(x+) + F(x-) times integral 0 to L/2 sine N+1/2 omega naught T divided by sine omega naught T/2 DT, so this quantity, this integral you can actually show that is actually 1, okay.

$$= \frac{1}{L}\int_0^{L/2} \Big[f(x+t) + f(x-t)\Big] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \Big[f(x+t) - f(x^+) + f(x-t) - f(x^-)\Big] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt + \frac{1}{L}\int_0^{L/2} \big(f(x^+) + f(x^-)\big) \cdot \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left(\frac{\big[f(x+t) - f(x^+) + f(x-t) - f(x^-)\big]}{t} \frac{t}{\sin\frac{\omega_0 t}{2}}\right) \sin(n+\tfrac{1}{2})\omega_0 t\, dt + \frac{1}{L}\big(f(x^+) + f(x^-)\big)\int_0^{L/2} \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0 t}{2}}\, dt$$

$$\lim_{t\to 0} \frac{f(x+t) - f(x^+)}{t} = f'(x^+) < \infty$$

$$\lim_{t\to 0} \frac{f(x-t) - f(x^-)}{t} = f'(x^-) < \infty$$

So we look at this integral plus this integral, so before we allow N goes to infinity, this is actually our SN, I first showed that I eventually I allow N goes to infinity, so before to do that, before doing that we will just take these 2 integrals, we will handle these 2 integrals separately, so we will start with the first this one, so one observe that, observe that this integral 0 to L/2 sine N+1/2 omega naught T DT divided by sine omega naught T/2, this value if you directly evaluate in terms of sines it may be little tough, so instead you write as sum, so this is simply, so you just, because this is an even function, because this is a sine function, this is a sine function, so it's an even function, so this is actually, because the integrand is an even function you can write from –L/2 to L/2 and this is going to be a sine same function, okay, omega naught T DT/sine omega naught T/2, and this is equal to 1/2 I integrate from L/2 to L/2, so this quantity integrand will just, this is actually DN(x) which I use a primitive expression that is finite sum, so that is partial sums -N to N, E power -IK omega naught X, okay, omega naught T DT.

Now you can see that, so once you have this exponential function which is running from –L/2 to L/2 whenever K is nonzero, whenever K is nonzero this integral value is actually equal to 0, okay, so when this is the nonzero only when K = 0, though which you have –N, -N, -1 and so on you keep on doing up to 0, 0 is the only contribution and K = 0 that is only contribution is left, and when K= 0 this is simply 1, other values so you should write like this, you can take this sum outside integral –L/2 to L/2 E power - IK omega naught T DT.

So now you look at the each of this when K is -N to N, so whenever this exponential function, when K is nonzero this is going to be 0, this integral is 0, so for that reason only contribution will be when K = 0, so when K = 0 so that is nothing but –L/2 to L/2, K = 0 that is 1 DT, so this is nothing but L/2, this is what it is, so you have this, this will be L/2 so you have finally this whole thing is 1/2 of the average value of, that is actually 1/2 of F(x+) + (x-) the average value.

$$S_n = \frac{1}{L}\int_0^{L/2}\left(\frac{[f(x+t)-f(x^+)+f(x-t)-f(x^-)]}{t}\,\frac{t}{\sin\frac{\omega_0 t}{2}}\right)\sin\left(n+\tfrac{1}{2}\right)\omega_0 t\,dt$$

$$+\frac{1}{L}\left(f(x^+)+f(x^-)\right)\int_0^{L/2}\frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t}{\sin\frac{\omega_0 t}{2}}\,dt$$

Observe that
$$\int_0^{L/2}\frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t\,dt}{\sin\frac{\omega_0 t}{2}} = \frac{1}{2}\int_{-L/2}^{L/2}\frac{\sin\left(n+\frac{1}{2}\right)\omega_0 t\,dt}{\sin\frac{\omega_0 t}{2}}$$

$$= \frac{1}{2}\int_{-L/2}^{L/2}\sum_{k=-n}^{n} e^{-ik\omega_0 t}\,dt$$

$$= \frac{1}{2}\sum_{k=-n}^{n}\int_{-L/2}^{L/2} e^{-ik\omega_0 t}\,dt$$

$$= \frac{1}{L}\int_{-L/2}^{L/2} dt = \frac{L}{2}$$

And we also define what is Q, let me define what is QN(x), it depends on N, so if you write this, now let's say not QN so it's just Q, Q(x) if I write it as that first integrand whatever in these brackets, this brackets F(x+t) – F(x+) + F(x-t) – F(x-)/T into T/sine omega naught T/2 that's what if you define like this then what happens to this integral, so you can see that it's a well-defined function, so Q(t) as rather, so let's write Q(t), so X is fixed, when we fix X should use the function of T it's well-defined for every, so as T goes to 0 this quantity is 1, sine X/X type and this quantity is anyway these derivatives are exist as T goes to infinity, no other T value as T between 0 to pi L/2 there is no issue so it's well-defined function, so QT is defined for every T is in 0 to L/2, okay.

If you do that what happens to your SN? SN is a QN, this is Q(t) that sine function, so let's write what happens to your SN, SN(x) is what do you have is 1/L 0 to L/2 Q(t) into sine N+1/2 omega naught T DT plus the other integral by the earlier calculation so this becomes 1/2 of F(x+) + F(x-) okay, this is what it is.

Let $$Q(t) = \frac{f(x+t) - f(x^+) + f(x-t) - f(x^-)}{t} \cdot \frac{t}{\sin \frac{\omega_0 t}{2}} \quad \forall\ t \in \left(0, \tfrac{L}{2}\right).$$

$$S_n(x) = \frac{1}{L}\int_0^{L/2} Q(t)\, \sin\left(n+\tfrac{1}{2}\right)\omega_0 t\, dt + \frac{1}{2}\left(f(x^+)+f(x^-)\right)$$

Now what happens to this? Now we look at this integral as N goes to infinity, so we'll take this limit, N goes to infinity so you have this limit N goes to infinity and this is anyway constant so that doesn't change with the limit, so we only have to worry about this N goes to infinity what happens to this 1/L 0 to L/2 Q(t) sine function sine N+1/2 omega naught T DT, okay, so this is equal to, so I'll write, so again so Q(t), Q(t) if you observe this T T goes, this is what is this one, this becomes, what happens this is odd function, if you see, replace T by -T numerator is same, here this this cancel so you have minus because of sine function, sine of -T so minus comes out so this is an odd function, so Q(t) is an odd function, sine T is an odd function, so together this is an even function, okay, so this integrand is an even function so again like earlier we can write a limit of 1/L rather so 2 /L times –L/2 to L/2, Q(t) sine N+1/2 omega naught T DT, so because it is an even function, so integrant is even so you can write 2 times that integral so you guys replace, you can extend it to negative side, so this is equal to limit N goes to infinity, so you can write now 1 by, rather half of it right so this is 1/2 times, so you have 1/4 times 2/L –L/2 to L/2, 2 2 cancels the same, okay, and you have Q(t), now you can expand this sine function so that is sine N omega naught T and cos of, cos omega naught T/2 times sine N+1/2 omega naught T +, and now Q(t) sine omega naught T/2 of this function you have cos N+1/2 omega naught T, so this DT, so this is what you have so this implies.
So what happens so as limit N goes to infinity, so what is this one? This is limit, N goes to infinity 1/4 so this is exactly Fourier coefficient of this function with sine and cosine, okay, sorry I think it should be, this is N, sine N, sine N omega naught, this is cos I expand it right so you have cos N omega naught T, so this is a Fourier coefficient of, sine coefficient of, Fourier coefficient, that is the coefficient of sine and omega naught T for this function Q(t) cos omega naught T/2, and we have seen by Riemann Lebesgue Lemma if Q(t) is well-defined function, so we'll start with whatever you have, F is piecewise periodic function, so F is differentiable

$$\lim_{n\to\infty} \frac{1}{L}\int_0^{L/2} Q(t)\,\sin\left(n+\tfrac{1}{2}\right)\omega_0 t\,dt$$

$$= \lim_{n\to\infty} \frac{1}{2L}\int_{-L/2}^{L/2} Q(t)\,\sin\left(n+\tfrac{1}{2}\right)\omega_0 t\,dt$$

$$= \lim_{n\to\infty} \frac{1}{4}\,\frac{2}{L}\int_{-L/2}^{L/2} \underbrace{\left(Q(t)\cos\frac{\omega_0 t}{2}\right)}\sin n\omega_0 t + \left(Q(t)\sin\frac{\omega_0 t}{2}\right)\cos n\omega_0 t\,dt$$

$$= \lim_{n\to\infty} \frac{1}{4}.$$

function, and F is, that means F is a continuous function, piecewise continuous, so it is integrable function, okay, and what you have seen is earlier if F is square integrable function or –L/2 to L/2 then its Fourier coefficient is, whose Fourier coefficient is actually is going to 0, okay, so I'll just show you what it is what I'm going to use, so if you just go back, so look at

$$\Rightarrow \sum_{n=0}^{\infty} a_n = \infty$$

$$\Rightarrow \boxed{\sum_{k=-\infty}^{\infty} |c_k|^2 \le \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\,dx.}$$

Corollary: If $\int_{-L/2}^{L/2} |f(x)|^2\,dx < \infty$, then $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$

$$\Rightarrow \lim_{n\to\infty} c_n = 0 \quad \text{(Riemann-Lebesgue Lemma).}$$

Theorem: If $f(x)$ is a piecewise differentiable periodic function with period L;

then $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+)+f(x^-)\right)$; $x \in \left(-\frac{L}{2}, \frac{L}{2}\right)$.

this, if you look at this Bessel inequality, so you look at the Bessel inequality and you see this one, so if F is a periodic function, periodic function which is piecewise differentiable that means the derivative is, the derivative exists that means this integral is finite, so integral mode FX DX which is finite, and once this is mode X is finite it is also square integrable, F square, F into F is also piecewise continuous function, F is piecewise continuous function, F square is also piecewise continuous function, that implies that is also integrable function, okay, so that way what you have is this quantity the right hand side this integral is finite, once this integral is finite this number series sigma CK square that means that is finite.

So once you have this by this corollary which is the Riemann Lebesgue Lemma the nth term has to go to 0, okay, so in that sense if you use this one, here for this function if F is such a thing F square is also integrable function, so if F is integrable, QN is also integrable you can see

$$= \frac{1}{L}\int_{-L/2}^{L/2} dt = \frac{L}{2}.$$

$$\text{Let} \quad Q(t) = \frac{f(x+t) - f(x^+) + f(x-t) - f(x^-)}{t} \left(\frac{t}{\sin\frac{\omega_0 t}{2}}\right).$$

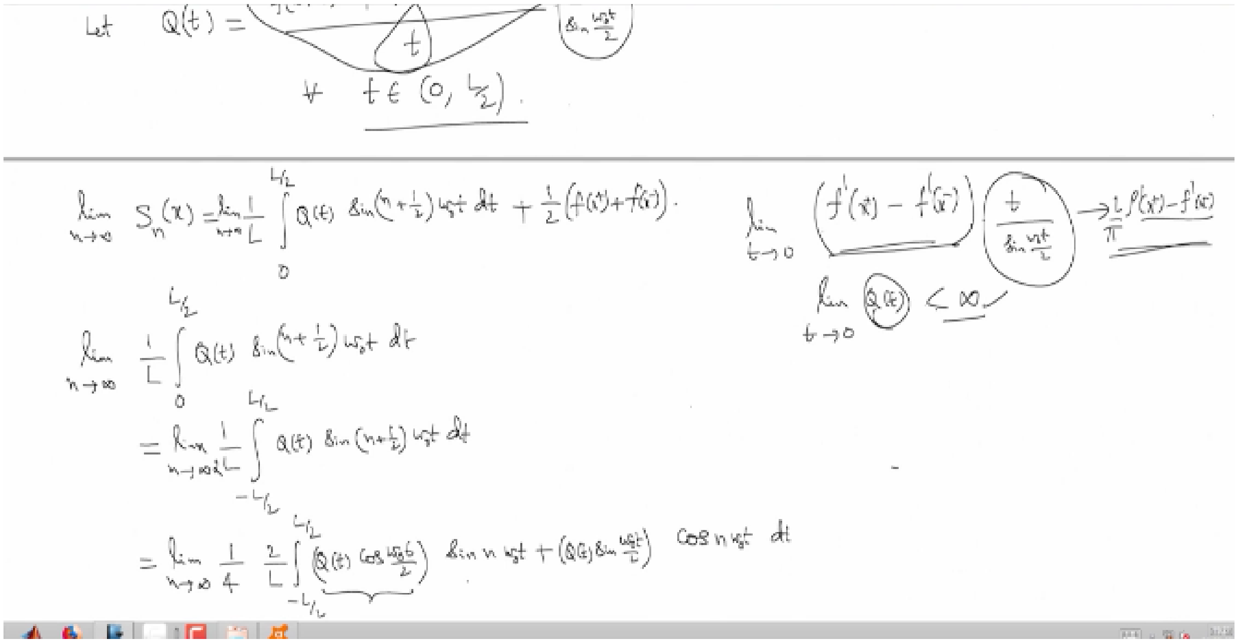
$$\forall \quad t \in \left(0, \tfrac{L}{2}\right).$$

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \frac{1}{L}\int_0^{L/2} Q(t)\, \sin\left(n+\tfrac{1}{2}\right)\omega_0 t\, dt + \frac{1}{2}\left(f(x^+)+f(x^-)\right).$$

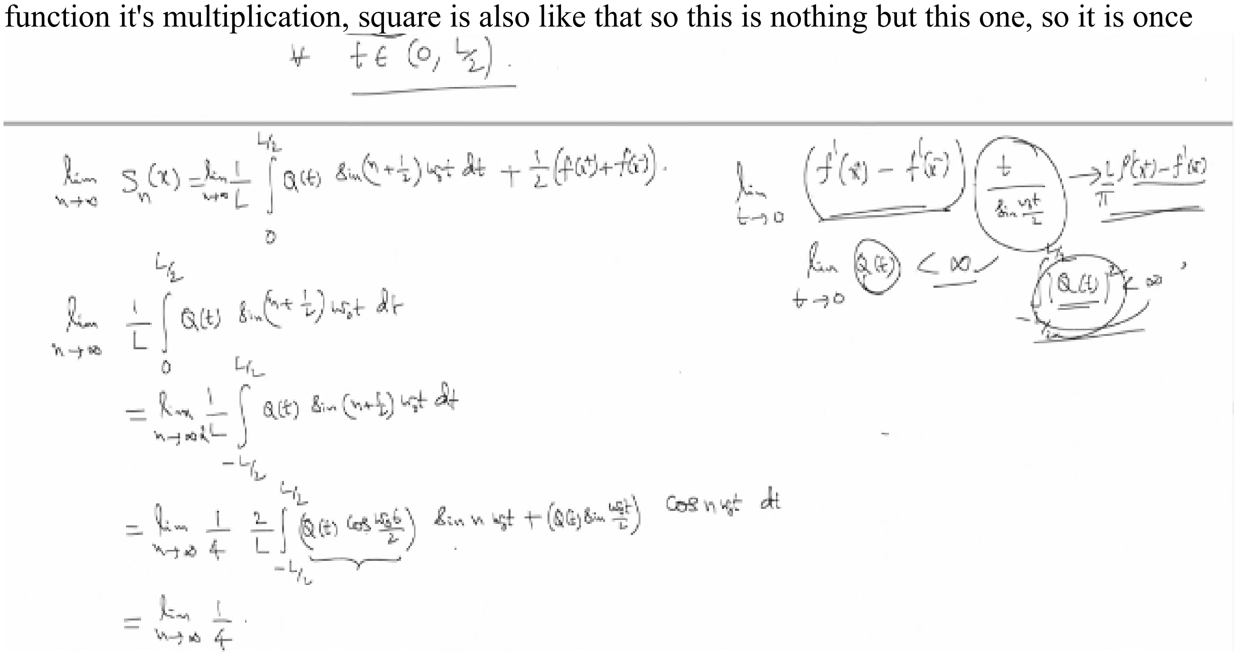
$$\lim_{n\to\infty} \frac{1}{L}\int_0^{L/2} Q(t)\, \sin\left(n+\tfrac{1}{2}\right)\omega_0 t\, dt$$

$$= \lim \frac{1}{L}\int^{L/2} Q(t)\, \sin\left(n+\tfrac{1}{2}\right)\omega_0 t\, dt$$

this one, so this will not really, see you can see that this quantity, this first term involving this and second term is involving this, these are a derivative kind of things, they are derivatives, these are F dash(x) - X+ and F dash(x-) that is the difference by T into, T divided by sine omega naught T/2, so after when you take the limit T goes to 0, this is what it is, so these are the derivatives, these are the things so this is a fixed quantity and you have T divided by this, as T goes to 0 so that is where only we have problem only when T = 0 this quantity is going to 1 so it is a constant, so and you have Q(t) that is well-defined for all 0 to L/2, even at 0 I don't have problems, it's not bounded, okay, what it shows is that the limit Q(t) only worried about T goes to 0, this is a finite number, because you look at this limit, as a limit this goes to, as T goes to 0, this is actually first term, these two terms will become this, and the other term is eventually becomes that itself it's 1, so it's not 1, 2/omega naught, okay, so F dash(x+) – F dash(x-) divided by 2/omega naught, okay, so 2, omega naught is 2 pi/L, so you have finally 2 2 goes L/pi, okay, so this is going to be L/pi, so it's well-defined.



So as T goes to 0 this function Q(t) at T = 0 it's a finite quantity and it is defined for all values, so it's a piecewise, so it's a piecewise continuous function implies Q(t) is integrable function that means this integral Q(t) is finite – L/2 to L/2, okay, that is one.
Basically that you composite with, you multiply with cosine or sine that is actually is a piecewise continuous function is finite, and once this is piecewise, Q is piecewise continuous function it's multiplication, square is also like that so this is nothing but this one, so it is once



this is finite by Bessel's inequality and its subsequent corollary that is Riemann Lebesgue Lemma you can see that the Fourier coefficient that CN CK there is for FK here this whole quantity, this is your CK for this function Q(t) into cos omega naught T/2, so this goes to 0 so this into this goes to 0, this goes to 0 + again 1/4 times, now 2/L into this quantity for this function this is your Fourier coefficient, this is –L/2 to L/2, so this is DT + 2/L, so this if you make it two brackets, so this integral is again a Fourier coefficient say BN, this is AN and this is BN, so all once CN goes to 0, AN BN also goes to 0 because AN's are simply nothing but

$$= \lim_{n\to\infty} \frac{1}{L} \int_{-L/2}^{L/2} Q(t) \sin\left(n+\tfrac{1}{2}\right)\omega_0 t \, dt$$

$$= \lim_{n\to\infty} \frac{1}{4} \left[\underbrace{\frac{2}{L} \int_{-L/2}^{L/2} \Big(\underbrace{Q(t) \cos \frac{\omega_0 t}{2}}_{g}\Big) \sin n \omega_0 t \, dt}_{} + \underbrace{\frac{2}{L} \int_{-L/2}^{L/2} \Big(\underbrace{Q(t) \sin \frac{\omega_0 t}{2}}_{}\Big) \cos n \omega_0 t \, dt}_{} \right]$$

$$= \frac{1}{4} \times 0 + \frac{1}{4} \times$$

CN and C-N divided by 2 and 2Y, okay, so because of that each of this quantity goes to 0, so by taking the limit so that is actually becoming this limit SN, SN(x) as N goes to infinity that is nothing but this Fourier series.

What happens? What is your Fourier series? That's going to be minus infinity infinity into F(x) that is CN, so you have CN E power IN omega naught X which is actually this converges, that is what happens, right, this limit is a partial sum and N goes to infinity this is what it is, and we have seen that this goes to limit SN is what we have seen is, if this quantity is going to 0 and finally this becomes is only this quantity, it is left is as N goes to infinity this integral is going to be 0, this integral is 0 though what I have shown just now, so what is left is only this one, so this is actually equal to 1/2 of F(x+) + F(x-) okay, so this is what you have shown. So now I can remove this one that is exactly what you want to show, so where is this one? If X belongs to –L/2 to L/2, if you choose only interior point of that open interval –L/2 to L/2, okay.

$$= \frac{1}{4} \times 0 + \frac{1}{4} \times 0$$

$$\boxed{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+) + f(x^-)\right), \quad x \in \left(-\tfrac{L}{2}, \tfrac{L}{2}\right)}$$

And now see since F is continuous function because you have chosen piecewise continuous function, piecewise a differentiable function so that means whose derivative exists at all points, interior points, so F(x) is same as F(x+) is same as F(x-) so that means 2 times F at X value, F(x) is defined, so this is exactly they were 2X, two times, 2 times FX and half of it so there you get F(x) okay, so it converges, this converges to F(x) whenever X is in here.

Now the question is what happens at the endpoints, if so you remember that you fixed X value so X you fixed and they worked with this, if X = -L/2 let us say, what happens you look at again this expression the limit SN, if now X = -L/2, okay, if X = -L/2, X is –L/2+T, so where are you? So that’s easy to work out, similarly you can work out for L/2, let's choose X = L/2, L/2 you have SN as this one, so because I wrote SN in terms of 0 to L/2, I have chosen X = L/2, if you want at X = -L/2 you rewrite, so when you split this into two parts 0 to L/2, 0 to L/2, what you make is you make this 0 to L/2 as from – L/2 to 0, so that eventually you will end up getting this SN as integral –L/2 to 0, and then you can fix X equal to, the same expression will get similar expression, but X = -L/2 you can work similarly, here for this type of expression for X equal to, argument is easier if X equal to, if you choose X = L/2, if you choose X = L/2, F(x+), F(L/2) + T that is you're going out of this integral, once you go out of this integral because it's a periodic, you're going to, so where is this function value? Once this function, so let me write this one, this function F(x+t) L/2, L/2 is the endpoint, L/2 + T because it is a periodic function repetitive, this is actually a repetitive so that means it goes back to, so you have – L/2 to L/2, so whatever the value, so here is actually same as a value here, because it's

$$= \frac{1}{L}\int_0 \left[f(x+t) - f(x^+) + f(x-t) \ldots \right] \ldots \frac{\ldots}{\sin\frac{\omega_0}{2}t}$$

$$+ \frac{1}{L}\int_0^{L/2} \left(f(x^+) + f(x^-)\right)\cdot \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

$$S_n = \frac{1}{L}\int_0^{L/2}\left(\frac{[f(x+t) - f(x^+) + f(x-t) - f(x^-)]}{t}\;\frac{t}{\sin\frac{\omega_0 t}{2}}\right)\sin(n+\tfrac{1}{2})\omega_0 t\, dt \checkmark$$

$$+ \frac{1}{L}\left(f(x^+) + f(x^-)\right)\int_0^{L/2}\frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0 t}{2}}\, dt$$

Observe that $\displaystyle\int_0^{L/2}\frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0 t}{2}}\, dt = \frac{1}{2}\int_{-L/2}^{L/2}\frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0 t}{2}}\, dt$

$$= \frac{1}{2}\int_{-L/2}^{L/2}\sum_{k=-n}^{n} e^{-ik\omega_0 t}\, dt$$

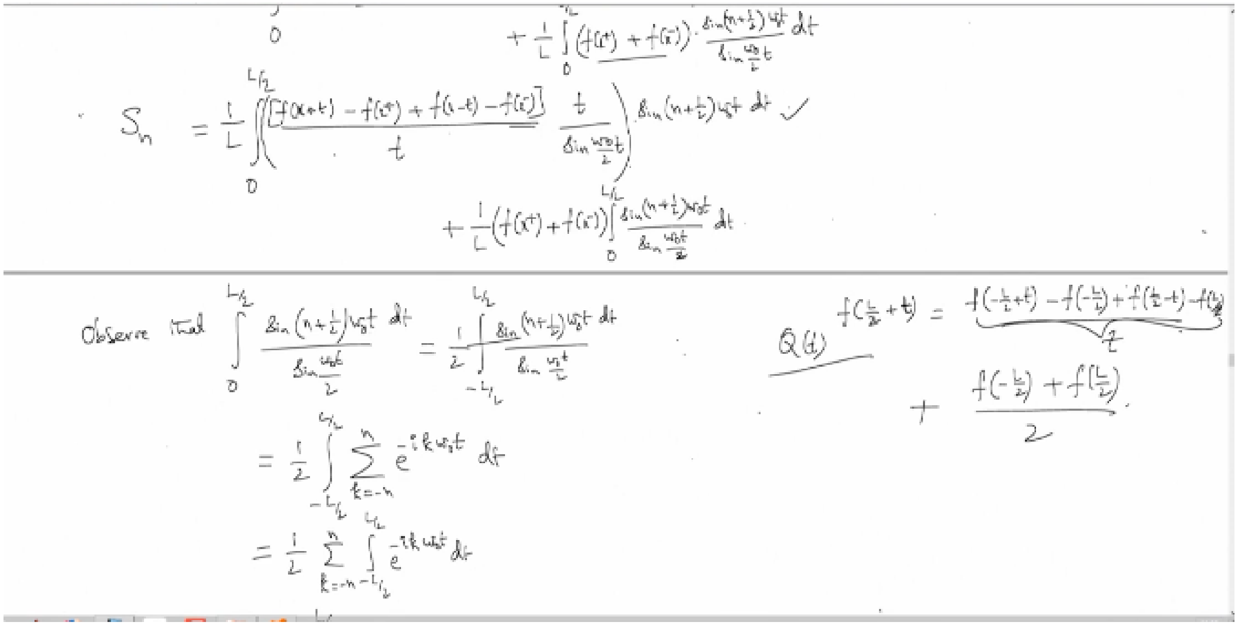
$$= \frac{1}{2}\sum^{n}\int^{L/2} e^{-ik\omega_0 t}\, dt$$

$f(\frac{L}{2}+t) = f(-\frac{L}{2}+t) - f($

repeating, its repetitive, if it is like this again it's like this, so what is this value? This value is here which is same as this here, okay. So this is actually kind of, you will see that F(L/2 + 2) is nothing but F(-L/2 + T), that's what you have, okay.

So you have this is -L/2 to L/2, L/2 + T is nothing but -L/2 + T, that's what you have here, okay, so this is same as this, so this one – F of, now L/2 so L/2 you have chosen, so you have this, this L/2 value, let us say if it is like this L/2 value that you keep it as it is, L/2+ right, where is that? Yeah here, so this is the one, so if L/2+1 F(x+) X is L/2+, so L/2 +1 means it's coming from, so it's again the same argument if it is from -L/2 to L/2 so you want this integral, so this limit of the function that is same as the limit of this function, because wherever the values if you want to look at it they are nothing but here, so as you approach here you are approaching right side means you are approaching here, because it is repetitive, so this is same as F(-L/2), okay.

And then what happens to the other one? That is going to be F(L/2 - T) that is same, anyway inside –F(L/2) that is also inside, that limit from this side so which is well-defined divided by T, so what is this one? So this one, when you write like this eventually so F(x+) is nothing but F (-L/2) so other part, other part is what you have is, if you do this one, so when you add and



subtract it you have to add with F(-L/2) + F(+L/2) so this anyway again the same argument, this divided by T into T divided by sine, when sine omega naught T/2 that whole as a Q(t) and then you have here, this part will be 1/L times so this is going to be half of this, so you will have that one Q(t) + half of, where Q now, wherever X is you replace with – L/2, and the second part you have +L/2, so with this argument if you do the same thing SN(t), SN(x) that is also going to 0, SN(x) is going to 0, first integral is going to be 0, second integral is now converged to this function, this value.

So when X = L/2, what you have is a limit SN, as N goes to infinity is nothing but when X is L/2 is actually equal to F(-L/2) + F(+L/2)/2, half of the value, okay. If X = -L/2 you want to look at it, you write this SN, not from 0 to L/2, you take it from minus, you replace over T/-T and you rewrite from –L/2 to 0, then you view it similar way, you can see that SN(x) as N goes to infinity, in both cases will be same as this, that you can take it as an exercise, okay. It's the same way so argument is same fix X and you look at its periodic thing, so if this is your periodic thing if you want to look at the values here they are here because it's repetitive, if you want to look at this side they are repeating here, so if you want here they are repeating here, so there's where is that, if you want to look at here so it's somewhere here, exactly half, okay, so

$$= \lim_{n\to\infty} \frac{1}{4} \left(\frac{1}{L} \int_{-L/2} \ldots \right)$$

$$= \frac{1}{4} \times 0 + \frac{1}{4} \times 0$$

$$\boxed{\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+) + f(x^-)\right) = f(x), \ x \in \left(-\tfrac{L}{2}, \tfrac{L}{2}\right)}$$

If $x = \pm \frac{L}{2}$, $\lim_{n\to\infty} S_n\left(\frac{L}{2}\right) = \frac{f\left(-\frac{L}{2}\right) + f\left(\frac{L}{2}\right)}{2}$ ✓

just add if you want wherever you want to look at some L + some A, A+L, you add up to L that is A+L, that is the value of L, okay, so that's how if you look at it you'll see that this is going to be this.

So the Fourier series that converges to CN E power IN omega naught X, this is actually converges to, if F is a piecewise differentiable function F(x) if X belongs to –L/2 to L/2 open interval, and if it's an endpoints this is going to be addition of, average value of the endpoints, half of this if X is equal to + or - L/2, so when this is our F(x), if it is –L/2 to L/2, if your function is like this if the value here, here both are same then it is function at the endpoints, okay, so this is what you have, so if F is same, if F(-L/2) is same as F(L/2) then average value becomes when at X = + or – L/2 this Fourier series CN E power IN omega naught X, N is from minus infinity infinity is F(x) for every X belongs to minus, including the points, okay, because if we include, so if such that is the case, if this is the case then we have this, this is the more general one, otherwise we have only this one, so only interior point you can always say converges to the function F(x) because F is continuous, here you're not sure, if your function is like this value here, here is both are different, so it converges at this point it converges to the average value and here also it converges to the average value, that's what it means, okay.

If $x = \pm \frac{L}{2}$, $\lim_{n\to\infty} S_n(\frac{L}{2}) = \frac{f(-\frac{L}{2}) + f(\frac{L}{2})}{2}$ ✓

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = \begin{cases} f(x), & x \in (-\frac{L}{2}, \frac{L}{2}) \checkmark \\ \frac{f(\frac{L}{2}) + f(-\frac{L}{2})}{2}, & \text{if } x = \pm\frac{L}{2} \checkmark \end{cases}$$

If $f(-\frac{L}{2}) = f(\frac{L}{2})$, then $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x)$, $\forall\, x \in [-\frac{L}{2}, \frac{L}{2}]$

So we'll see, we have used Bessel's inequality and Riemann Lebesgue Lemma the corollary of it, and we have shown this convergence of the Fourier series to the function F itself and it is piecewise differentiable function, periodic piecewise differentiable function, we will see the corollary that is this such, if your Fourier coefficients are same then you have unique Fourier series, okay, if you can always define if it is a piecewise periodic function, piecewise continuous a periodic function if we choose you can construct your Fourier coefficients in place you have a Fourier series, so far we have not seen if it is a piecewise continuous function we have not seen the convergence of this Fourier series to the function F, but if it is a piecewise differentiable continuous, differentiable function, piecewise differentiable periodic function then you have just now seen that the Fourier series, Fourier coefficients again you can easily define, and using the Fourier coefficients you can make a Fourier series that actually converges a point wise, that means you fix X in the Fourier series that converges to as a number series, it converges to F(x), if it is continuous for, because anyway continuous function within the open interval it converges to F(x) at the endpoints it converges to the average value of the endpoints, function value at endpoints, that is this okay.

We will see it’s a Fourier series is unique, if your Fourier coefficients are uniqueness theorem so we can write the uniqueness of the Fourier series, so if I choose two functions F(x) and G(x) be two piecewise differentiable function, differentiable periodic functions and with, you have two such piecewise periodic differential functions, then you can define Fourier coefficients for each of them, let's call this FN and so it's earlier we have defined CN, CN is F(x) into some integral, right integral with exponential function that is your CN, so I am calling it for corresponding to F, I am calling it a FN, so FN with FN being Fourier coefficients, with Fourier coefficients let's write with Fourier complex, Fourier coefficients FN is same as GN, this is true for every N, okay.

$n=-\infty$

$\left\{ \frac{f(\frac{L}{2})+f(-\frac{L}{2})}{2}, \text{ if } x = \pm\frac{L}{2} \right.$

$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x), \ \forall x \in [-\frac{L}{2}, \frac{L}{2}]$

Uniqueness of Fourier Series: Let $f(x)$ and $g(x)$ be two piecewise differentiable periodic function with Fourier coefficients f_n, g_n such that $f_n = g_n, \forall n$.

So let this is the case, okay, Fourier coefficients FN such that FN = GN for every N this is what it have, then what happens? Then F(x) = G(x) that means F(x) is nothing but we have seen just now seen from this theorem that F(x) is of limits of Fourier series of F, and G(x) is a Fourier series of G, and we are seeing that Fourier series are also same if the Fourier coefficients are same, that is what is the meaning of this, so a proof is easy, if FN = GN for every N then CN which is FN - GN which is equal to 0, so some coefficients are given which is 0 and we know that CN, so what is the CN? CN is a Fourier coefficient of, CN is a Fourier coefficient of F-G, okay, if FN is a Fourier coefficient of F, so FN is 1/L, what is the Fourier coefficient? 1/L the complex Fourier coefficient which is 1/L –L/2 to L/2 F(x) E power -IN omega naught X DX. Similarly GN, F,G is nothing but G,F so you replace G F with this, these are the Fourier coefficients GN FN, so wherever you have G you put G instead of F okay, so FG so FGN like that, if you do these are the coefficients, so what happens to FN – GN? This is a Fourier coefficient of –L/2 to L/2 for F-G(x) E power -IN omega naught X DX, so a Fourier coefficient of F-G, okay, so because F and G are continuous function, piecewise differentiable functions F-G is also piecewise differentiable function so you have a, whose Fourier coefficients are CN E

Then $f(x) = g(x)$

Proof! If $f_n = g_n \;\forall n$, $\quad f_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ (g_n, g)

then $c_n = f_n - g_n = 0 \quad c_n = f_n - g_n = \frac{1}{L}\int_{-L/2}^{L/2} (f-g)(x)\, e^{-in\omega_0 x}\, dx$

$\sum c_n e^{i}$

Uniqueness of Fourier Series: Let $f(x)$ and $g(x)$ be two piecewise differentiable periodic function with Fourier coefficients f_n, g_n such that $f_n = g_n$, $\forall n$.

Then $f(x) = g(x)$ wherever f & g both are continuous.

Proof! If $f_n = g_n \;\forall n$, $\quad f_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ (g_n, g)

then $c_n = f_n - g_n = 0 \quad c_n = f_n - g_n = \frac{1}{L}\int_{-L/2}^{L/2} (f-g)(x)\, e^{-in\omega_0 x}\, dx$

$$0 = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x) - g(x)$$

$$\Rightarrow f(x) = g(x), \quad \forall x \in \left(-\frac{L}{2}, \frac{L}{2}\right).$$

power IN omega naught X, N is from minus infinity infinity should converge to by the theorem, it has to converge to F-G which is equal to, what happens to this? Which is right F(x) – G(x), and we know that CN is 0, so that means the whole thing is 0, so this implies F(x) = G(x) for every X in –L/2 to L/2, that is where F is continuous, okay, so we have to write for every X wherever, if wherever F and G both are continuous, okay, so that means we have seen so the

Uniqueness of Fourier Series: Let $f(x)$ and $g(x)$ be two piecewise differentiable periodic functions with Fourier coefficients f_n, g_n such that $f_n = g_n$, $\forall\, n$.

Then $f(x) = g(x)$ wherever f & g both are continuous.

Proof: If $f_n = g_n$ $\forall\, n$, $\quad f_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ (g_n likewise with g)

then $c_n = f_n - g_n = 0$ $\quad c_n = f_n - g_n = \frac{1}{L}\int_{-L/2}^{L/2} (f-g)(x)\, e^{-in\omega_0 x}\, dx$

$$0 = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x) - g(x)$$

$$\Rightarrow f(x) = g(x), \quad \forall\, x \in \left(-\tfrac{L}{2}, \tfrac{L}{2}\right).$$

piecewise differential function means let us say wherever period functions in –L/2 to L/2, okay, L/2 with Fourier coefficients for every X, for all X at which F and G are continuous.

If they are piecewise differentiable functions, and this means obviously it should be continuous for all X in this, okay, in the open interval that shows that this is a unique Fourier, so you get a unique Fourier series once you have unique Fourier coefficients this is one corollary, one result

Uniqueness of Fourier Series: Let $f(x)$ and $g(x)$ be two piecewise differentiable periodic functions in $\left[-\frac{L}{2}, \frac{L}{2}\right]$ with Fourier coefficients f_n, g_n such that $f_n = g_n$, $\forall\, n$.

Then $f(x) = g(x)$, for all x at which f and g are continuous.

Proof: If $f_n = g_n$ $\forall\, n$, $\quad f_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$ (g_n likewise with g)

then $c_n = f_n - g_n = 0$ $\quad c_n = f_n - g_n = \frac{1}{L}\int_{-L/2}^{L/2} (f-g)(x)\, e^{-in\omega_0 x}\, dx$

$$0 = \sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x} = f(x) - g(x)$$

$$\Rightarrow f(x) = g(x), \quad \forall\, x \in \left(-\tfrac{L}{2}, \tfrac{L}{2}\right).$$

which immediately can say after this convergence after this theorem and you can also say one more result that if F is, if F dash(t), if F dash(x) is piecewise continuous, then limit N goes to infinity, N times Fourier coefficient FF you call this FN, it's going to be 0, okay, so what does it mean? That is FN of, FN behaves like order of 1/N power 1 + alpha, alpha is positive, right, otherwise this is not going to be 0, this limit, right, N power alpha then it N into FN will be order of 1/N power alpha, alpha is positive so as N goes to infinity that goes to 0, so this is true. So how do we prove this one? This is from Riemann lebesgue lemma, so if F(x) is piecewise continuous that Fourier coefficient is, Fourier coefficient of its derivative DF/DX F dash N

Fourier coefficient is defined as 1/L –L/2 to L/2 F dash(x) E power I omega IN omega naught X DX, you do the integration by parts, what you gain is, E power so what you get F(x) integration by parts if we do IN omega naught X, okay, from –L/2 to L/2 1/L - integral 1/L –L/2 to L/2 F(x) you do the integration here, so you'll get IN omega naught into E power itself IN omega naught X DX, so 1/L of this is a Fourier coefficient so you'll get and this one will be 0, so both will be 0 you can see, okay, this will not be contributing so what you get is -IN omega, omega naught is 2 pi/L, 2 pi/L square it's a constant, anyway this is your FN, so you can easily see that, this quantity NFN by Riemann Lebesgue lemma we know that if it is a Fourier coefficient of a function F dash is, which is piecewise continuous implies is Fourier coefficients define, and implies it is integrable and it is square integrable implies it is, whose Fourier coefficient, Bessel inequality is true so from that from the Bessel inequality you have a Riemann Lebesgue Lemma that shows that this is going to be 0.

What is this FN dash? This is nothing but N times FN into minus some constant, as N goes to infinity equal to 0 so that constant is if you write –IN 2/L square = -IN 2/L square this constant times this, because of this is 0 and this has to be 0, NFN as N goes to infinity so this means this has to go to 0, this limit NFN, N goes to infinity this is 0, that shows that so alpha is actually 1, so is equal to 1, okay, strictly speaking this is like order of 1/N square that is exactly what you get in the corollary, okay.

$$\Rightarrow \quad f(x) = g(x), \quad \forall\, x \in \left(-\tfrac{L}{2}, \tfrac{L}{2}\right).$$

Cor: If $f'(x)$ is piecewise continuous. Then

$$\lim_{n\to\infty} n\cdot f_n = 0. \quad \text{ie, } f_n \sim o\left(\frac{1}{n^2}\right).$$

Proof:

$$f'_n = \frac{1}{L}\int_{-L/2}^{L/2} f'(x)\, e^{in\omega_0 x}\, dx = \cancel{\frac{1}{L} f(x)\, e^{in\omega_0 x}\Big|_{-L/2}^{L/2}} - \underbrace{\frac{in\omega_0}{L}\int_{-L/2}^{L/2} f(x)\, e^{in\omega_0 x}\, dx}$$

$$= -\frac{in\,2\pi}{L^2}\cdot f_n.$$

$$0 = \lim_{n\to\infty} f'_n = \lim_{n\to\infty} n f_n \left(\frac{in2}{L^2}\right) = \left(-\frac{in2}{L^2}\right)\lim_{n\to\infty} n f_n$$

$$\Rightarrow \quad \lim_{n\to\infty} n f_n = 0.$$

So we can go on doing like this if F dash itself is a piecewise differentiable function that means F double dash is piecewise continuous implies N square FN is 0 and so on, okay, you can go on doing like this recursively.

So what is that we have done so far? So if F is a piecewise, a differentiable periodic function the Fourier series converges to the function itself because F is continuous in the interval – L/2 to L/2, open interval this converges to F(x) if the endpoints are same the Fourier series actually converges to the function F itself F(x) when X is the endpoints, if endpoints are different average values of the function value at the endpoints is the limiting value of the Fourier series at the endpoints, okay, this is what we have seen, this so far we have used only differentiability of the function F, piecewise differentiability, piecewise differentiable function F, okay, so these are actually sufficient conditions for the Fourier series convergence to the function here, so

we'll define what is a delta function and we will prove the sufficient conditions in the next video. Thank you very much.
[Music]

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