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Transform Techniques for Engineers

Evaluation of Infinite Sums by Z-Transforms

With

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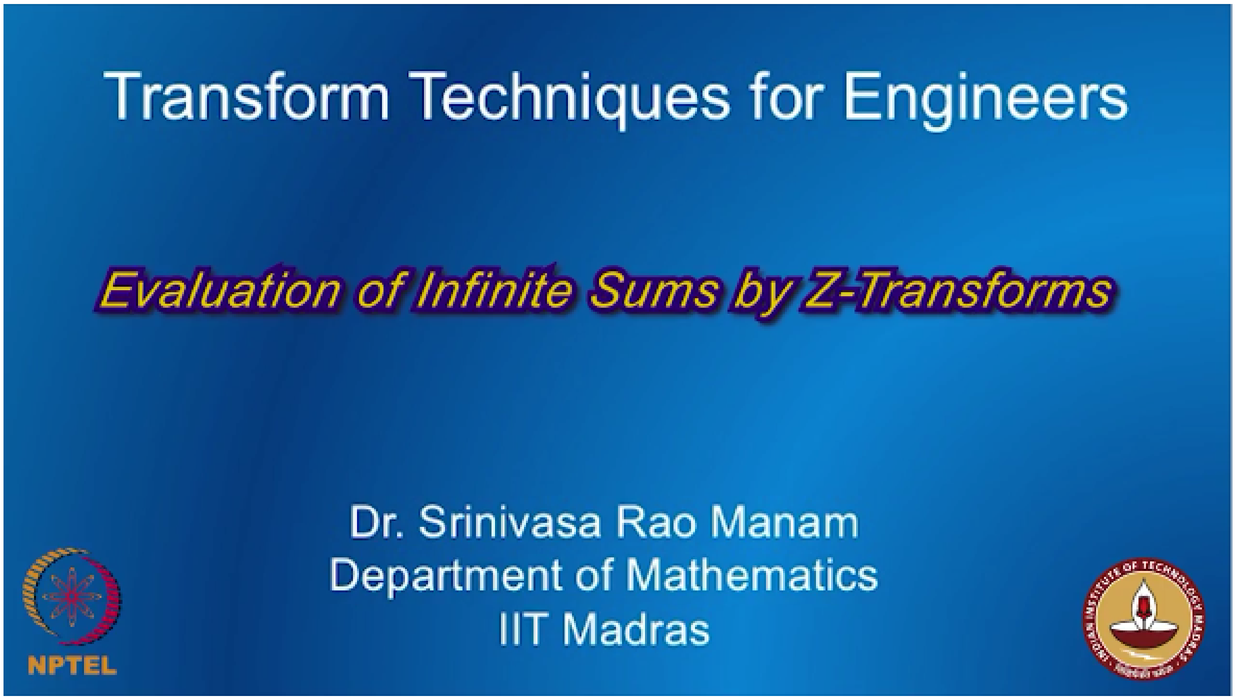

Welcome back, in the last video we have seen the application of Z transforms to solve difference equations, we have seen many examples of first order and second order difference equations, and we've shown the method of solving we just used only Z transform and it's inverse transform, mostly some of the properties to find the inverse transform finally to see the solution of the difference equation.

And this video we will have another application of Z transform that is to evaluate certain infinite sums when it has a parameter which is a discrete parameter, let us say N so that you can apply Z transform to it, this is similar to the evaluating certain integrals involving a parameter by using Laplace transforms that we have seen earlier, okay.

So let’s discuss how to evaluate these infinite sums, before we do that we will give one important property so one or two properties of Z transforms that we have left out that maybe useful to, in this application, so let me state that and prove and then we will do these examples of evaluation of these infinite products.

Let’s look at this property, a property of Z transform that says so if F(z) is, capital F(z) is the Z transform of F(n) okay, so this is small Z, okay, and then you have Z transform of let's say F or F(n)/N as a function of Z, as a function of Z this is equal to integral Z to infinity, this capital F(z)/Z DZ also Z transform of F(n)/N+M as a function of Z this is also from Z to infinity you have Z power M comes out times Z to infinity capital FZ DZ, F over Z power M+1 DZ, so these are the two results that we will see now, okay.

Property of Z-transform:

If $F(z) = Z(f(n))(z)$, then

(i) $$Z\left(\frac{f(n)}{n}\right)(z) = \int_z^\infty \frac{F(z)}{z}\,dz$$

(ii) $$Z\left(\frac{f(n)}{n+m}\right)(z) = z^m \int_z^\infty \frac{F(z)}{z^{m+1}}\,dz$$

So this is the transform, this is a property of 1/N, product of 1/N or 1/T times the signal, time signal F(t) for which if you take the Laplace transform, we have similar property in terms of the integrals, so this is analogues to that, and let's see we prove this, this I think straightforward manner we can use, since F(z) is this so we use the definition of this Z transform and look at what is this Z to infinity integral for capital F(z)/Z DZ, so if you see this one, if I make use of this integral Z to infinity for the F(z) you write the definition of the Z transform that is 0 to infinity F(n) Z power -N divided by Z DZ, so this is same as integral Z to infinity, sigma N is from 0 to infinity, F(n) times Z power -N -1 DZ.

If $F(z) = Z(f(n))(z)$, then

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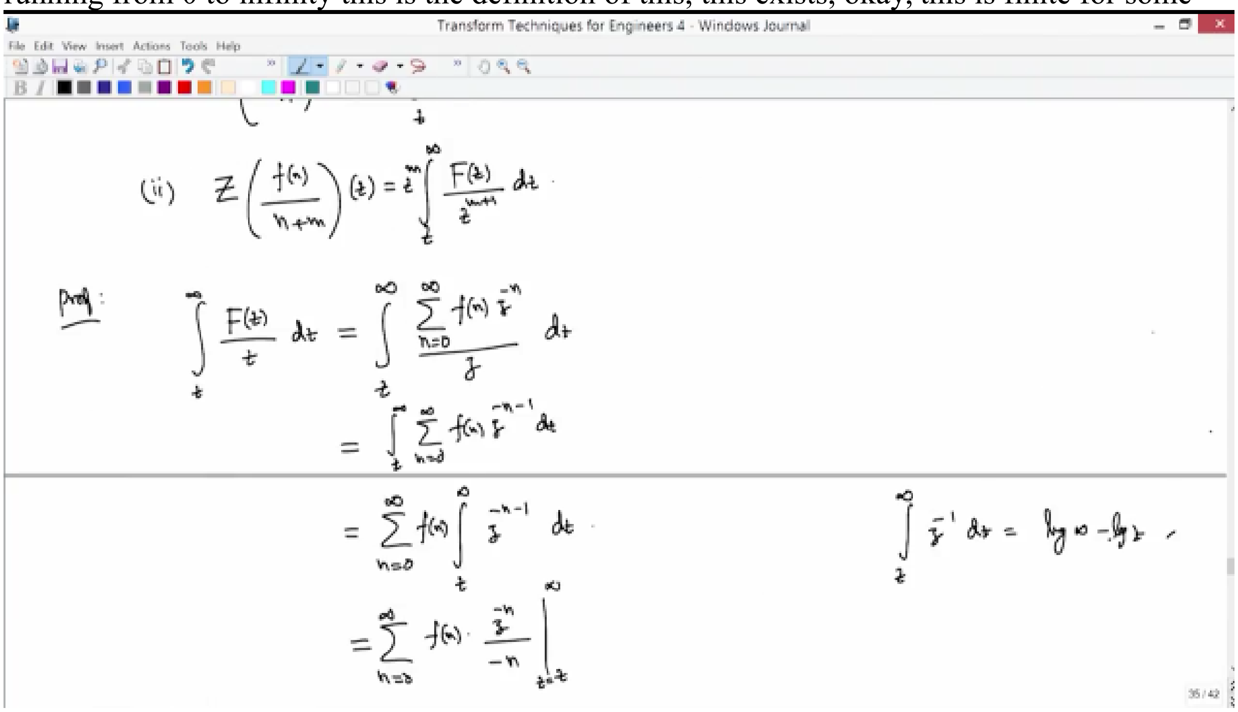
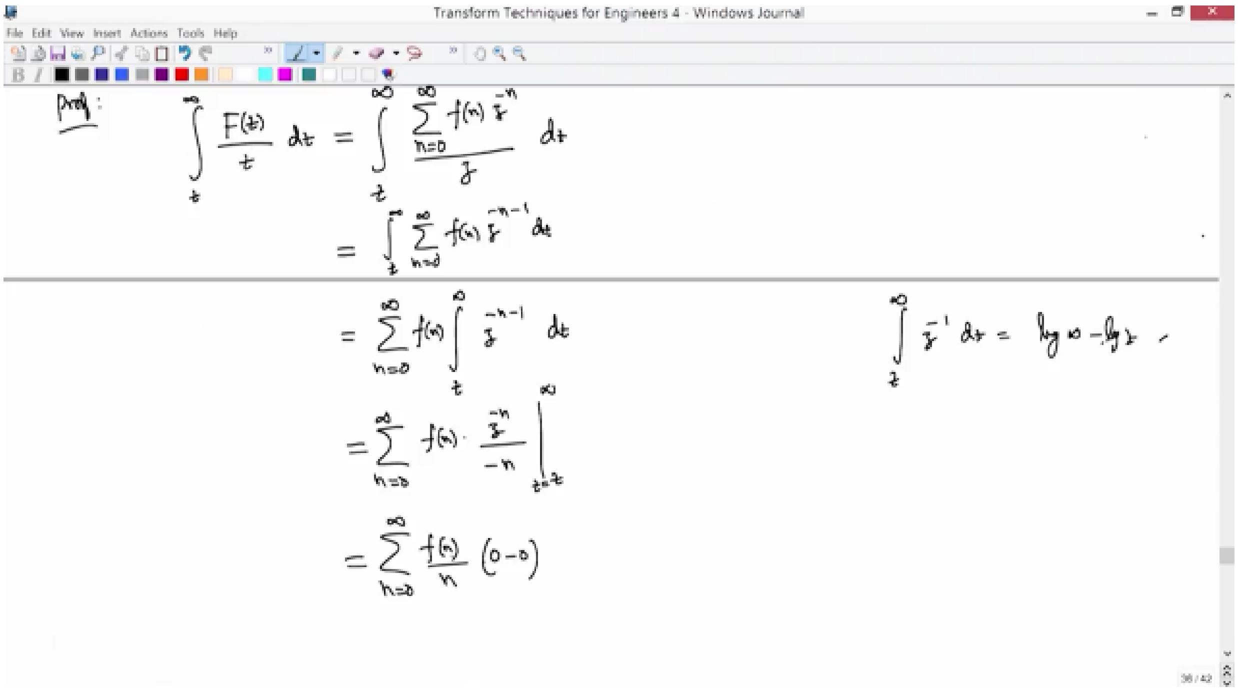
Proof:

$$\int_z^\infty \frac{F(z)}{z}\,dz = \int_z^\infty \frac{\sum_{n=0}^\infty f(n)\, z^{-n}}{z}\,dz$$

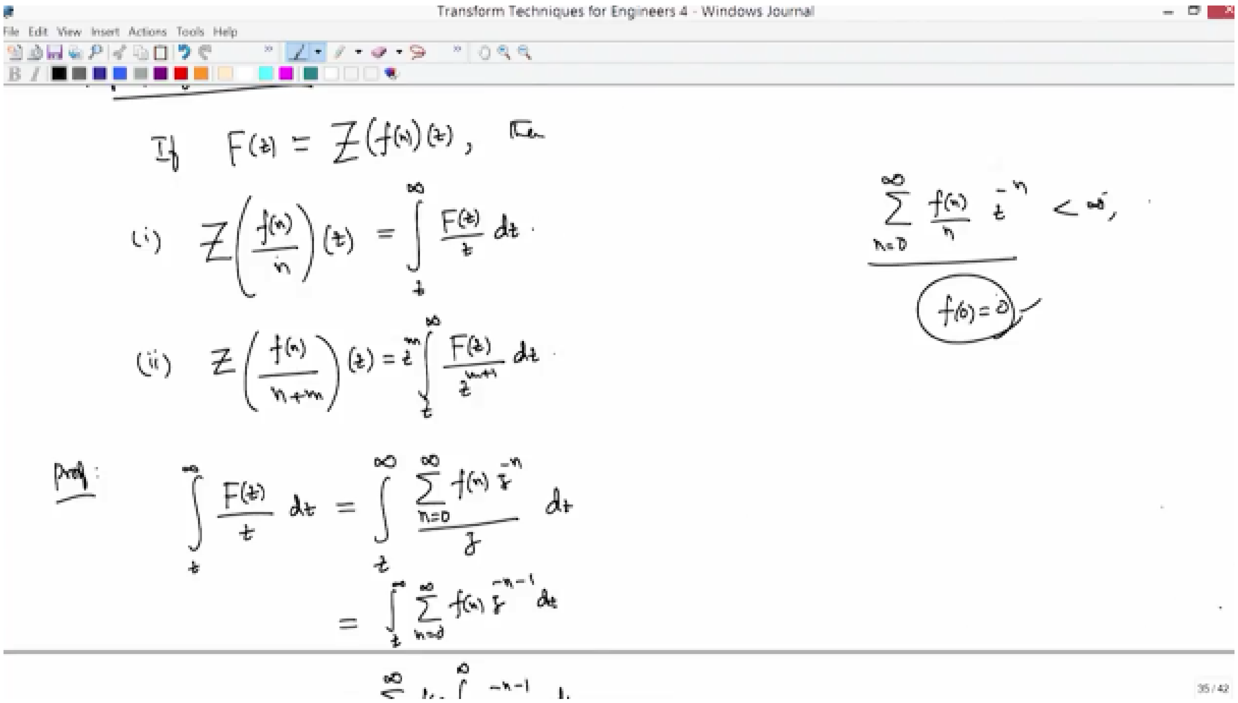
Now because this is, if you assume that this is finite, if this is finite and that means this is finite, so once this is finite these two sums can be interchanged and both should be same, okay, so what you have is N0 to infinity, so under those assumptions when F, capital F(z)/Z is integrable from Z to infinity complex integration if it is, if it's finite then this double sum this one is the integral, other one is a sum, both you can interchange, so that is the kind of theorem that we have already seen as a double integrals iterate integrals you can interchange order of integration when you have a double integral is finite, okay, so the similar thing here.
So now if I interchange this sum and integral and what you have is F(n) comes out if I interchange, F(n) there's nothing to do with Z, so this DZ, and now you have Z power -N -1, so this is nothing but N is from 0 to infinity F(n) and this one if you integrate Z power -N -1 +1 so that is this divided by –N, and now you substitute these limits, Z = Z to infinity, so if you substitute you can easily see what is this one, so this is F(n) infinity because it's 1/Z power N that goes to 0 for every N = 0, okay, so N = 0, what happens N = 0, N = 0 case you have Z power -1 integral, Z to infinity DZ, this is log Z okay, this is log Z and you substitute log infinity – log Z, so that is, so you see that this is not finite thing, okay, so but then we have assumed that you have Z transform of this which is running from, so you have this should exist okay to see that this exists left hand side you have a definition is F(n) divided by N, N is running from 0 to infinity this is the definition of this, this exists, okay, this is finite for some



domain in the complex plane, so that means F(n)/N that means F(0) is 0, so it's like 0/0 form okay, or F(n) so if this exists means if N = 0 because you are division with 0, F(0) this is Z power -0 that is 1, F(0), F(0) has to be, so F(0) has to be 0, if this exists F(0) has to be 0, so in that case F(0)/N, okay, and you have Z power –N, so that is Z power 0 which is running from infinity -1 so that is 0 - 0 okay, so you have this.

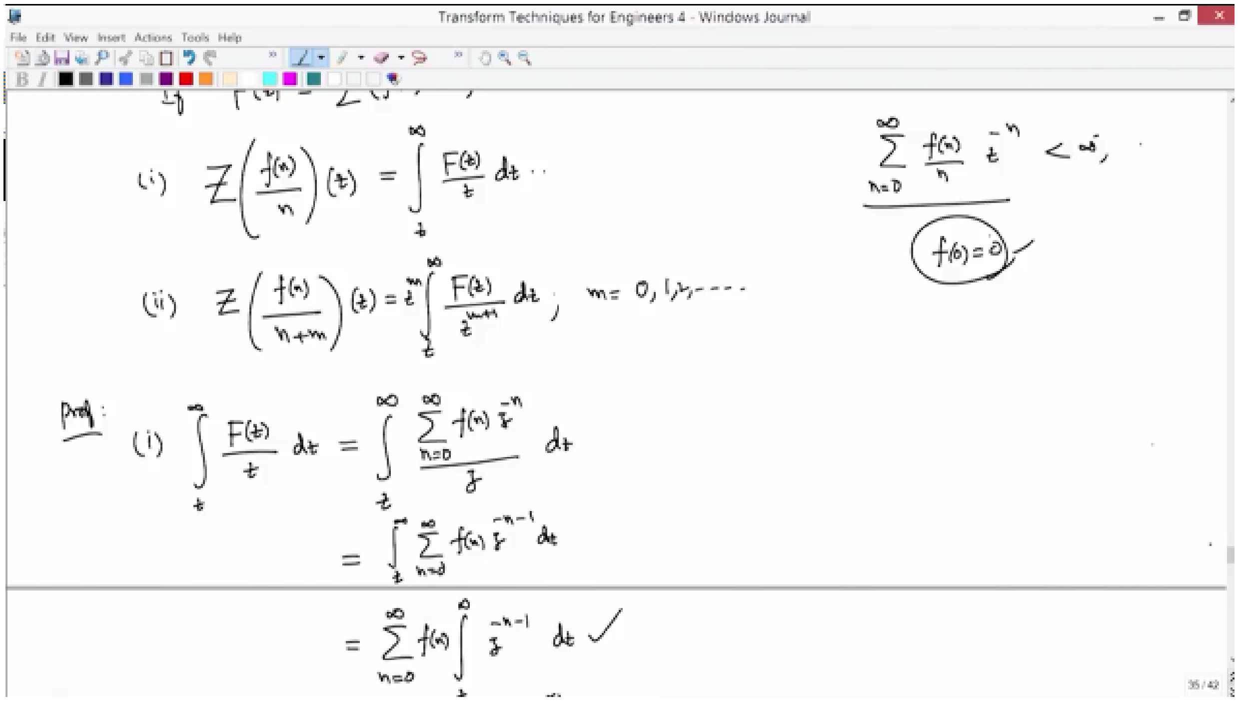


And you have a similar N = 0 you have already 0/0 form, so when N = 0 you have 0/0 form times, what is left is Z power -N that is Z power 0 that is simply 1, okay, Z power –N that is if you do that integral will be Z power, so when you see this one, this part when you put, we can take this N out F(n) should be 0 if it exists, and you have 0/0 form times, and this one so this means this should be finite quantity and something indeterminate so this quantity should be finite if it exists, so F(0)/0 this any finite quantity times this you have N, when you put N = 0 this is simply 1, so when you substitute the limits 1 – 1 that is 0, so finite quantity times 0, so N = that is 0, and what is left is N is from 1 to infinity, F(n)/N and what you have at infinity Z power –N, and N is running from 1 to infinity that is 0 and you have a minus minus plus and you have that, that will become Z power -N and you have seen that N = 0 this quantity is 0 so what you have is you've already got 0 quantity whatever there so you can include that part here, so this is nothing but Z transform of F(n)/N as a function of Z, so wherever you have the, this exists, this exists that is in the same domain this also, the definition of this Z transform of F(n)/N so Z belongs to the same domain wherever this Z transform of F(n) okay, whatever be the domain here, it should be the domain of this here.



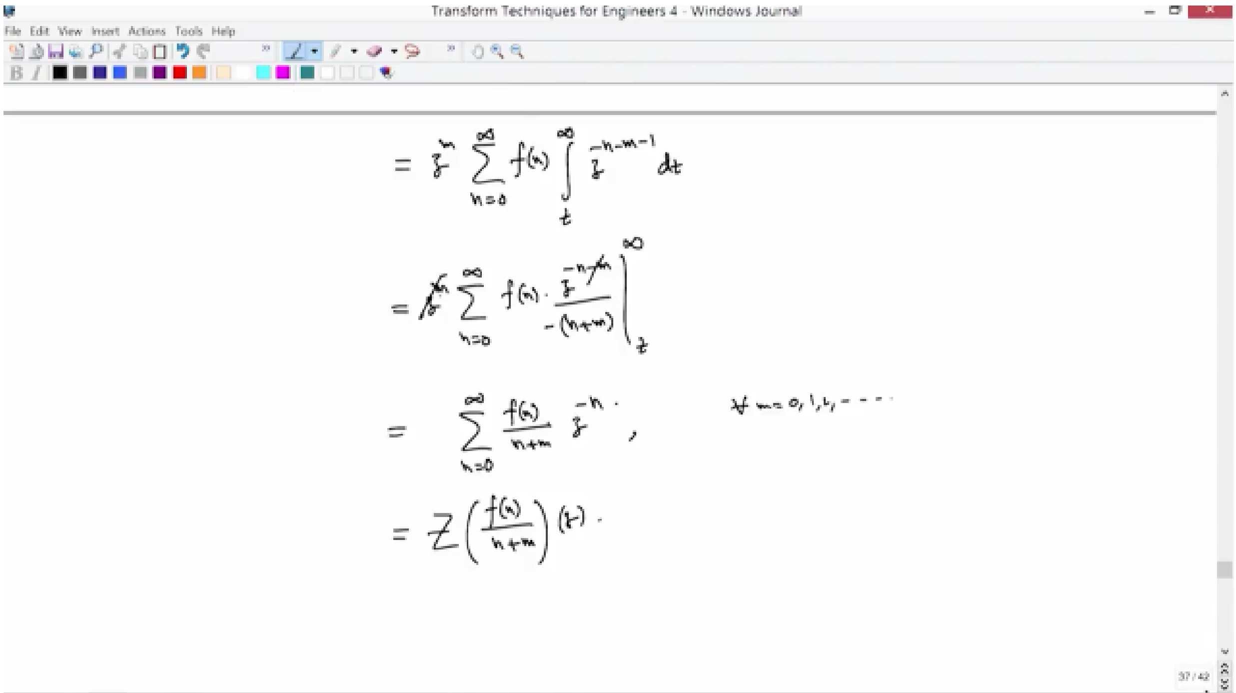
So I’ll just verified right hand side is equal to left hand side, you can do the same way or to show the second part, second point if you do the right hand side if you consider Z power N times integral Z to infinity capital F(z) DZ/Z power M+1 if you take this is equal to Z power M times integral Z to infinity this is summation N is running from 0 to infinity, F(n) times Z power –N DZ/Z power M+1 that you can put it here so that it will become N-M-1, so if you actually do this Z power M so this is nothing to do with this, you can take it inside so you cancel this with this -N so that you have simply this, so you can take this summation outside, you can integrate F(n) times the same technique F(n) Z power -N -1 DZ, so you have already seen that this is nothing but F(n)/N that is F(n)/N right, so F(n), but you want N+M to get this so you have, let me I think I have made some mistakes so you have Z power M let me use, you have Z power when you do this 1/Z power so you have -N -M -1 DZ.

So let me do this, this is equal to Z power N times, Z to infinity or now I interchange this sums F(n) this integral Z to infinity, Z power -N -M -1 DZ, so if you do this Z power M times, sigma N is from 0 to infinity F(n), this will become Z power -N –M divided by N + M –M, -N+M so if you substitute these limits this integrand is this anti-derivative is this, if we substitute you simply end up getting Z power M times, N is from 0 to infinity, now you can remove this, this this goes so you can remove F(n) times, F(n) divided by N+M at infinity that is becoming 0 as you have seen earlier F(n) has to be 0 the same case, and N is running from what is M? M is



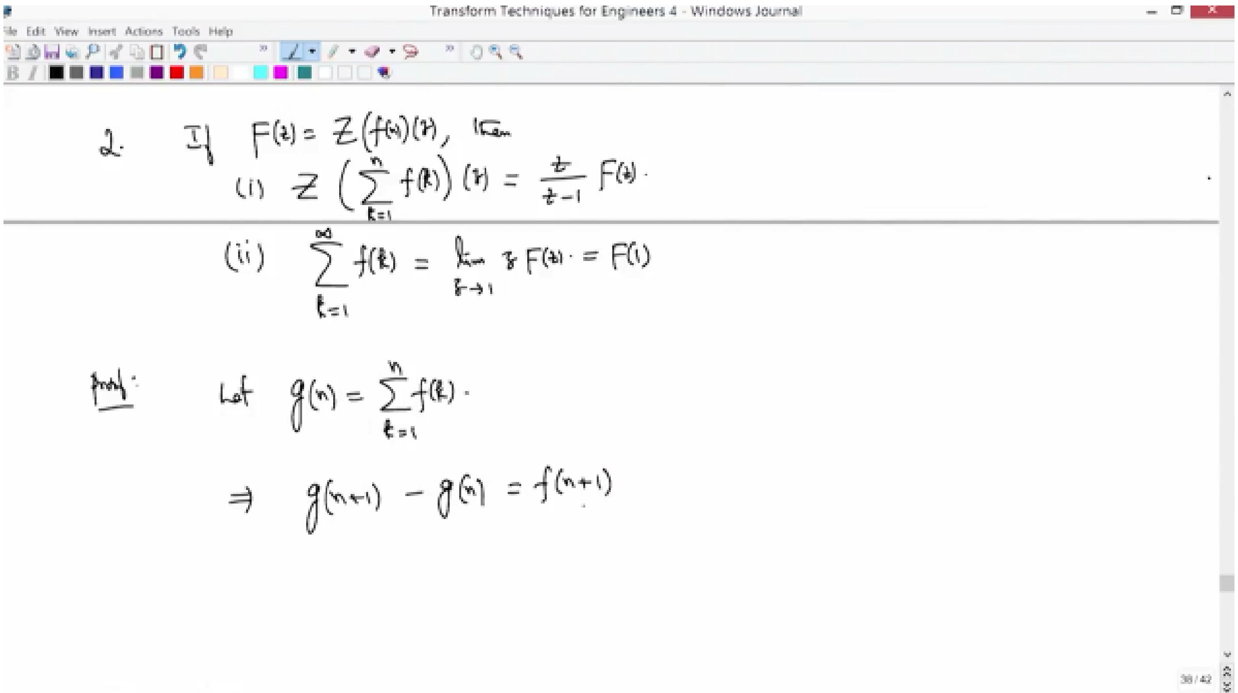
running from 0, 1, 2, 3 onwards, if it is 0 this is the, it's a case of 1, this you have seen already when for all cases when M is 0, 1, 2 whatever be the case if N is 0, if M = 0 case, if M = 0 this N = 0 you have 1/0 form, so in that case F(n) has to be 0, that's what we have seen already, and you have 0/0 form that's a finite quantity Z power –M when you substitute, when you put N = 0 that is a simply 1, so when you substitute the limits 1-1 that is 0, and M is not equal to 1, not equal to 0 that is M is running from, M is either 1 or 2 or 3 and so on you can easily see that this is going to be, this comes out and you simply have you can see that N is, even at N = 0, you have 0 by some nonzero quantity times 1 – 1, okay.

So even in that case M is not equal, for all M 0, 0 case we have already seen when it is 1, 2, 3, and so on, in that case N+M will be nonzero, but F(0) is 0 and Z power -0 that is 1, so 1-1 that quantity is 0 into 0, so there is no issue N = 0 the whole contribution is 0, N = 1 onwards so anyway this is becoming 0, infinity contribution at Z = infinity that is 0, and minus of minus, this minus you make it simply Z power –N, so this is nothing but your Z transform of F(n)/N+M as a function of Z.



So again Z transform of this, definition of this Z transform, the domain of this Z transform is same as a domain of Z transform of F(n) okay, so these two properties we use to evaluate certain infinite sums, let's do one more, some example so summation of infinite series, so we get this infinite sums, so first I'll start with before I do this let me do some more result, so this is one, and another one is another property that we have is, so this is one property that we use and another property is F(z) is capital F(z) is the Z transform of F(n) as a function of Z then this sigma K is from 1 to N, F(k) this finite sum is same as Z inverse rather Z transform of this finite sum as a function of Z is same as is actually equal to Z/Z-1 F(z), okay, so again the definition of Z is domain of capital F(z) and this is one.

And one more thing we have a result is if you look at this infinite sum, K is from 1 to infinity, F(k) this is actually limit Z goes to 1, Z into capital F(z), so this is nothing but capital F(1), because it's analytic function both this is, Z is analytic function, capital F(z) is analytic function in the domain wherever it is defined, so if you approach in that domain 0 to 1 this is what happens, okay, so to do that one should be in the part of the domain of definition of F(z), so that is, you keep that in mind, let's give the proof, so what I do is I consider this as a new discrete signal that is, let's call it some G(n) as this summation K is from 1 to N F(k), if I do this then what I get is G(n+1) – G(n) is same as F(n) right, or G(n+1) is actually equal to G(n)+ F(n+1) right that is F(n+1), so it's you're adding 1 to N+1 for this in that you are removing 1 to N so you have, you're left with this.



So now you can use you can also rewrite this as G(n) – G(n-1) = F(n) I can rewrite so this is running from, N is from here if I do this it's because G is this so it's running from 1, 2, 3 onwards right, it's 1, 2, 3 onwards, so if I want Z transform I should include 0 also, right, you have this so here if I include N = 0, it's fine just leave it so you have G(0) is 0, if I take G(0) which is equal to 0 from this, okay, G(0) is 0 because F of, in that case F(0) F is running from, K is from 1 to 0, so it's not defined, okay, so let us not worry, so simply you have this is the case N is from 1, 2, 3, onwards this is still true, assume that G(0) is 0 if you do this G(0) is 0, okay, to include let me instead of taking K is from 1 to N let me add here 0, 0 to N so that what is your G(0)? G(0) in that case will be simply F(0) so it's well-defined, okay, so if I take this one and what you get is N is running from 0, 1, 2, 3 onwards, okay, F is also, but F is running from 1, 2, onwards, so this you can write you can rewrite this as 0, 1, 2, 3 onwards, of course negative G of negative things which is 0.

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(ii) $\sum_{k=1}^{\infty} f(k) = \lim_{z \to 1} z F(z) = F(1)$

Proof: Let $g(n) = \sum_{k=0}^{n} f(k)$,

$\Rightarrow g(n+1) - g(n) = f(n+1), \quad n = 0, 1, 2, \ldots$

$\Rightarrow g(n) - g(n-1) = f(n); \quad n = 0, 1, 2, \ldots$

So if I put N = 0, G(-1) and that has to be 0 okay, you can consider this, under this now you apply Z transform, application of Z transform to the above differential equation, difference equation gives what we get this is Z transform of G, let's write capital G(z) and here negative when it is G(n-m) what is happening is Z- power -1 simply G(z), okay, and then this is equal to capital F(z) so this is what you have, so you have a G(z) times 1 -1/Z that is simply Z-1/Z is equal to F(z), so this gives me G(z) as Z over Z-1 times capital F(z), so what we require is G(Z transform) so Z transform of this is equal to, so it's as a mistake it should be from 0 to N okay, it's from 0 to infinity then it's all makes sense, so if I consider this is from 0 to N then I can do this, I can assume this and get the difference equation, apply the Z transform and get this.

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Let $g(n) = \sum_{k=0} f(k)$.

$\Rightarrow g(n+1) - g(n) = f(n+1), \quad n = 0, 1, 2, \ldots$

$\Rightarrow g(n) - g(n-1) = f(n); \quad n = 0, 1, 2, \ldots \quad (g(-1) = 0)$

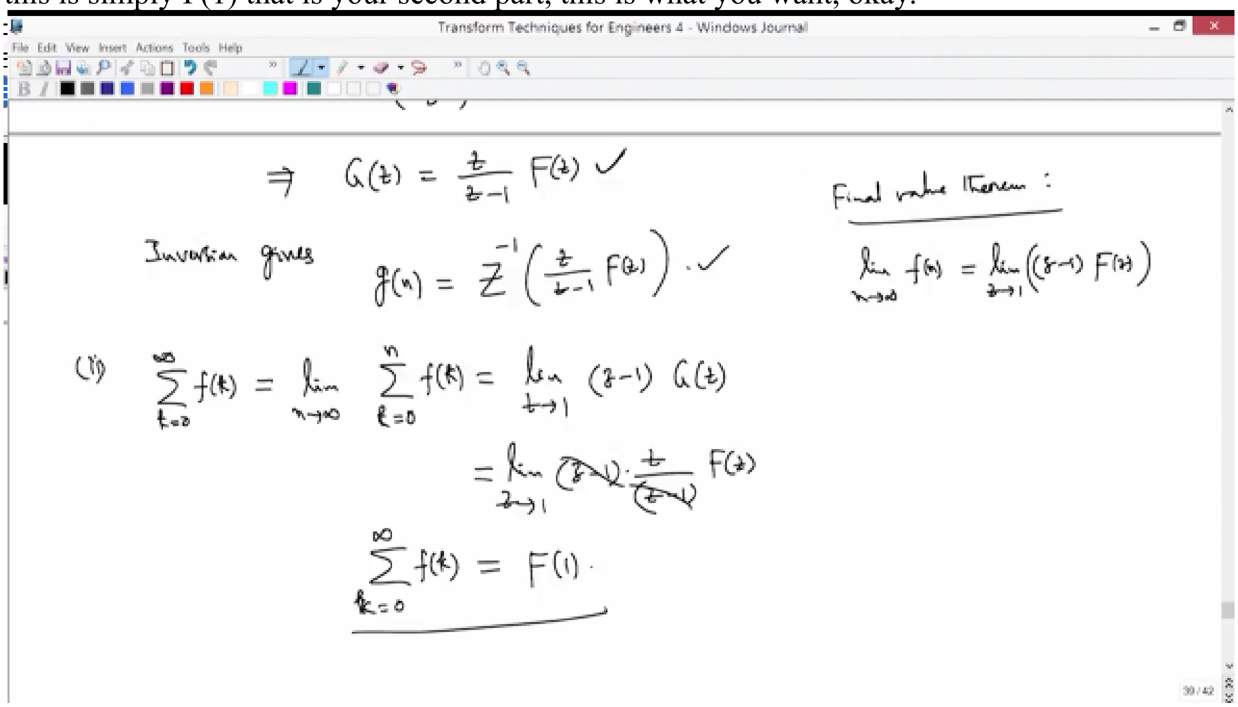
Application of Z-transform to the above difference equation gives

$G(z) - z^{-1} G(z) = F(z)$.

$G(z) \left(\frac{z-1}{z} \right) = F(z)$.

$\Rightarrow G(z) = \frac{z}{z-1} F(z)$

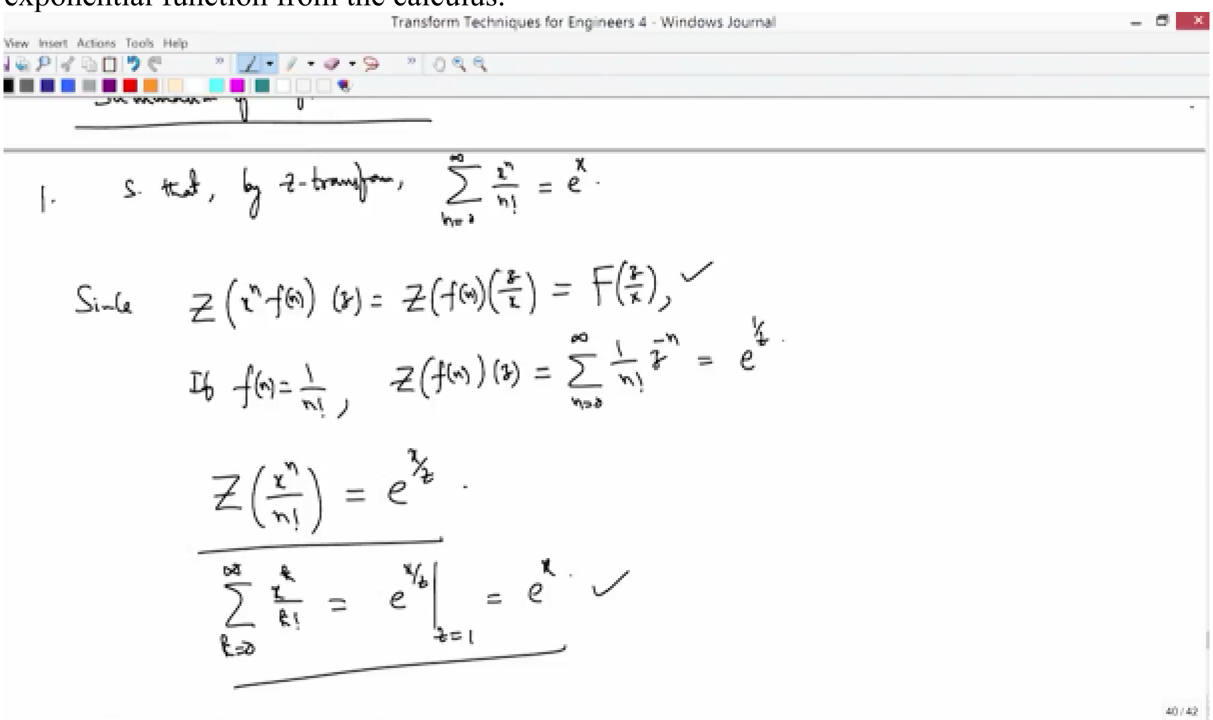
Now inverse transform gives, inversion gives G(n) that is what you want, which is summation this is exactly what you have, that is your first part, this is Z transform of this G(n) is equal to Z over Z/N, so inversion gives so if you actually do the inversion you get G(n) = Z inversion of Z/Z-1 times F(z) okay, this is one result, what we need as your first part.
The second part, to see the second part, the second part is straightforward so what we do is I allow this, so G(n) is simply N is from 0 to, K is from 0 to N F(k) equal to, so if I look at this this is your Z transform of this, let me write so this is, if I take this N goes to infinity so that is actually your full sum, K is from 0 to infinity F(k) this is equal to this.
And now this is equal to limit N goes to infinity, what is that we use here? I think this is where we use a final value, final value theorem if we use here so can we recall what is the final value theorem, so if I recall final value theorem that one of the properties that we have seen earlier, if F of, capital F(z) is the Z transform of discrete signal F(n), sequence F(n) then limit N goes to infinity F(n) this is same as limit Z goes to 1, and Z-1 times capital F(z), so this is what is the final value theorem, so if I use here for this G, okay, this is this, a limit of this is same as a limit of Z goes to 1 instead of F(n) I have this G(n), so limits Z goes to 1, Z-1 capital G(z) so this is same as limits Z goes to 1, Z-1 times G(z) I use here, Z divided by Z-1 times capital F(z), so this gets cancel what you have is simply capital F(1), so this sum is K is from 0 to infinity, F(k) this is simply F(1) that is your second part, this is what you want, okay.



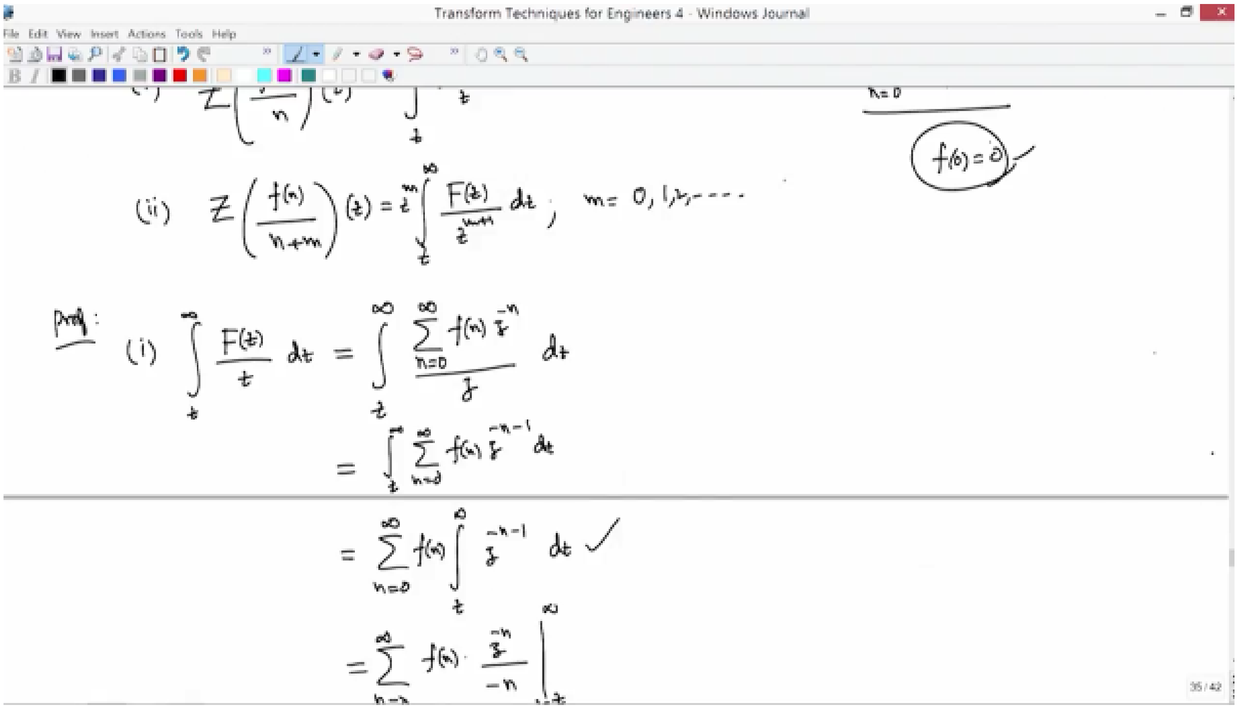
So with this we will try to get this infinite summations, let's evaluate some infinite sums, simple sums to start with the simple sums the summation of infinite series, this is what we want to do. So to start with these examples let me start with, show that by Z-transform N is from 0 to infinity X power N/N factor we already know this standard sum, infinite sum this Taylor series is simply E power X, so we'll show this by Z transform, so if you actually apply we make use of one of the properties of Z transform that is Z transform of X power N times F(n) if you see this one this is actually equal to Z transform of, this is actually Z transform of F(n) for, instead of Z you have Z/X, so this is same as capital F of, Z transform of F(n), capital F, argument is Z/X, so if I use this this is straight forward, this is nothing because N is from 0 to infinity X power N, F(n) Z power -N by definition, so you can include here you have Z/X power –N, if I remove this, this is same as this, this is nothing but capital F(z/x) okay, so its straightforward so by definition you can get this property, since because of this so what is my F(n)? If F(n) is 1/N factorial so what is the Z transform? Z transform of F(n), if F(n) is this Z transform of F(n) of Z of is simply N is from 0 to infinity, F(n) is 1/N factorial, Z power –N, so this becomes this is simply 1/E power 1/Z okay, so we know already that this is true.

So what happens to Z transform of X power N/N factorial, from this, this is nothing but E power because of Z transform of F(n), this argument Z/X, so if you replace Z by Z/X so it's going to be X/Z okay, Z transform of this is this.

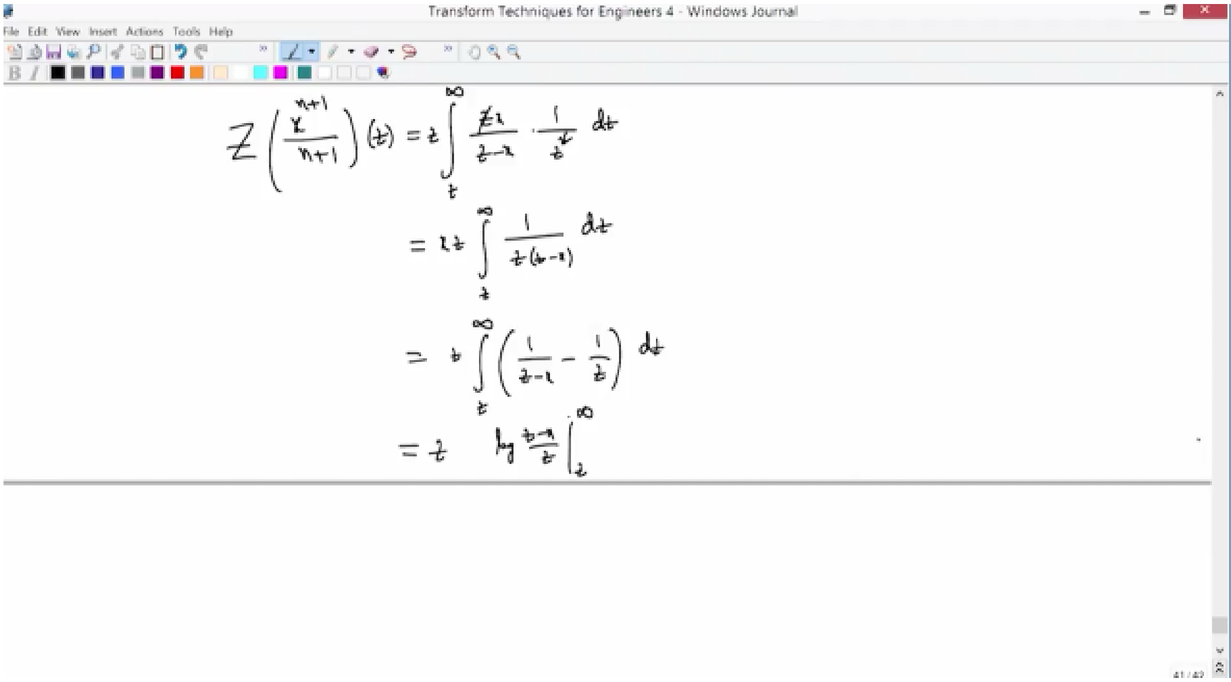
Now if I use the second part of this, this is the second part of this property, the summation of F(k), K is running from 0 to infinity is Z transform of F(k) at Z =1, so I know what is the Z transform of X power N/N factorial that is E power X/Z so you have this summation K is running from 0 to N, 0 to infinity, F(n) so that is X power K/K factorial, okay, so that is this, F(k) is X power K/K factorial equal to Z transform of this that is E power KX/Z at Z = 1, so this is simply E power X, so this is how we can prove that this infinite sum is a standard exponential sum that is exponential function that is also happened to be the Taylor series for this exponential function from the calculus.



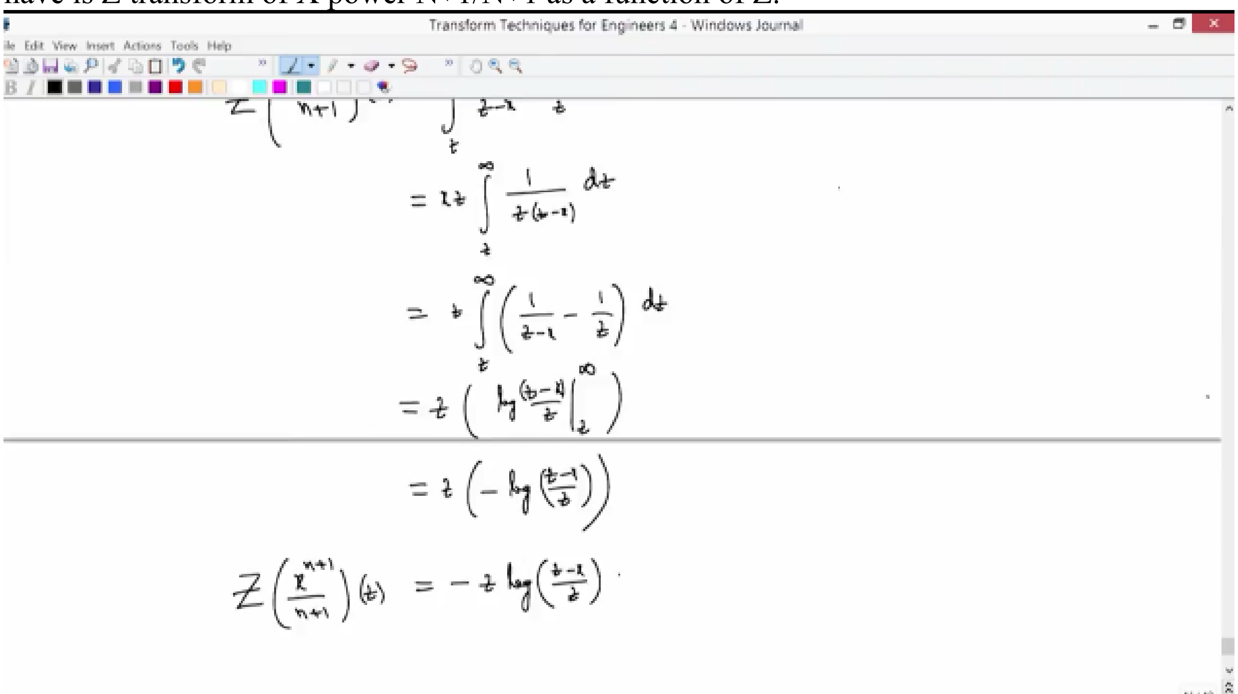
Let's do some not so easy problems, evaluation of this infinite sums, let's do find show that let me write N is from 0 to infinity, -1 power N X power N+1/N+1 = log 1+X, so how do I show this infinite sum? X is the parameter, okay so when you have this X parameter you have this one, so what we do is this solution, Z transform of X power N+1 let me consider, okay, Z transform of X power N+1 is equal to same as X comes out, Z transform of X power N, this is the function of Z, this is simply X times Z/Z-X, I've seen A power N Z/Z-A so that's what I have used so you have ZX/Z-X is this Z transform.



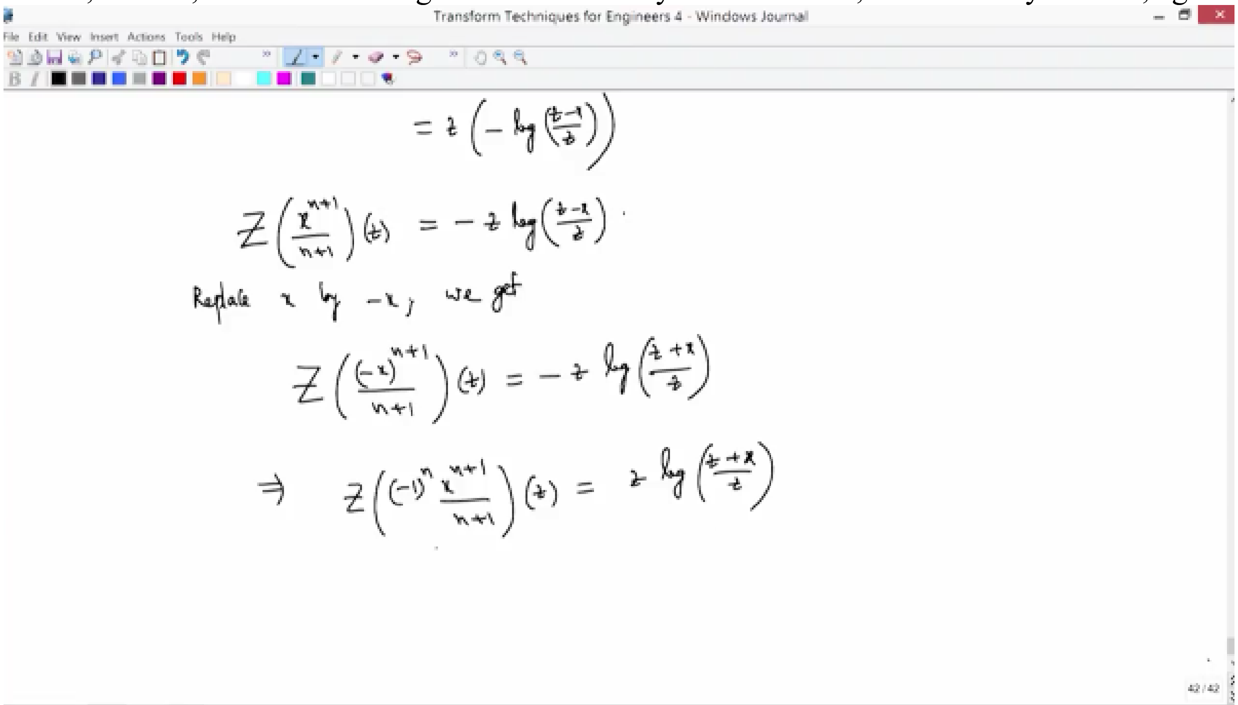
Now you can use the property, a first property that we have just proved, we can make use of a second property here, if you use this here if I use this here the second property you know F(n), let me see this, if I use that property Z transform of X power N+1/N+1 as a function of Z this is equal to and you have integral and you have Z, Z power M, M is 1 and so Z times, Z to infinity capital F(z) that is the Z transform of this function that is ZX/Z-X divided by 1/Z power M+1, M is 1 so you have Z square DZ, so this is equal to simply you have X comes out so you have XZ and you have Z to infinity and Z comes out because nothing to do with this integration variable Z okay, so you have Z Z goes here you have simply 1/Z into Z-X DZ, so this is nothing but XZ, Z to infinity you can write as partial fractions here that becomes 1/Z-X – 1/Z if you do this I have it that becomes X by, so you can include here X, X, you can take this X let me write this okay what is this one? This is actually X/Z into Z-X, but you have only, so instead of this that you can write it here, you can include this one, X divided by this becomes, this is actually equal to X divided by Z into Z-X, so that is that I am replacing its DZ, so you have Z times this is simply if you take the log Z-X – log Z so you can write log Z-X/Z and you include this one,



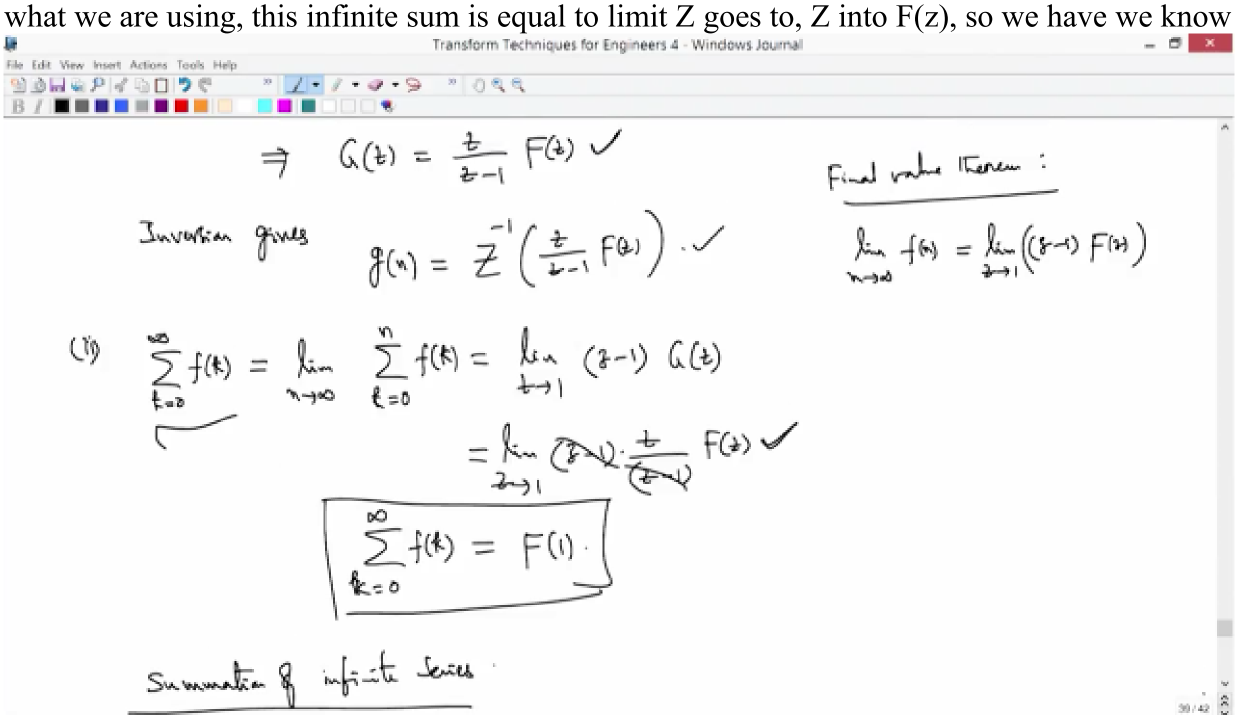
you substitute these limits for this function so you get only for this, okay, so Z times at infinity log 1 as Z goes to infinity, Z-X, X is finite so that is 0, and you have minus, so this is -log Z – X/Z that is exactly what we have, so this is going to be -Z times log Z – X/Z, this what you have is Z transform of X power N+1/N+1 as a function of Z.



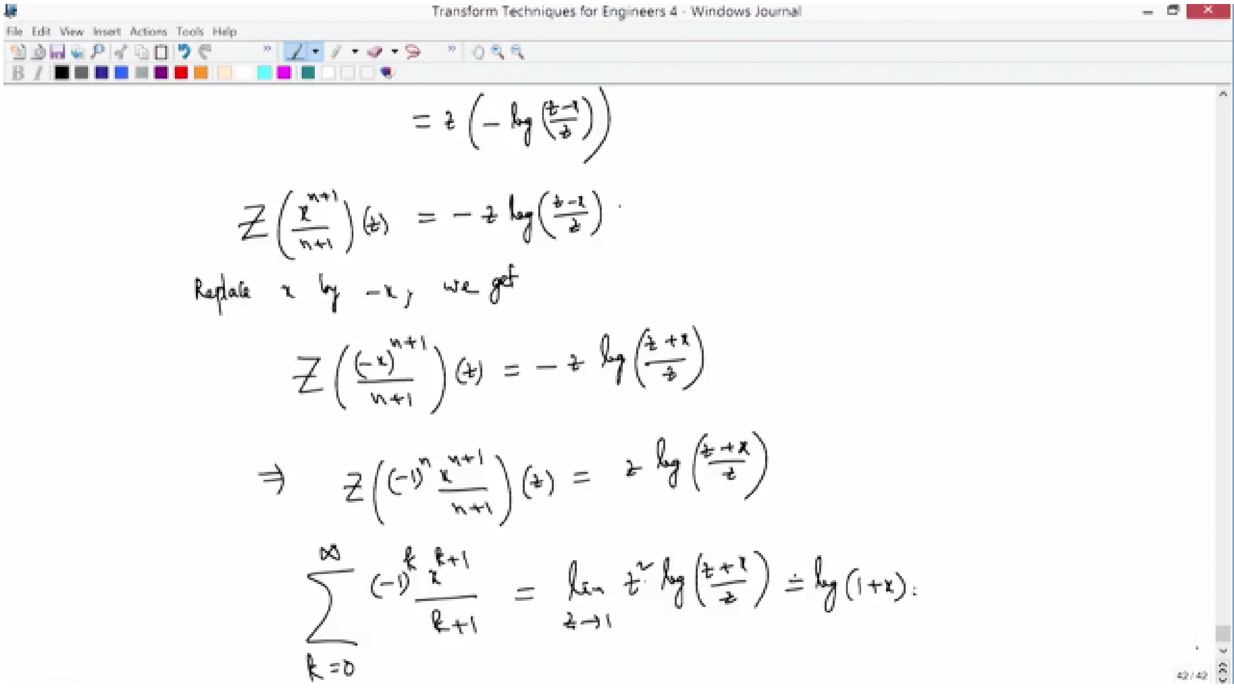
I replace here X by, X can be anything here so far, okay, so if I replace X by -X here, replace X by -X if you do we get, what we get is Z transform of –X, -X will be -1 –X power N+1/N+1 for this transform you get -Z log Z + X/Z, okay, but this is nothing but Z transform of minus you can take it out this is going to be constant, right, so you have a -1 power N+1, so that N you put it inside and put 1-, so you have -1 power N, X power N+1/N+1 for this, this is same as –Z log Z+1/Z, Z+X/Z, so minus minus goes both sides so you have this is, this is what you have, again



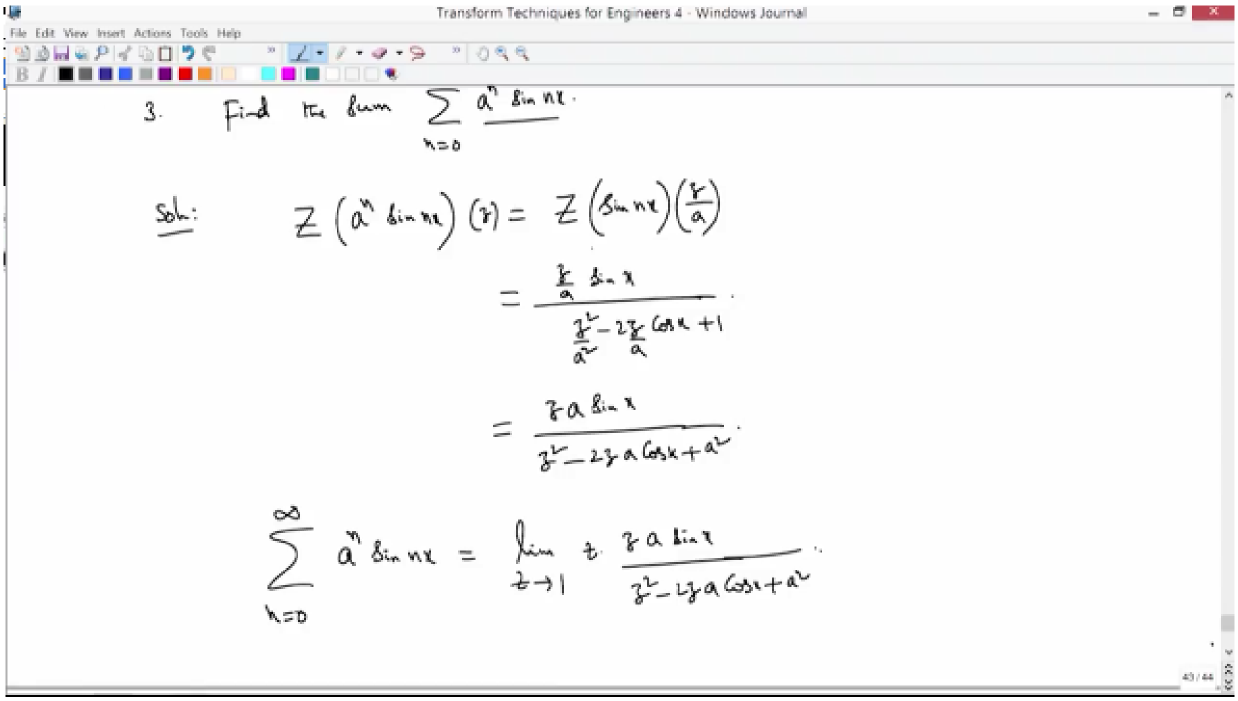
you can apply this infinite sum, this is what you know and now you can have this infinite sum and K is from 0 to infinity -1 power K X power K+1/K+1 this infinite sum is nothing but limit Z goes to 1, Z-1 times Z into, so I have seen, what is this one? Z into this, if you see this, this is what we are using, okay, F(1) is nothing but Z into F(z) okay, Z into F(z) so basically this is what we are using, this infinite sum is equal to limit Z goes to, Z into F(z), so we have we know



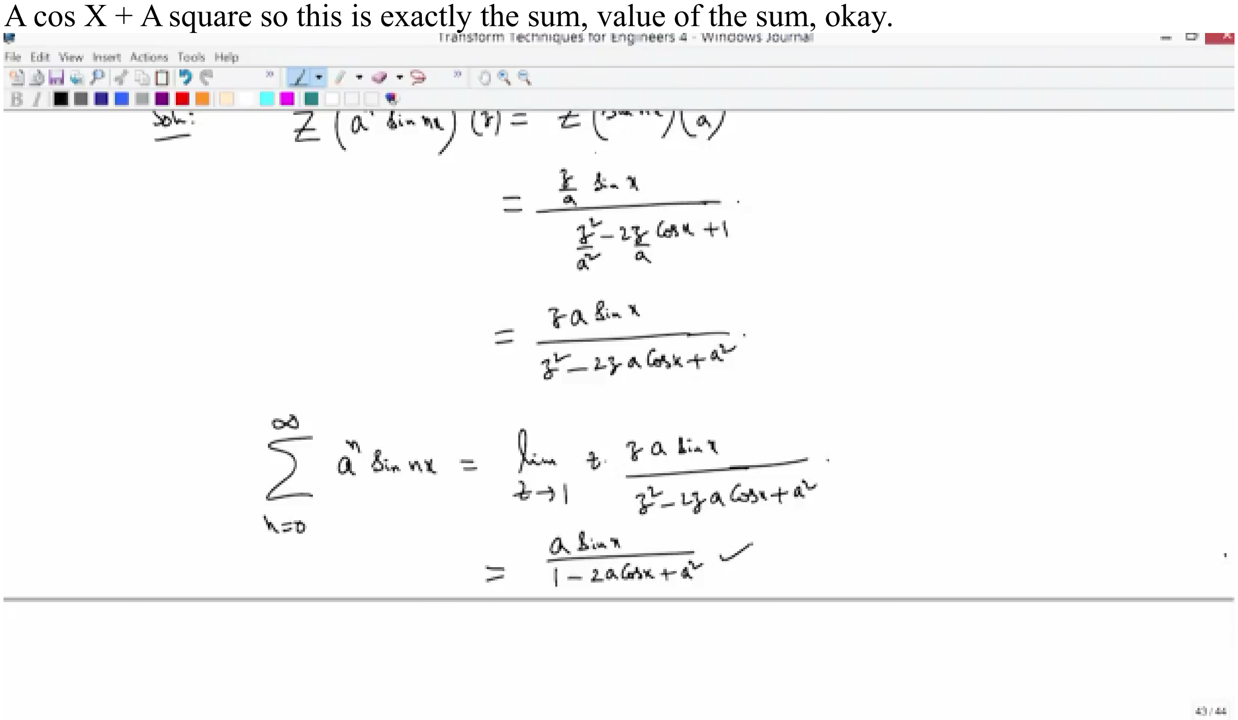
what is capital F(z) so you have Z into, Z into capital F(z) is this one, that is Z square log Z+X/X, Z+X/Z, so as Z goes to 1 so what you get is this is simply log 1 + X, this is what we need to show that is exactly what we have, okay. So this is how we can apply the Z transforms and find these infinite sums like this.



So let me do one more example, let me call it 3, this example is to find the sum, find the sum, infinite sum, sigma N is from 0 to infinity A power N sine NX, X is a variable, so again you can follow the same technique, so solution Z transform of F(n) which is sine NX, let's take the sine NX if you know the sine NX I can take this A power N sine NX, same as Z transform of, there is a function of Z this is same as Z transform of sine NX. Now instead of Z you have Z/A okay, this argument Z you have Z/A, so this is so if you know what is, you know already what is Z transform of sine NX that is Z times sine X/Z square -2Z cos X+1, in the place of this you replace this is, instead of Z you have Z/A, so you have Z/A square, Z/A and so on, so what you have is simply A A gets cancelled so it goes up 1A, so you have ZA sine X divided by Z square -2ZA cos X + A square, so this is what you know, those Z transform of this argument, now you can apply that property, so that you can get the sum, N is from 0 to infinity A power N sine NX is simply a limit Z goes to 1, the final value theorem if you use Z goes to 1 into Z into F(z), so Z into F(z) is this, so we have this simply we have this here, so ZA sine X/Z square -2ZA cos X



+ A square, so as Z goes to 1 what is this value? Simply 1 so you have A sine X divided by 1-2 A cos X + A square so this is exactly the sum, value of the sum, okay.



So this is how you can get the infinite sums, certain infinite sums you can evaluate using this Z transform, okay, so with this we end this Z transforms, Z transforms inverse Z transforms and their applications we have given 2 applications of these Z transforms you may also find some other places where you can use these Z transforms wherever you see some sequences, when you want to evaluate certain nth term of the sequence you can apply, you can try to apply that Z transform based on this inverse Z transforms you can get back your F(n) okay, so I can use this technique, I try to find other applications in other areas.

So thank you for watching, this is the end of the course, so far we have seen maybe let me conclude properly, so in this course what we had is a Fourier series to start with periodic signal and then we find the infinite series for that as a Fourier transform, inverse Fourier transform, Fourier transform will be simply Fourier coefficients they are so you have those Fourier coefficients are defined over only those periodic part, so that is a Fourier transform, inverse transform is basically Fourier series and then you extend it this to infinite periodic, non periodic signal that is fully -infinity infinity signal over the full real line and you ended up getting using this periodic signal we defined, we get this Fourier transform for a non-periodic signal that is, that gives the Fourier transform and it's inverse transform over a full real line, so the some byproducts of these are other Fourier transforms such as sine Fourier transform, sine transform, cosine transform and also you can have a different versions of these Fourier transforms, that's what you have seen in the Fourier transform, and then using this Fourier integral theorem the same definition of the Fourier transform inverse transform, the theorem based on which we have defined a Fourier transform and inverse Fourier transform we made use of this to extend the class of functions for which you can define your Fourier transform, so that you ended up getting into a complex plane, that's where you have seen different transform that is a Laplace transform and it's inverse transform are the results, and you have seen all its properties and its applications in differential equations other areas, evaluating some integrals, so you have seen many applications for both Fourier and Laplace transforms.

And if you take the signal which is a discrete signal we can have a discrete version of the Laplace transform that is exactly is your Z transform, that is what you have seen last few videos, so again we have given how to find the Z transforms, inverse transforms and certain properties based on which we give the application of the Z transforms to solve difference equations, and then evaluating some infinite sums, okay, so this is a comprehensive course of these 4 transforms. Thank you.

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