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Transform Techniques for Engineers

Conditions for the Convergence of Fourier Series

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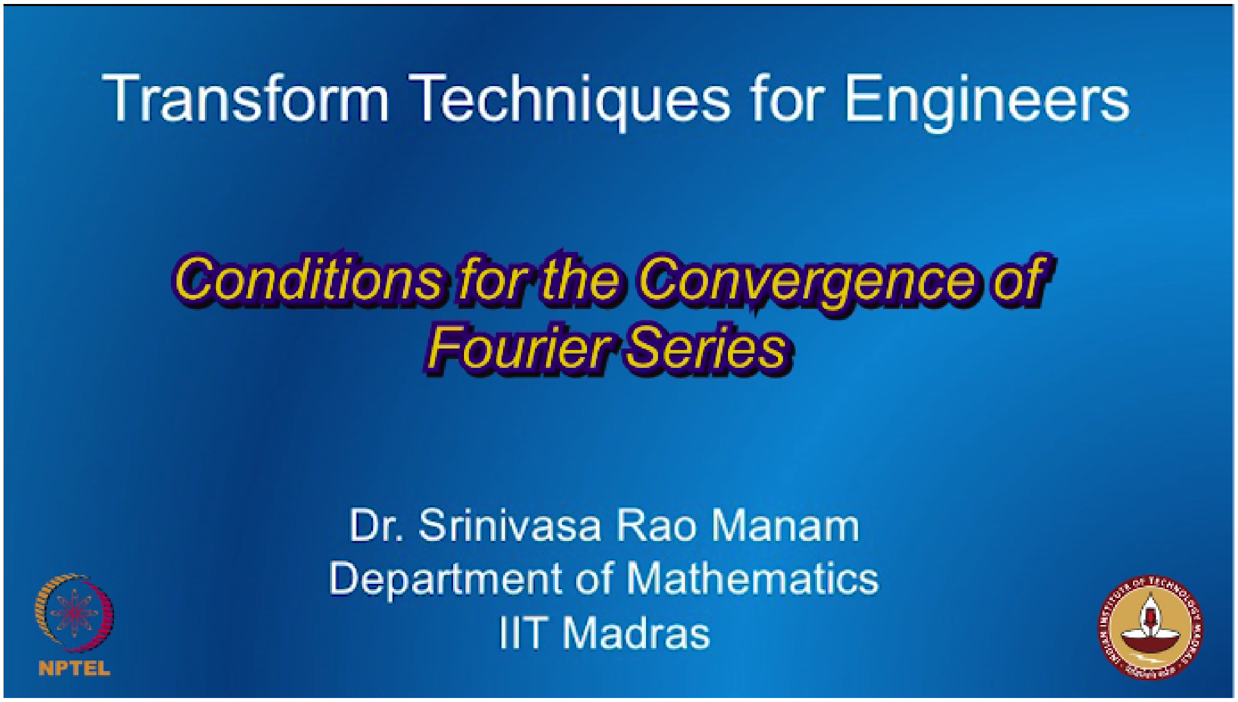


Welcome back, in this video we will see sufficient conditions on the function F(x) so that it's Fourier series converges point-wise to the function F(x), okay. So there are many sufficient conditions, so we'll start with one sufficient condition on the signal, that is a piecewise differentiable function, that means it is a differentiable function, F(x) is differentiable that means F-(x) exists, and that F-(x) is a piecewise continuous function.

Theorem:

Let $f(x)$ be piecewise continuous periodic function with period L.

Then $$\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 \, dx. \quad \text{(Bessel's inequality)}$$

Proof:

So let's take, so if we have such a conditions on the signal F(x) so we will just show that the Fourier series composed of Fourier coefficients and fundamental signals in terms of cosines and sines so that Fourier series composed of all this, use a Fourier series, we will show that this series, this function series converges to the function F(x) point-wise, that means you fix X then the function series becomes a number series, that number series converges to the value of the function at X, so to do this if you have such a function if F(x) is piecewise differentiable function then it has certain properties so we'll do that properties first, so let’s consider let F(x) be piecewise continuous period.

So let's start with the piecewise continuous function, so one of the sufficient conditions we have is a piecewise differentiable function, so if it is piecewise differentiable function implies it's a continuous function, so for such a function piecewise continuous, actually periodic function, so we only consider periodic functions, periodic function with period, period let us say L then, then what you have is this result, so this is called Bessel inequality, N is from minus infinity to infinity, this is your CN complex, Fourier coefficients, this is less than or equal to 1/L – L/2 to L/2 absolute value, square of the absolute value of this signal with DX, this is what you have. So this is called Bessel's inequality.

So we’ll first prove this as a theorem, let's say let's call this theorem and if you proof us, will first prove this and use this and the sufficient conditions to show the convergence of Fourier series of F.

So we’ll start with, so you have a, once you have this CN’s as a complex Fourier coefficients, so you have the Fourier series so the Fourier series is, Fourier series, this N is from minus infinity to infinity, and you have CN, E power IN omega naught X, omega naught is 2 pi/L because it's a, signal is of period L, so this is your Fourier series so let's consider the partial sum of this, partial sums let's represent SN, SN(x) you take the partial sum of the series there is you take it from - N to N, it depends on N so if I change this index as K so it's running from minus N to N, CK E power IK W naught X, so this is your partial sum.

Theorem:

Let $f(x)$ be piecewise continuous periodic function with period L.

Then. $\sum_{n=-\infty}^{\infty} |c_n|^2 \leq \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\, dx$. (Bessel's inequality)

Proof: Fourier Series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$

Let $S_n(x) = \sum_{k=-n}^{n} c_k e^{ik\omega_0 x}$.

So once you assume this is your partial sum CN is 1/L integral -L/2 to L/2, F(x) E power - IN omega naught X DX, okay so if you integrate, integral – L/2 to L/2, F(x) – SN(x) this you multiply with E power - IN omega naught X DX, what is this value? This value is actually, this is integral – L/2 to L/2, you expand it, so this is F(x) E power - IN omega naught X DX - integral –L/2 to L/2, SN(x) this you write it as K is from - N to N, CK E power IK omega

Proof: Fourier Series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$, $c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x}\, dx$.

Let $S_n(x) = \sum_{k=-n}^{n} c_k e^{ik\omega_0 x}$.

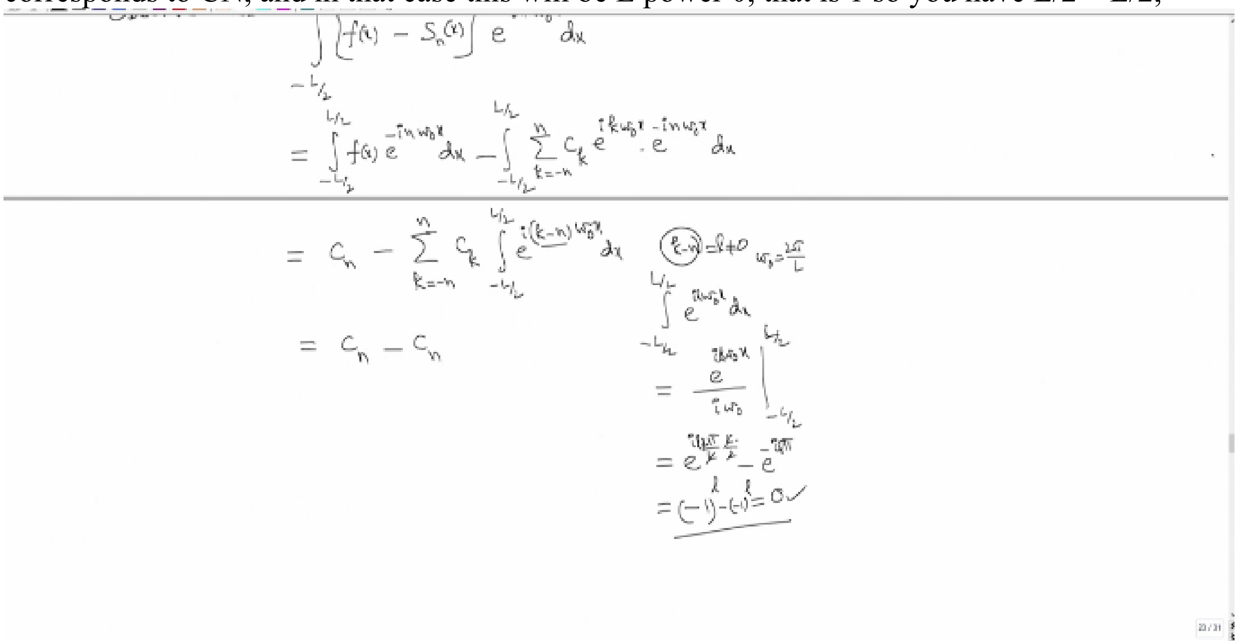
Observe that

$$\int_{-L/2}^{L/2} \left[f(x) - S_n(x)\right] e^{-in\omega_0 x}\, dx$$

$$= \int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x}\, dx - \int_{-L/2}^{L/2} \sum_{k=-n}^{n} c_k e^{ik\omega_0 x} \cdot e^{-in\omega_0 x}\, dx.$$

naught into E power - IN omega naught X, this with DX, so this is nothing but, this is your CN and here this is your, so you can, this is a finite sum you can take it out, K is from -N to N CK this is the integral – L/2 to L/2, and what you are left with is E power I K-N omega naught X DX, so what happens when K and K is from minus N to N, and K is N only when K equal to N or K - N is nonzero, and KN – N is nonzero you can see that this quantity is the same, so it is going to be 0.

So when it is done it's only when it is equal to 0 so that is actually CN -, so let us choose this is equal to 1, for example this is always number so let us take as 1, what happens to – L/2 to L/2? E power I omega naught X DX, what is omega naught? Omega naught is 2 pi by period is L, so this is equal to I omega naught E power I omega naught X for which you take this L/2 to L/2 limits you apply you get E power I times, omega naught is 2 pi/L and X is at L/2, so 2 2 goes L by so E power I pi - E power, again - I pi, so what is this one? This is actually E power I pi is, cos pi is -1 and here also minus off, cos pi is – 1, so it's going to be + 1 which is 0, if it's 2, if I take any N value here, so if K – N is some number, let us say some L which is nonzero, so what you have is IL omega naught, so you have IL omega naught, so you still have IL omega naught IL pi, so when L is integer, so this is still – 1 power, – N power L, okay, into minus of minus 1 power L, so it's any case it is 0, so you can see that when K – N is a nonzero integer, this is always 0, so this quantity is 0 so everything else will go except when K = N, so that is corresponds to CN, and in that case this will be E power 0, that is 1 so you have L/2 – L/2,



$$\int_{-L/2}^{L/2} \{f(x) - S_n(x)\} e^{-in\omega_0 x} dx$$

$$= \int_{-L/2}^{L/2} f(x) e^{-in\omega_0 x} dx - \int_{-L/2}^{L/2} \sum_{k=-n}^{n} c_k e^{ik\omega_0 x} \cdot e^{-in\omega_0 x} dx$$

$$= c_n - \sum_{k=-n}^{n} c_k \int_{-L/2}^{L/2} e^{i(k-n)\omega_0 x} dx$$

$$= c_n - c_n$$

$$(k-n) = l \neq 0 \quad \omega_0 = \frac{2\pi}{L}$$

$$\int_{-L/2}^{L/2} e^{il\omega_0 x} dx = \left. \frac{e^{il\omega_0 x}}{il\omega_0} \right|_{-L/2}^{L/2} = e^{il\pi} - e^{-il\pi} = (-1)^l - (-1)^l = 0$$

that's going to be, of course so this is not exactly CN, CN is L into CN, so you have L into CN this is actually, by definition this is the Fourier coefficient L into CN, 1/L of this is CN, so you have L into CN, and here CN, and K = N it contributes integral – L/2 to L/2 and K = 1, this is 1 DX that is L, so finally it is 0, okay.

So we see that, we observe that this is the case F(x) - SN multiplied with E power - IN omega naught X, that quantity is 0, so because of that we note that – L/2 to L/2 F(x) – SN(x), so

Proof: Fourier Series $\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 x}$, $\checkmark\, c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx$.

Let $S_n(x) = \sum_{k=-n}^{n} c_k e^{ik\omega_0 x}$ $\checkmark$

Observe that

$$\int_{-L/2}^{L/2} \left[f(x) - S_n(x)\right] e^{-in\omega_0 x}\, dx \quad \checkmark$$

$$= \int_{-L/2}^{L/2} f(x)\, e^{-in\omega_0 x}\, dx - \int_{-L/2}^{L/2} \sum_{k=-n}^{n} c_k e^{ik\omega_0 x} \cdot e^{-in\omega_0 x}\, dx$$

$$= Lc_n - \sum_{k=-n}^{n} c_k \int_{-L/2}^{L/2} e^{i(k-n)\omega_0 x}\, dx \qquad (k-n) = l \neq 0 \quad \omega_0 = \frac{2\pi}{L}$$

$$\int_{-L/2}^{L/2} e^{il\omega_0 x}\, dx$$

$$= Lc_n - c_n L = 0 \quad \checkmark$$

because you multiply any of this quantity, okay, so what is your SN? SN is this, you take any of this quantity that is 0, okay, so instead of, so you take a linear combination of them, so that is SN, I have chosen minus of this, so SN is with plus so you can just take SN bar that will have exponential with - I sine, so SN(x) conjugate of this DX this is nothing but sigma, K is from - N to N, so SN's you can replace with CK bar, so integral – L/2 to L/2, F(x) – SN(x) that is as it is into E power - IK omega naught X DX and this is exactly what we have seen that it is 0, okay.

$$= Lc_n - \sum_{k=-n}^{n} c_k \int_{-L/2}^{L/2} e^{i(k-n)\omega_0 x}\, dx \qquad (k-n) = l \neq 0 \quad \omega_0 = \frac{2\pi}{L}$$

$$\int_{-L/2}^{L/2} e^{il\omega_0 x}\, dx$$

$$= Lc_n - c_n L = 0 \quad \checkmark$$

$$= \frac{e^{il\omega_0 x}}{i\omega_0}\Bigg|_{-L/2}^{L/2}$$

$$= e^{il\frac{2\pi}{L}\frac{L}{2}} - e^{-il\pi}$$

$$= (-1)^l - (-1)^l = 0 \quad \checkmark$$

Note that

$$\int_{-L/2}^{L/2} \left[f(x) - S_n(x)\right] \overline{S_n(x)}\, dx$$

$$= \sum_{k=-n}^{n} \overline{c_k} \int_{-L/2}^{L/2} \left(f(x) - S_n(x)\right) e^{-ik\omega_0 x}\, dx$$

So this simply in the place of N you can put anything K so it doesn't matter, so you have, because this quantity is 0, so what you have is every time it is 0, whatever may be K value it is

$$\Big|_{-L/2} \quad \text{(}i\omega_0\text{)}$$

$$= e^{i\frac{2\pi}{L}k\frac{L}{2}} - e^{-ik\pi}$$

$$= (-1)^k - (-1)^k = 0$$

Note that $\int_{-L/2}^{L/2} [f(x) - S_n(x)]\, \overline{S_n(x)}\, dx$

$$= \sum_{k=-n}^{n} \overline{C_k} \int_{-L/2}^{L/2} (f(x) - S_n(x))\, e^{-ik\omega_0 x}\, dx$$

$$= 0$$

0, this quantity is 0 so that means is completely 0. So we first note this one, then we observe that if you simply integrate from – L/2 to L/2, F(x) – SN(x) and this is a complex quantity and you have F(x) bar – SN(x), the conjugate of this whole thing that is actually absolute value square, DX this is always positive, okay.

So and what is this one? This is actually equal to – L/2 to L/2 and because this with SN bar that is 0 so you are left with only F(x) – SN(x) into F(x) bar DX which is, that is always, this is same as this because the other part will be is shown to be 0 earlier here, okay, so this is equal to integral – L/2 to L/2 this is F(x) into F bar that is mode F(x) whole square DX -, – L/2 to L/2 SN(x) F(x) bar DX, okay.

And you can also observe here this gives, we have noted that this quantity is 0, okay, this quantity is 0 implies – L/2 to L/2 by 2 F(X) SN(x) bar DX is nothing but – L/2 to L/2 SN(x) bar modulus square DX, okay, so that is exactly we will write here, so this is equal to – L/2 to L/2 F(x) bar modulus square DX - this is L/2 to L/2 SN(x) square DX, and clearly this quantity is positive quantity, okay.

$$0 \le \int_{-L/2}^{L/2} \left(f(x)-S_n(x)\right)\overline{\left(f(x)-S_n(x)\right)}\,dx = \int_{-L/2}^{L/2}\left(f(x)-S_n(x)\right)\overline{f(x)}\,dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2\,dx - \int_{-L/2}^{L/2} S_n(x)\,\overline{f(x)}\,dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2\,dx - \underline{\underline{\int_{-L/2}^{L/2} |S_n(x)|^2\,dx}}.$$

And what is this one? So integral – L/2 to L/2 mode SN(x) or rather SN(x) into SN(x) bar DX, if you write this you actually see that this is quantity, K is from - N to N CK E power I, so what is this SN? SN would define as E power IK omega naught X into its bar, so that is CK bar, so rather this into, now I change this is different that index you can write in a different way or you can use the same index - N to N CK bar E power - IK omega naught X DX this is what it is, this is SN(x), this is a SN bar, this is equal to so only contribution is when they are equal, when they are same, that is and also integral from – L/2 to L/2 any exponential function, and this one, and this is different, okay. So rather let's use that different K, different, so I use this instead of this index I use a different index that is let us say M, so this M, M is from - N to N see this is a product of this finite sums as well as so, as long as this exponentially multiply this with this you get E power I K–M, and only when K = M that is becoming E power 0, so that is I times 0, that is the only contribution you will have, otherwise that integral is 0 that is what you have seen earlier.

$$0 \le \int_{-L/2}^{L/2} \left(f(x)-S_n(x)\right)\overline{\left(f(x)-S_n(x)\right)}\,dx = \int_{-L/2}^{L/2}\left(f(x)-S_n(x)\right)\overline{f(x)}\,dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2\,dx - \int_{-L/2}^{L/2} S_n(x)\,\overline{f(x)}\,dx$$

$$= \int_{-L/2}^{L/2} |f(x)|^2\,dx - \underline{\underline{\int_{-L/2}^{L/2} |S_n(x)|^2\,dx}}. \checkmark$$

$\underline{\underline{k=m}}$

$$\int_{-L/2}^{L/2} S_n(x)\,\overline{S_n(x)}\,dx = \int_{-L/2}^{L/2} \underline{\sum_{k=-n}^{n} c_k e^{ik\omega_0 x}} \cdot \underline{\sum_{m=-n}^{n} \bar{c}_m e^{-im\omega_0 x}}\,dx.$$

$$= \sum_{k=-n}^{n} |c_k|^2 \int_{-L/2}^{L/2} dx = \underline{L \cdot \sum_{k=-n}^{n} |c_k|^2}.$$

So what you get is sigma K is -N to N CK mode square only when M = K that is what is the result, and it will become 0 so that is exactly you'll get so that integral will be simply L, so each one so you have this is – L/2 to L/2 DX this is nothing but L times sigma K is from - N to N modulus of CK square, this is what you get, so this implies sigma K is from - N to N modulus of CK square is you can replace this quantity here minus of that, so if you use this inequality 0 less than or equal to integral – L/2 to L/2 mode F(x) whole square DX, in the place of this now I'll replace this that is actually L times - L times sigma K is from - N to N CK square, this is true for every N okay, so this implies, you can bring this this side so you get sigma CK whole square, K is from -N to N is always less than or equal to 1/L integral – L/2 to L/2 mode of F(x) square DX, so this is true for every N okay, so this implies, because this is true for every N, you can allow this to be N goes to infinity, so you can see that this is actually true for every N so that means you can allow N goes to be infinity so you have, so K is from minus infinity to infinity mode CK square is less than or equal to 1/L – L/2 to L/2 mode F(x) square DX, so this is exactly what you want to prove that is Bessel's inequality.
So that you can see here, this is your Bessel's inequality, so this is what we are going to use to show that if the signal is piecewise a differentiable function, and we'll show the Fourier series actually converges, so the process and the proof of that we will use this inequality, so one important corollary of this, one implication of this result is that, so if this quantity is, the right hand side quantity is finite that means if F(x) is, or if – L/2 to L/2 if it is square integrable function DX, if this is finite, okay, then this quantity N is from minus infinity to infinity, I am just changing the index here, so CN square is always finite that means, okay, so what does this mean? So this means that implies, it's clear right this is a straight forward, if this is finite, this is finite, that is what I have written, so this implies a limit of CN, as N goes to infinity, this is

$$0 \le \int_{-L/2}^{L/2} |f(x)|^2\,dx - L\sum_{k=-n}^{n} |c_k|^2, \quad \forall n.$$

$$\Rightarrow \sum_{k=-n}^{n} |c_k|^2 \le \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\,dx, \quad \forall n.$$

$$\Rightarrow \boxed{\sum_{k=-\infty}^{\infty} |c_k|^2 \le \frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2\,dx.}$$

Corollary: If $\int_{-L/2}^{L/2} |f(x)|^2\,dx < \infty$, then $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$

$$\Rightarrow \lim_{n\to\infty} c_n = 0.$$

going to be 0, so this is straight forward from the number series if a limit, if sigma AN, N is from 0 to infinity or minus infinity infinity, if this quantity is finite then this limit of AN has to go to 0, if it's not see the proof is why is this so? So we'll use contrapositive argument that if limit of this is nonzero, if suppose this limit AN is not equal to 0, okay, then what happens? So this converges to some nonzero quantity some L okay. Then what is the result? AN - L you can put them in some arbitrary small quantity whenever N is bigger than some big N okay, so that means AN is always lying between L - epsilon to L + epsilon, for every N bigger than or equal to N, so you can choose epsilon as 1, so you can choose any epsilon, given epsilon there is a big

N that depends on it, so given epsilon there is an N beyond which this AN's will strictly lying here, so if we choose epsilon as 1, your AN's will be lying so for some N, N is this capital N is actually depending on what you give as epsilon, here we have given as 1.

$$0 \le \int_{-L/2}^{L/2} |f(x)|^2 dx - L\sum_{k=-n}^{n} |c_k|^2, \quad \forall n.$$

$$\Rightarrow \sum_{k=-n}^{n}|c_k|^2 \le \frac{1}{L}\int_{-L/2}^{L/2}|f(x)|^2 dx, \quad \forall n.$$

$$\Rightarrow \boxed{\sum_{k=-\infty}^{\infty}|c_k|^2 \le \frac{1}{L} \int_{-L/2}^{L/2}|f(x)|^2 dx}$$

Corollary: If $\int_{-L/2}^{L/2}|f(x)|^2 dx<\infty$, then $\sum_{n=-\infty}^{\infty} |c_n|^2<\infty$

$$\Rightarrow \lim_{n\to\infty} c_n = 0.$$

If $\sum_{n=0}^{\infty} a_n<\infty$, then $\lim a_n = 0$

Proof: Suppose $\lim a_n = L$

$|a_n - L|<1$, $n \ge N$

$\Rightarrow L-1<a_n<L+1$; $\forall n\ge N$

If L is nonzero let's say you can think of L as, L is a positive quantity so let's choose positive quantity so assume that L is positive, so I can choose epsilon as L/2, okay, so if I choose L/2 there exist corresponding some N beyond which you have L + L/2, so this is, because L – L/2 this is always positive, so this implies sigma AN, because these are all positive quantities you are adding up every time, okay, so for N is running from 0, so A is 0, A1 every time you are adding a positive quantity all the time so this has to go to infinity, but that is not the case, but

$$0 \le \int_{-L/2}^{L/2} |f(x)|^2 dx - L\sum_{k=-n}^{n} |c_k|^2, \quad \forall n.$$

$$\Rightarrow \sum_{k=-n}^{n}|c_k|^2 \le \frac{1}{L}\int_{-L/2}^{L/2}|f(x)|^2 dx, \quad \forall n.$$

$$\Rightarrow \boxed{\sum_{k=-\infty}^{\infty}|c_k|^2 \le \frac{1}{L} \int_{-L/2}^{L/2}|f(x)|^2 dx}$$

Corollary: If $\int_{-L/2}^{L/2}|f(x)|^2 dx<\infty$, then $\sum_{n=-\infty}^{\infty} |c_n|^2<\infty$

$$\Rightarrow \lim_{n\to\infty} c_n = 0.$$

If $\sum_{n=0}^{\infty} a_n<\infty$, then $\lim a_n = 0$

Proof: Suppose $\lim a_n = L$ $\quad L>0$

$|a_n - L|<\frac{L}{2}$, $n \ge N$

$\Rightarrow 0<L-\frac{L}{2}<a_n<L+\frac{L}{2}$; $\forall n\ge N$

$\Rightarrow \sum_{n=0}^{\infty} a_n = \infty$

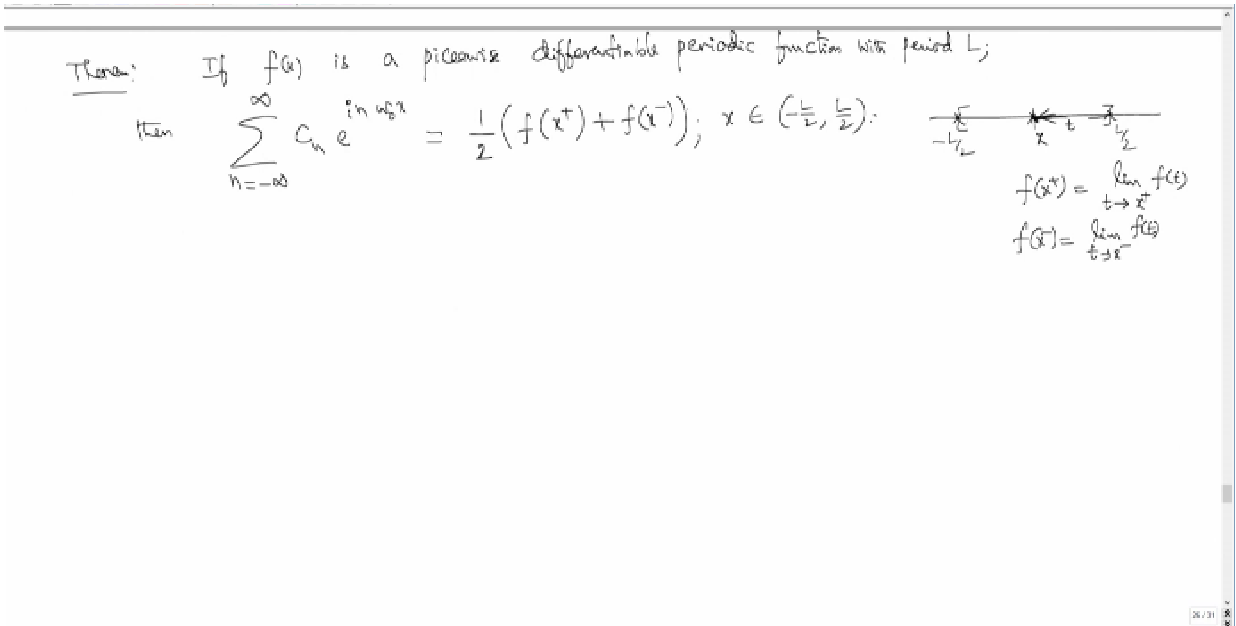
we know that this is actually finite, so that means this has to go to 0, this is a small result in the on sequences on series of real numbers, so using that you can see that this is our limit quantities has to go to 0, so this is a Riemann Lebesgue Lemma, this is Riemann Lebesgue, okay.

Corollary: If $\int_{-L/2}^{L/2} |f(x)|^2 \, dx < \infty$, then $\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty$

$$\Rightarrow \underline{\lim_{n \to \infty} c_n = 0} \; \checkmark \; \text{(Riemann-Lebesgue Lemma)}.$$

25 / 31

So now we give, we state this sufficient conditions as a theorem main result that if F(x) is a piecewise differentiable function, periodic function with period L, then this what you have is this Fourier series, complex Fourier series which I'm writing CN E power IN omega naught X, omega naught is the fundamental frequency which is 2 pi/L, okay, this is equal to, it is actually converges to the value, once we fix this value X this converges to 1/2 of F(x+) that means if you have, this is your X on the real line between – L/2 to L/2, X+ is a limit of F(x+) is a limit of F(t), now T is going to X from the positive side that is the meaning, so the limit of this function as T, if T is here, T is approaching X right hand side. Similarly F(x-) is a limit of F(t), T is going to X negative side so this is the meaning of this two function values, F(x+) F(x-), so whose average value is what this series, converges to that, if you fix your X, X I fix may be inside, okay, because it's a differentiable function which is a continuous function, implies it is a continuous function so F(x) so you can always choose like that, okay. As of now I am choosing X in between here, so not at the endpoints, so if it is endpoints is actually, we will see what it is okay, so X belongs to – L/2 to L/2.



So let me give you the proof now, this is the sufficient condition for this Fourier series converges to function, because it's a differential function is actually you can say that this is exactly equal to F(x), because this is nothing but F(x) because it is of its continuity both are same, so average value is itself, so X belongs to –L/2 to L/2, so proof is again we will consider the partial sums of this left-hand side, so partial sums we represent SN(x), okay, as in the earlier case, so you have K is from - N to N CK E power IK omega naught X, so this is equal to sigma K is from -N to N CK, so CK I'm going to define so what it is? CK is a Fourier coefficient that is 1/L –L/2 to L/2 F(t) E power - IK omega naught T that is my CK into, this I write it as it is so this is with DT and E power minus this power this is as it is, that is E power IK omega naught X, so this will give me, I take this sum inside so you have 1/L – L/2 to L/2, F(t), and now I write the sum here, finite sum, because it's a finite sum I can take it inside, so what you have is E power - IK omega naught T - X DT, so what is this one? This is equal to 1/L, so you can see this one, you can just replace, you can break this integral into two parts, first from 0 to L/2 F(t) this finite sum, K is from - N to N E power minus IK omega naught T – X DT + 1/L – L/2 to 0, F(t) sigma, K is from -N to N E power –IK omega naught, T – X.

Theorem: If $f(x)$ is a piecewise differentiable periodic function with period L;

then $$\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 x} = \frac{1}{2}\left(f(x^+)+f(x^-)\right);\quad x \in \left(-\frac{L}{2},\frac{L}{2}\right).$$
$$= f(x), \quad x \in \left(-\frac{L}{2},\frac{L}{2}\right).$$

$$f(x) = f(x^+) = \lim_{t\to x^+} f(t)$$
$$f(x) = f(x^-) = \lim_{t\to x^-} f(t)$$

Proof: Let $$S_n(x) = \sum_{k=-n}^{n} C_k e^{ik\omega_0 x}$$
$$= \sum_{k=-n}^{n} \frac{1}{L}\int_{-L/2}^{L/2} f(t)\, e^{-ik\omega_0 t}\, dt\; e^{ik\omega_0 x}$$
$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\cdot \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}$$

So what I do here is, so let's call this some quantity which is messy, so let's define let DN be, I'll call it some DN(x) this is a function of X okay, so let me define what is this DN? DN(x) if I define it as K is from - N to N, E power -IK omega naught X, if I define it like this, then what is this quantity? This quantity is actually equal to so E power IN omega naught X, when I put K equal to - N + E power IN - 1 omega naught X and so on, we'll go on 0 that is 1, 1 also come here somewhere comes in between and then it's becoming negative, when I put K = N is finally end up with IK, K is N, IN omega naught X, so this is equal to if you take E power IN omega naught X out, so you have 1 + E power -I omega naught X, and the E power -I to omega naught X and so on, finally E power -I 2N omega naught X, that is what is that thing, so this sum, this is in GP so you can just have it, so E power you can write that sum with A is 1, into 1- R is E power -I omega naught X, power 2N+1, so you have 2N+1, so N is 2N here, so N+1 is 2N+1 divided by 1 - E power -I Omega naught X, that is what it is.

$$= \frac{1}{L}\int_{-L/2}^{L/2} f(t)\cdot \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}\, dt$$
$$= \frac{1}{L}\int_{0}^{L/2} f(t) \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}\, dt + \frac{1}{L}\int_{-L/2}^{0} f(t) \sum_{k=-n}^{n} e^{-ik\omega_0 (t-x)}$$

Let $$D_n(x) = \sum_{k=-n}^{n} e^{-ik\omega_0 x} = e^{in\omega_0 x} + e^{i(n-1)\omega_0 x} + \cdots + e^{-in\omega_0 x}$$
$$= e^{in\omega_0 x}\left(1 + e^{-i\omega_0 x} + e^{-i2\omega_0 x} + \cdots + e^{-i2n\omega_0 x}\right)$$
$$= e^{in\omega_0 x}\,\frac{1 - e^{-i\omega_0 x(2n+1)}}{1 - e^{-i\omega_0 x}}.$$

So if I now take it inside, so this is E power, this is a product for everything, so if you take this inside E power IN omega naught X - E power - IN + 1, E power IN+1 omega naught X divided by 1 - E power - I omega naught X, what I do is I take, the denominator also I rewrite like this, I multiply E power IN omega naught by 2, E power I omega naught by 2X, if I multiply and divide, IN+1 omega naught X divided by, so you also do it here, so if I do both sides, so you take it inside the denominator one you take it inside here, so you have E power I omega naught 2X - E power - I omega naught/2X this is what it happens, it becomes.

Let $$D_n(x) = \sum_{k=-n}^{n} e^{-ik\omega_0 x} = e^{in\omega_0 x} + e^{i(n-1)\omega_0 x} + \dots + e^{-in\omega_0 x}$$

$$= e^{in\omega_0 x}\left(1 + e^{-i\omega_0 x} + e^{-i2\omega_0 x} + \dots + e^{-i2n\omega_0 x}\right)$$

$$= e^{in\omega_0 x} \cdot \left(\frac{1 - e^{-i\omega_0 x(2n+1)}}{1 - e^{-i\omega_0 x}}\right)$$

$$= \frac{e^{in\omega_0 x} - e^{-i(n+1)\omega_0 x}}{1 - e^{-i\omega_0 x}} \checkmark$$

$$= \frac{e^{\frac{i\omega_0}{2}x}\left(e^{in\omega_0 x} - e^{i(n+1)\omega_0 x}\right)}{\left(e^{i\frac{\omega_0}{2}x} - e^{-i\frac{\omega_0}{2}x}\right)}$$

Now take this one inside, so what you see is E power I (N+1/2) omega naught X - E power is actually minus, - I (N+1/2) this one so – 1, -I, +I become that's – 1/2, so that's what happens, omega naught X divided by, this is actually nothing but this is as it is, so I omega naught/2X - E power - I omega naught /2X, so what is this one? This is exactly equal to sine 2I sine of (N+1/2) omega naught X divided by sine omega naught/2X, this is what is my DN(x), if I define like this as DN, DN is this so clearly DN(-x) is actually equal to DN(x), this is an even function, even function, so if you use this one what happens to your SN? SN is this one, so what happens to your SN? SN(x) is 1/L integral 0 to L/2, so what happens to F(t), DN of, what is your DN(t-x), DN(t-x) DT + 1/L – L/2 to 0, F(t) DN(t-x) DT, that is also same, right, so you can see that both sides are same, so you just split that integral, so this is what it is.

Now put T-X as some X dash in this, in this integral if you do this you have DT is DX dash and that becomes, and here T-X as, first of all put T = -T, so in both the integrals you can put it as T-X as X dash, then what happens to SN(x)? SN(x) is 1/L integral 0 to, so if I put T = 0, and this is going to be minus, so T-X is, so you need not split it, before splitting itself you can do, you can split it later so minus L/2 to L/2 this is what is this SN, you put this one, so you can see that it's going to be –L/2 –X, because X is fixed to +L/2 –X, because it's a periodic, the translation will be same as again –L/2 to L/2, so both will be same, so this will not change because it's a periodic function, so - L/2 to L/2, F(t) is X+X dash or DN(x dash) DX dash, this is what it is.

Now you can split this into two parts so 1/L –L/2 to 0, F(x+ x dash) DN(x dash) DX dash + 1/L times 0 to L/2, F(x+ x dash) DN(x dash) DX dash, so what happens here, so here you change X dash by, put some X dash by –X, -T if you say if you do so you get 1/L, this becomes L/2 to 0, F(x-t) DN(-t) which is, DN is an even function, so DN(-t) is DN(t) and you have – DT, so that is minus, this minus so you can change the limits, so L/2, so T you replace as it is, so X dash itself, actually this is what, okay let's write DT only, so DT and this one is +1/L integral 0 to L/2 F(x+t) DN(t) DT, this is what you have, okay, so this is exactly it has become my SN(x), so partial sum of the Fourier series.

$$= \frac{1}{L}\int_{-L/2} f(x+x')\, D_n(x')\, dx'$$

$$= \frac{1}{L}\int_{-L/2}^{0} f(x+x')\, D_n(x')\, dx' + \frac{1}{L}\int_{0}^{L/2} f(x+x')\, D_n(x')\, dx'$$

$$x' = -t$$

$$S_n(x) = \frac{1}{L}\int_{0}^{L/2} f(x-t)\, D_n(t)\, dt + \frac{1}{L}\int_{0}^{L/2} f(x+t)\, D_n(t)\, dt$$

So the next step is I'll just write what is your DN(t), that is 1/L, 0 to L/2 you can combine these two integrals there is going to be F(x+t) + F(x-t) this one I wrote first and into, this plus this or DN(t) is common, so DN(t) we have seen already that this is sine N+1/2 omega naught X divided by sine omega naught X/2, okay, this is what we have, this is a function of T so you write T, DT. So what I do is, I do small trick here, I just add and subtract because so I am going to assume that this quantity is one function, okay, this is one function if I take this as one function so you will see that, so what we are going to show is eventually maybe I'll see in the next video that I will show that if I choose, if QN, so let's call this some QN of, let's say a quantity QN(t) is F(x+t) + F(x-t), X is fixed, so that's why I'm calling it T, function of T and divided by sine omega naught T/2, so if I call this as my function this is, this has to be well-defined, okay, you see that T is actually is, T belongs to 0 to L/2, so that is what is the integral is between, so as T goes to 0 I have a trouble here, so this is 0 and this quantity, so something divided by 0, so that is why to avoid this something some quantity F(x), 2 times F(x) divided by 0 that is going to be infinity, so to avoid that so I will just rewrite this Q as F(x+t) – and I have X a positive side function, okay, and then + F(x-t) I will also subtract a negative side function, divided by, I divide with X so that I have to multiply X divided by this sine omega naught T/2.

$$S_n(x) = \frac{1}{L}\int_0^{L/2} f(x-t)\, D_n(t)\, dt + \frac{1}{L}\int_0^{L/2} f(x+t)\, D_n(t)\, dt$$

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) + f(x-t)\right] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

Is $\quad Q_n(t) = \dfrac{f(x+t)+f(x-t)}{\sin\frac{\omega_0 t}{2}} \qquad t \in [0, \frac{L}{2}]$

$$= \frac{f(x+t) - f(x^+) + f(x-t) - f(x^-)}{x} \cdot \frac{x}{\sin\frac{\omega_0 t}{2}}$$

Okay, this is what I will do, okay, so this integrand value this is what I saw, because I added and subtracted, so I will have to add, I will have to take, I’ll have some addition that is F(x+) + F(x-) because I subtracted both times, XX I don't use divided by sine omega naught, I have

$$= \frac{1}{L}\int_0^{L/2} \left[f(x+t) + f(x-t)\right] \frac{\sin(n+\frac{1}{2})\omega_0 t}{\sin\frac{\omega_0}{2}t}\, dt$$

Is $\quad Q_n(t) = \dfrac{f(x+t)+f(x-t)}{\sin\frac{\omega_0 t}{2}} \qquad t \in [0, \frac{L}{2}]$

$$= \frac{f(x+t) - f(x^+) + f(x-t) - f(x^-)}{x} \cdot \frac{x}{\sin\frac{\omega_0 t}{2}}$$

$$+ \frac{f(x^+) + f(x^-)}{\sin \omega_0}$$

subtracted only this one, right, so okay so we'll see maybe in the next video we don't have time, so from here we will go on next, we'll see in the next video, we're trying to prove that the Fourier series converges to the function F when it is piecewise differentiable function, so we are halfway through so we have taken the partial sum of this Fourier series, and we manipulated up to some point, so the remaining part to show that this actually converges to the function F(x) in the next video. Thank you very much. [Music]

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