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Transform Techniques for Engineers
Solving first order PDE’s by Laplace
Transform
With
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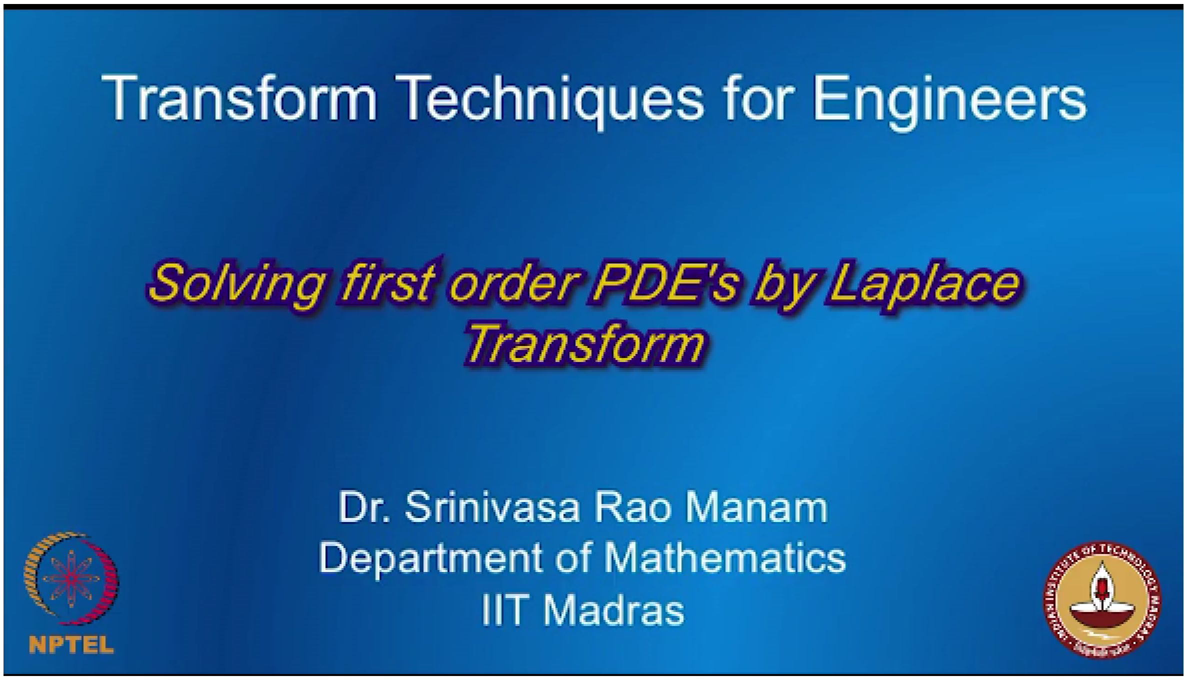
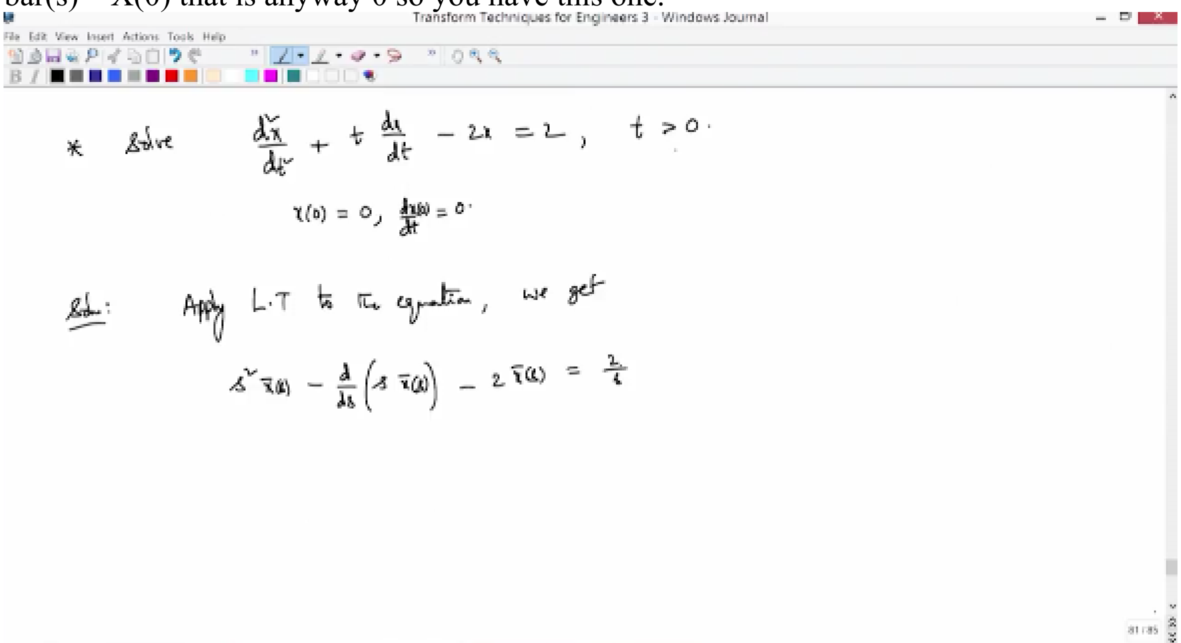


Welcome back, we were discussing about the applications of Laplace transform, and we have solved second order linear ordinary differential equations, and also a system of linear ordinary differential equations.

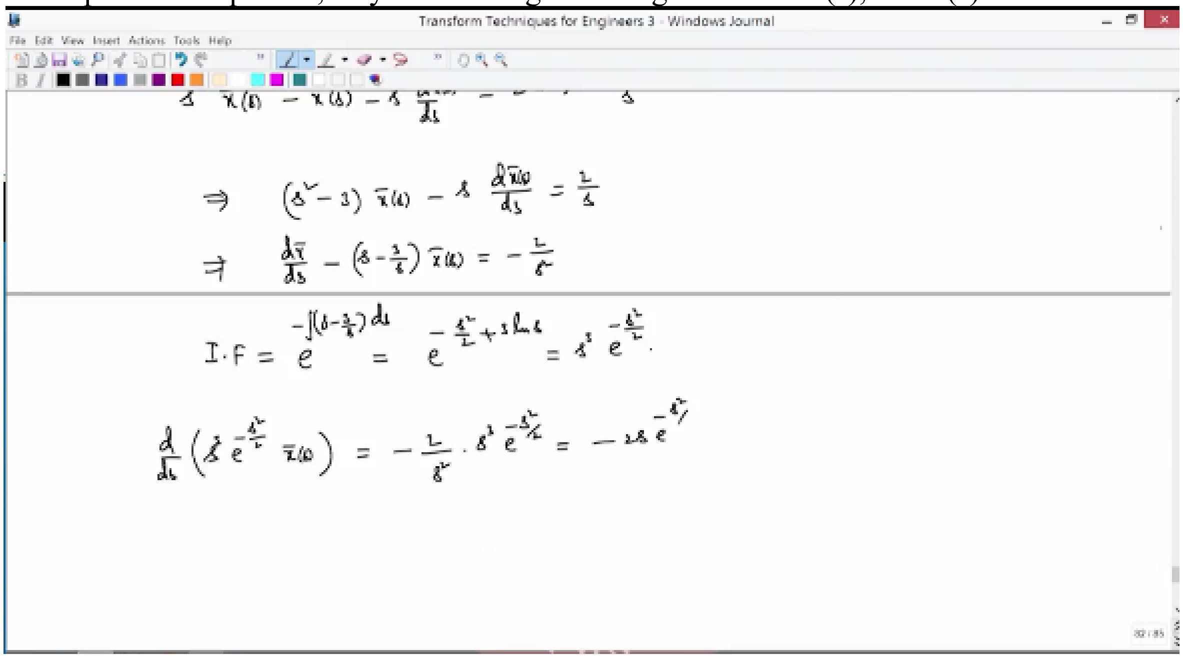
We have solved how to solve, we have seen how to solve these problems, so let's solve, we have also solved some special equations it's called Bessel equation with boundary conditions rather initial conditions you have seen how to get that solution by making use of initial value theorem.

We will also look at similar problem, so let's do one more problem in this video and along with we will solve, we'll also solve partial differential equations along with some more equations okay. In this video to start with some second order ordinary differential equation of similar type like what I have done in the last video, it's like a kind of equation, second order equation with variable coefficients and then with some initial conditions how do we solve it, so here also we may have to use the initial value theorem, and then we'll move on to solve other applications for example solving partial differential equations, we start with the first order, we will try to solve one or two problems, and then we move on to a typical linear equations which is a hyperbolic wave equation, heat equation, and Laplace equation, we’ll try to solve and then now as an applications okay.

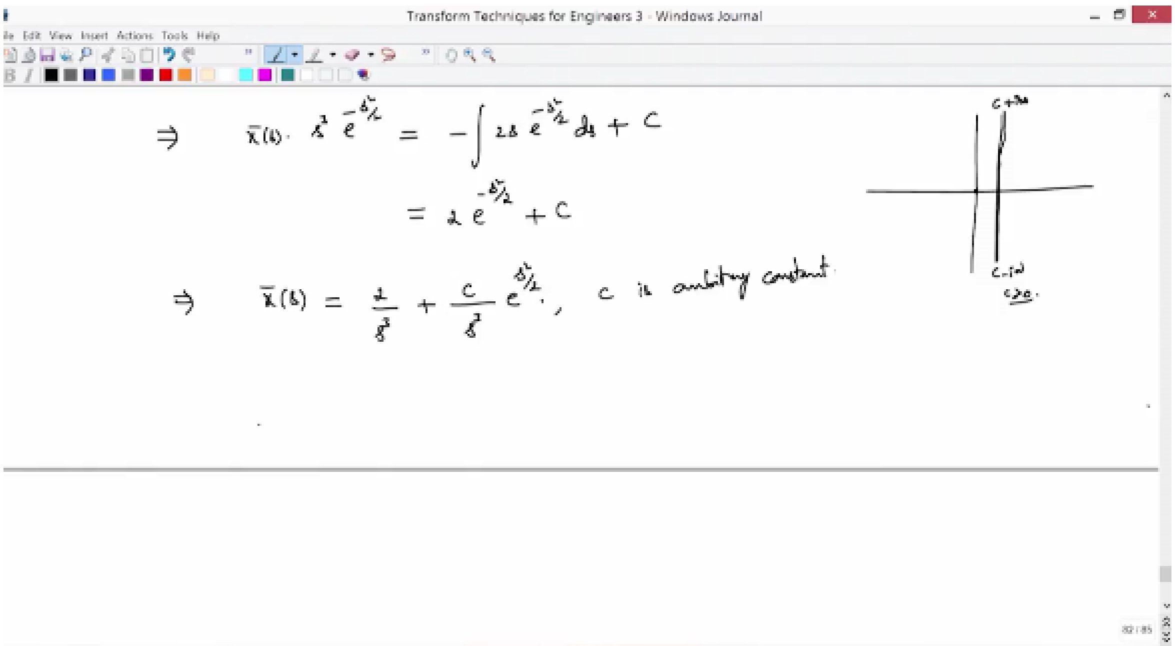
And then we'll solve, we will give applications to how to solve, as an application of this Laplace transform we can also solve integral equations this is what we will see, some special type of integral equation which you will see in the next one or two videos, okay.
So let us start doing this one more example so that is a solve D square X/DT square + T DX/DT - 2X = 2, so this is what you have to solve and T is positive domain is 0 to infinity, and you have this initial conditions, this is 0 and also X DX/DT (0) this is also 0, okay, so these are the two initial conditions, so you can look at the solution, solution is apply Laplace transform, same Laplace transform to get to the equation we get Laplace transform of two derivatives that is S square X bar(s) - S times X(0) - X dash(0), so these two are the conditions which I'm using here so that makes it 0, so this is what you have for first term. And second term you have –D/DS of Laplace transform S times, Laplace transform of DX/DT, sorry –D/DS of Laplace transform of this one when T is multiplied with this, if you want to have a whole transform of T times this one, what it is? Is –D/DS you have seen when we were doing properties of the Laplace transform, I've seen the derivative of Laplace transform of this is –D/DS of Laplace transform of coefficient of T that is Laplace transform of DX/DT that we know already that this is S X bar(s) – X(0) that is anyway 0 so you have this one.



And then – 2 X bar(s) = 2 times and you have the Laplace transform of 1, so that is 1/S, so you have 2/S we really don't know what is the domain of this S, okay, so S is because that really we don't know because that is an unknown function, and only because we know only the domain T positive we're just applying it here along with these initial conditions we have used, and this is what we have, so you have X bar(s) if what you get is okay let me use this S square and then -X bar(s) and then -S times D/DS of X bar(s) - 2X bar(s) = 2/S, so this becomes S square -3 times X bar(s) - S times DX bar(s)/DS, this is equal to 2/S, so this you can rewrite as DX bar/DS - S-3/S of X bar(s), I divide with S both sides so that you have, this is going to be -2/S square, so we need to solve for X bar(s), so this is a linear equation, linear ordinary differential equation with S variable, so one can solve by multiplying an integrating factor as E power integral this, this is as your P that is - S-3/S DS so this is your integrating factor, so you get E power –S square/2 times + 3 LN(s), simply you have, you get it as S cube times E power -S square/2, so if you multiply this both sides of this equation you get S cube E power -S square/2 times X bar(s) whose derivative is nothing but after multiplying this left hand side, that's your left hand side and you already multiplied this side with S square times S cube E power -S square/2 which is -2S E power -S square/2, so you can integrate and get this X bar(s), X bar(s) times S cube times

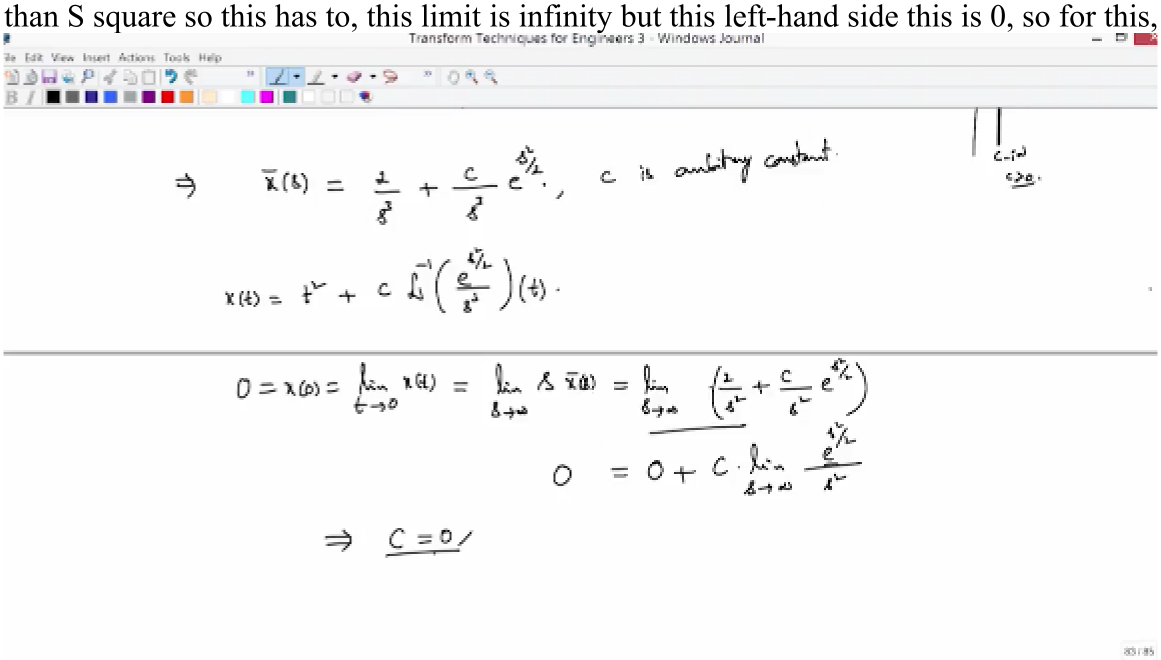


E power -S square/2 is actually equal to -integral 2S E power -S square/2 DS + an arbitrary constant C, integration constant so this one you can easily see that this is going to be E power minus S square/2, so 2 times 2, 2 comes out and this is exactly what you have, so this is your anti-derivative for this integral, so you have integral values is this +C.
So if you now from this you can get your X bar(s) as 2/S cube and then +C/S cube times E power S square/2 this is what you get as a solution X bar(s). Now you can invert this, as you can see how already use your initial conditions, both the initial conditions we have used here, so there's nothing to use, but still we have an arbitrary constant, C is an arbitrary constant, so this we may have to use, we have to find out, so as you can see this is, I have only 0 is the singular point in your S plane, so 0 is the same so you can use your C-I infinity C infinity, C+I infinity that inversion any curve with C positive, okay, so otherwise E power S square/2 is analytic function, only S cube having a singular point at 0, so all this side, so this side you can see that this function is analytic function, so that's how you choose your this line C, C value so that you can get your X(t) from your definition of inverse Laplace transform, so otherwise you

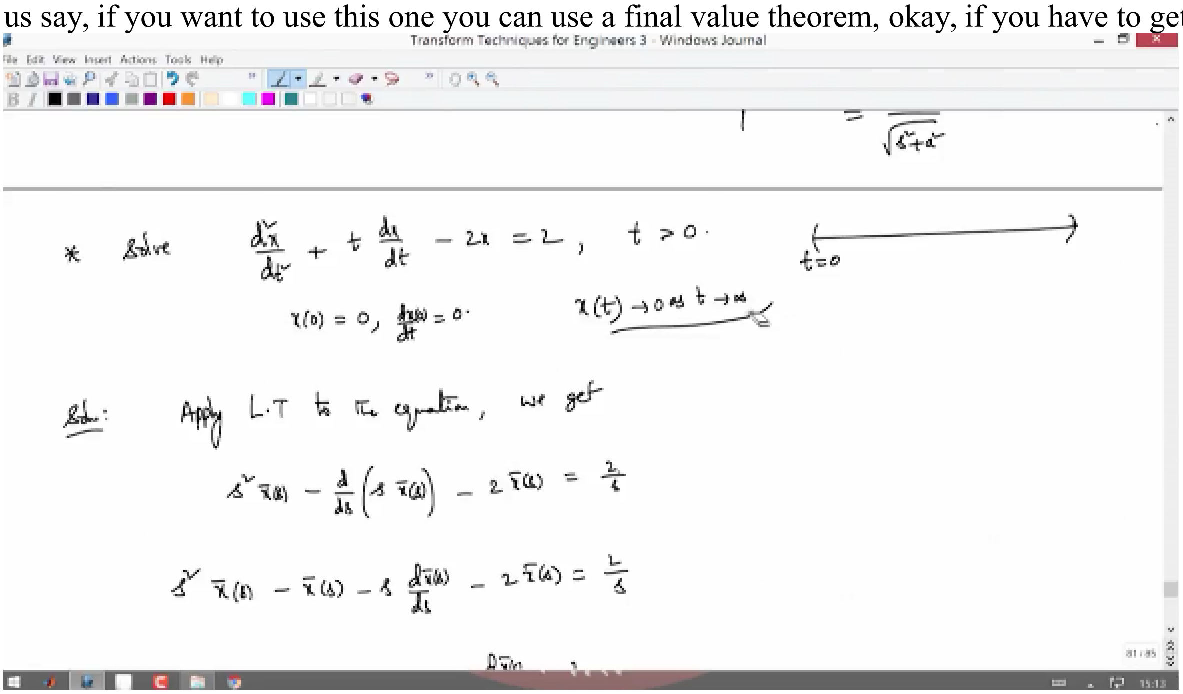


can get X(t) directly, so 2/X cube is actually, is your T square, so Laplace inversion of this is your T square + C times Laplace inversion of E power S square/2 /S cube okay, so this is a function of T, right.

So you know that if you want to use your initial value, initial value theorem X(0) that is 0, so you can see that this is 0 is X(0) this is a limit of X(t) as T goes to 0, so X(t) is satisfying the differential equation, so it's differentiable, so it's limit exists so this is the one which is equal to limit of S times X bar(s) as S goes to infinity, so what is this one? This is equal to limit S goes to infinity, S times X bar(s) is 2/S cube + C divided by S cube E power S square/2, so this if you want this, this if you see this is going to be if you remove I have S square, here also S square, so this is going to be anyway first term is going to be 0 + second term this is going to be the limit, S goes to infinity by S square E power S square/2, exponential function grows bigger than S square so this has to, this limit is infinity but this left-hand side this is 0, so for this, this



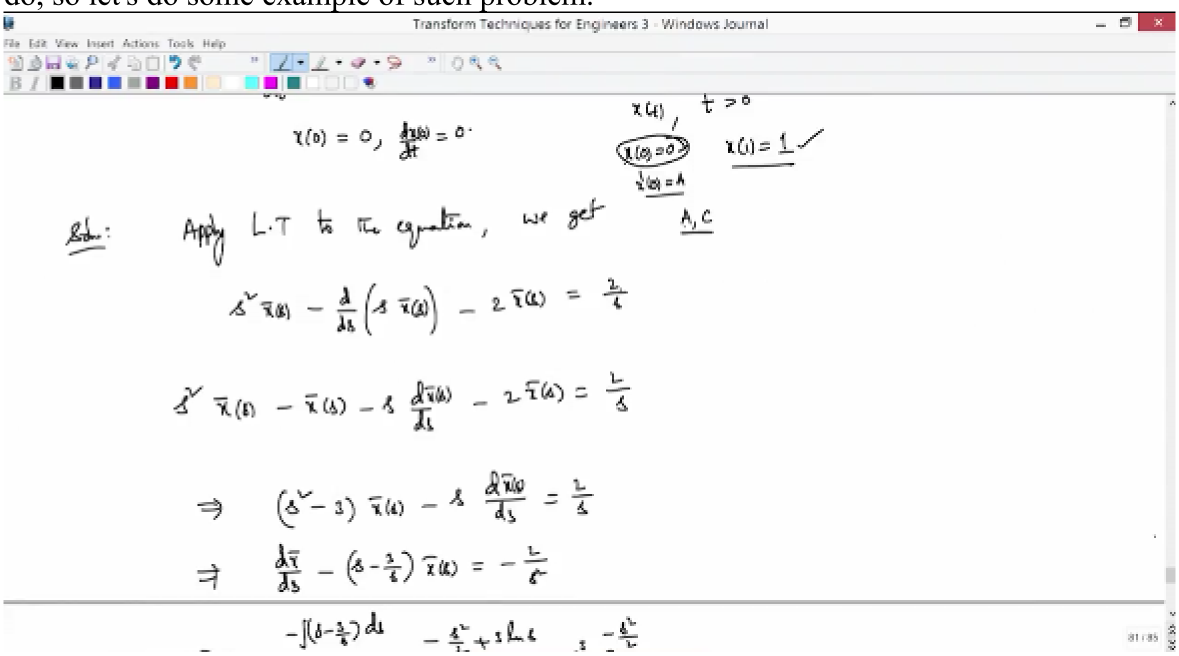
implies C has to be 0, so this gives me because of this once we find C is 0 so this implies the solution X(t) is simply T square, so this is your solution of this initial second order differential equation with the initial data, so X is a non-homogeneous equation in fact, T square if you substitute you can verify T square if you differentiate initial condition is satisfied, T square if you do it here 2T, and so here 2 and if you put it here T times so 2+T times 2T, -2 times T square, so this goes to 0 and this is left with 2, so that is exactly satisfying, so equation is satisfied along with this initial condition, so the solution is required solution is only T square, so this is how you solve a second-order equations even if it is non-homogeneous equation you don't have to worry about non-homogeneous term separately, you can just use Laplace transform directly get its inverse and you can get this, you can use that's why so, some places you may have to use your, if you get an arbitrary constant you can use your initial value theorem or final value theorem depending on what is given, if it is given as X(0) that is initial value theorem you can use, if you have X(infinity) if you have a boundary condition.
So for example what is the boundary here? So this is T = 0 and this is at infinity, so at infinity if you provide 2 boundary conditions let us say X infinity, X(t) goes to 0 as T goes to infinity let us say, if you want to use this one you can use a final value theorem, okay, if you have to get



rid of the arbitrary constants involved in your Laplace transform at the end, okay, so instead of this if you are given this still you can solve as though this is an unknown so when you are applying Laplace transform for the second derivative you will get DX/DT (0) that you take it as some constant A, so A will be there throughout, that A you can get it by applying this boundary condition to get rid of this A, and if it still, if it in the inversion when you solve this equation if you still get some another arbitrary constant that you can use it either with this initial value theorem because of this condition or with this final value theorem you can use, you can use to get rid of that arbitrary constant involved in the solution of the ordinary differential equation of, solution of ordinary differential, so the first order differential equation of in, for X bar, okay, so you can use that if it is, if you want to use this one you have to use final value theorem, if you want to use this initial condition first point is this boundary condition if you want to use you can use the initial value theorem, if you want to use this boundary condition to get rid of that arbitrary constant that is coming at the end, for example this here this you can use this arbitrary constants, this boundary, this boundary condition if you want to use you have to use final value theorem, okay.

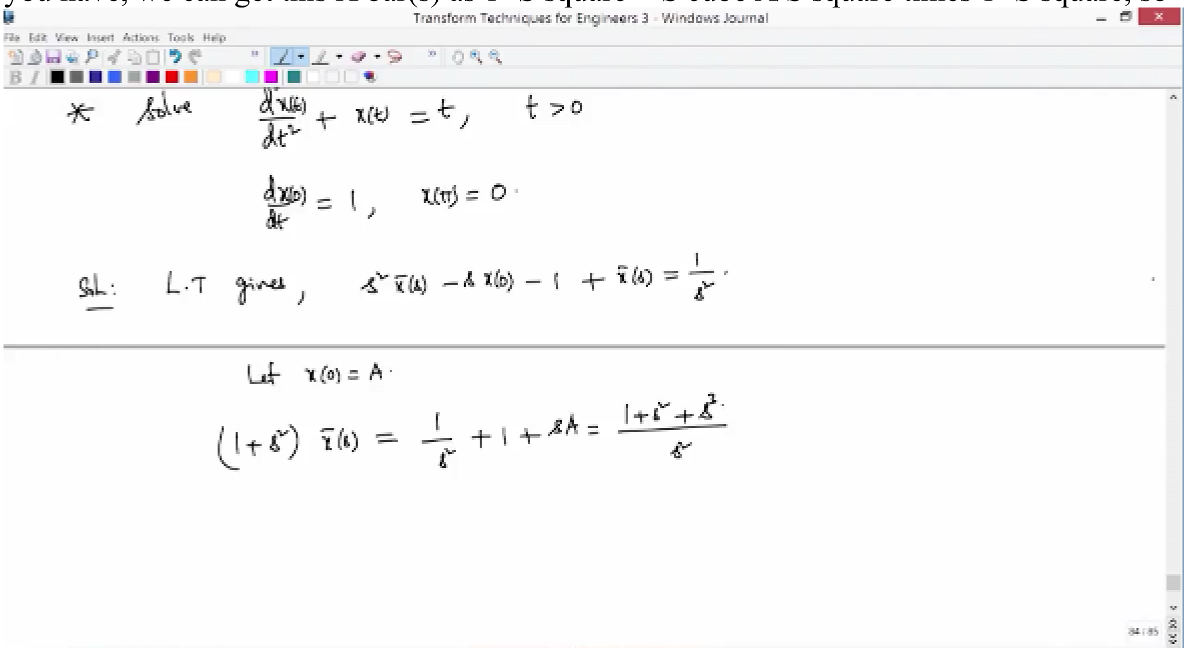
So that's how you make use of those two simple results that I have explained few videos, last few videos, initial and final value theorems they're also called Tauberian theorems, so they're useful in somewhere else but for our purposes these are the place you can use these, those two initial and final value theorems.

Let's move on to solve not only so as I explained you can, you need not be given two initial conditions X(0) and derivative at X = 0 if it is, you can also get some, if your domain is only, if your domain is this and if you need, you can get this boundary condition, if you have a boundary condition like this still you can get this as a solution, okay. For example if you still can get, if your domain is this you're looking for X(t) for all T positive but what is given is X(0) is 0 one condition, and suppose you want to give somewhere else, some other condition, let us say some at 1, then X(1) if it is given say let us say 1, then still as though you know that X dash(0) is some arbitrary constant which you use, so once you use this you will see that X(s) this differential equation will have this arbitrary constant A that along with that once you solve this differential equation that A and C you will get an arbitrary constant of integration you will get, you get 2 arbitrary constants A,C, A you can get it from this and C you can get it by initial value theorem or, basically initial value theorem because you know X(0) is 0 so like that we can do, so let's do some example of such problem.

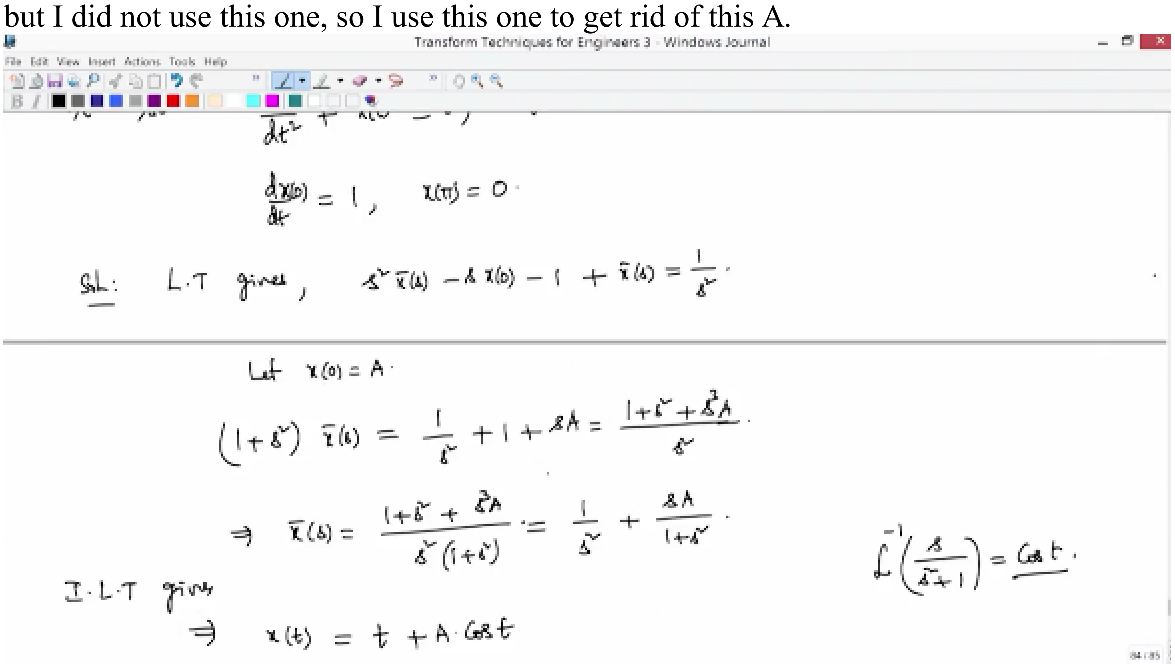


Let me use some simple example solve Y double dash so that is DX, let me use DX okay, DX/DT D square X /DT square + X(t) this is X(t) okay, this is X = T, and this is T positive what is given is X DX/DT (t) = 0 that is given as 1, and X(pi) is given as 0, so suppose you have, so as I explain this you have the T is positive 0 and you have up to infinity, you can get it up to 0 to infinity and what is given is somewhere in between you have at pi, at pi you have the value of your unknown function, so let's try to attempt to get this solution.

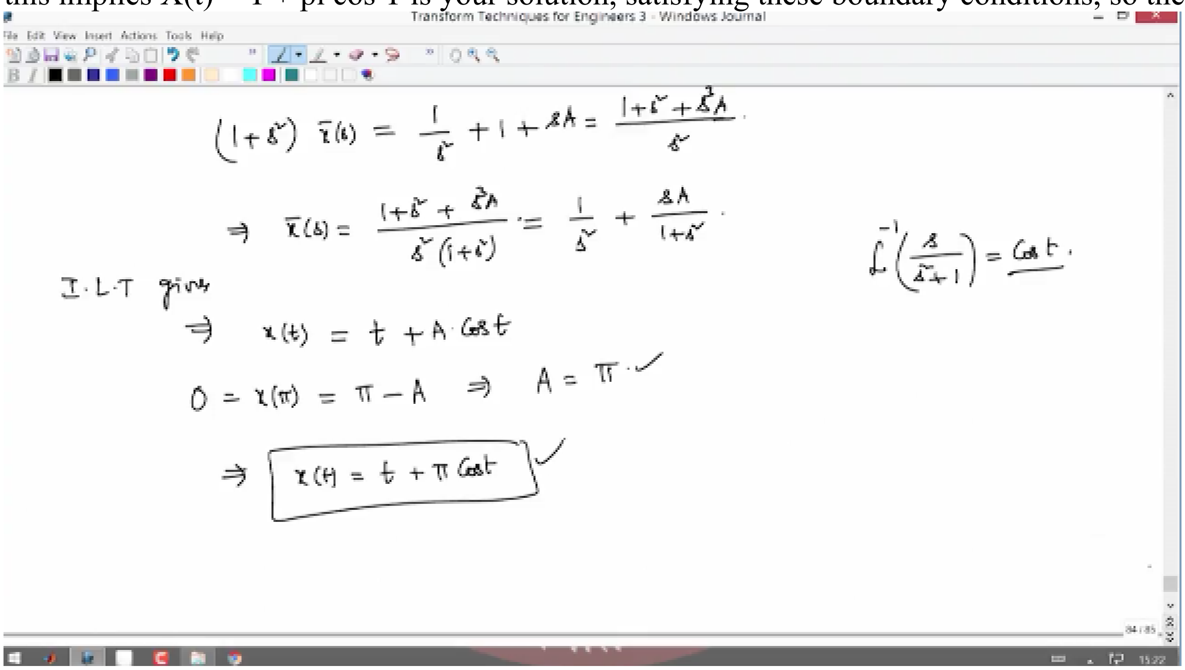
So apply Laplace transform as ,usual Laplace transform gives from gives, what you get you can write this as S square X bar(s) - S times X(0) and then you have -X dash(0) that is given as 1, so let me put it here, and +X bar(s) that is a Laplace transform of this second term, and Laplace transform of T is 1/S square, so I used this condition so far, I have not used this one, so this is still unknown X(0), so let's say this you call it let X(0) is unknown that is A, okay, so you have S square, so you have 1+S square times X bar(s) and then you have a minus is equal to 1/S square + 1+S times A, X(a) is A, so this is actually equal to S square 1+S square + S cube A and you have, we can get this X bar(s) as 1+S square + S cube A/S square times 1+S square, so



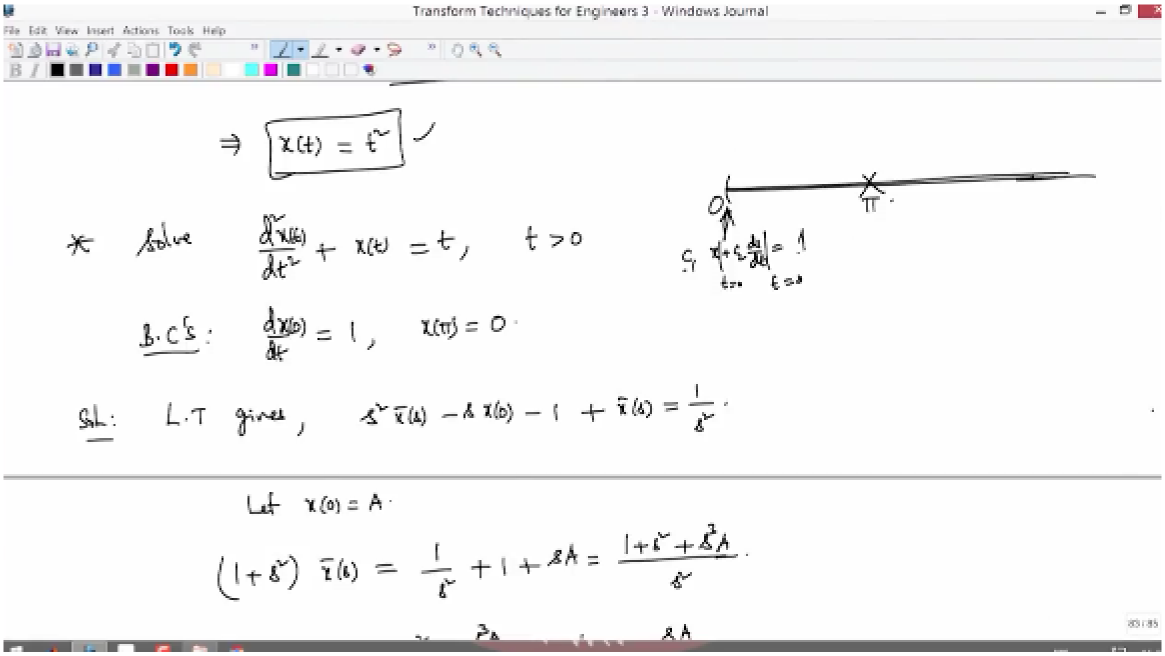
this is, you can get it as 1/S square first term, and second term you can get it as SA/1+S square, okay, so you can easily get your inversion, inverse Laplace transform gives now X(t) and 1/S square is simply T, and this one will be A times, and you have S/1+S square S square +1 that is of cos or sine, cos, right, so cos T, so cos T is S divided by S square + 1 the Laplace inversion of this is cos T, so I've used this, so this is the solution I have derived so far I used only this one but I did not use this one, so I use this one to get rid of this A.



And also you can also use, okay, so initial value we cannot use, initial value theorem we cannot use because X(0) is not known, still it is an unknown, okay. So let me use, so if I use let me see what is the initial value if I use X(0) which is A, which is equal to limit S goes to infinity, S times X bar(s), what is X bar(s)? That is simply 1/S square + SA/1+S square, so if you calculate this that is going to be 0, and here S square by this that is simply A, so A = A you don't gain your A, okay if you apply, so you cannot use this initial value theorem, and but you don't have you have, because you don't have X(0) is 0, so you can use this other condition to get this X pi, so use this X pi which is taken as 1, right, so we have that is 0, you have consider this to be 0 this is equal to pi times + A times cos pi that is -1, so you have minus that makes it A = pi, so this implies X(t) = T + pi cos T is your solution, satisfying these boundary conditions, so these



are your boundary conditions not initial conditions alone, so you have a boundary conditions, boundary conditions you can give because it's a second-order equation you can give up to first order, so you can boundary conditions can be combination of X and DX/DT at T = 0, C1 + C2 can be 0 at T = 0, X(t) at T = 0, so this is the boundary condition it can be here with C1 C2 known, not both of them are 0, if you have both of them 0 there is no boundary condition, so one of them has to be nonzero, so here in this case you have C1 is 0, C2 = 1, and of course this is given as a non-homogeneous boundary condition which we used here, okay.

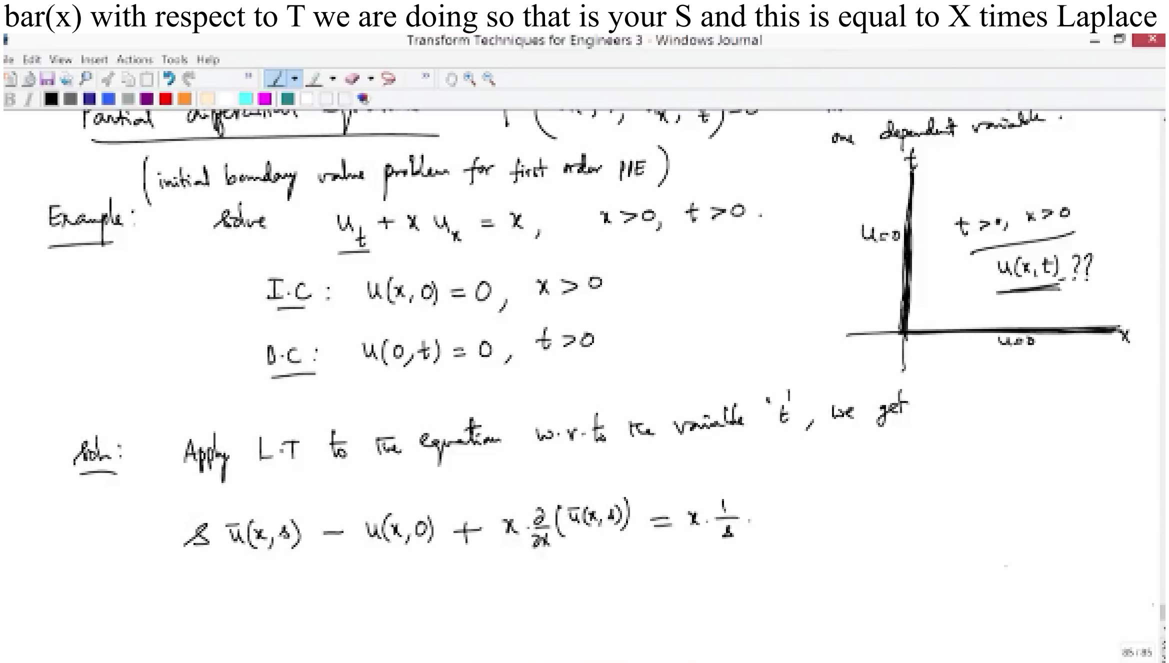


So other places we have similarly X pi = 0, so you have a boundary condition at T = pi that is here, and C2 is 0 and C1 so this is equal to 0, this is the boundary condition C1 = 1 and C2 is 0 in this case at T = pi, so that's how you have provided these boundary conditions and you get the solution just by applying this non-homogeneous second order linear equation with boundary conditions you can derive Laplace transform, by Laplace transform technique you could derive its solution directly okay, so this is how we can solve these differential equations, ordinary differential equations especially, we will get assignments to solve similar problems.

Let's move on to find the solutions of partial differential equations, so we can also use this Laplace transform to solve initial boundary conditions for partial differential equations of first order, so to start with we'll use, we will solve some first-order partial differential equation, one can solve these partial differential equations of course some specific partial PDE's you can solve, first order means you have only, first of all what is a partial differential equation you have a dependent variable of 2 variables, 2 variables 2 independent variables one or more okay, so you have more than one independent variable, you should have more than one independent by XT or independent variables and only one dependent variable, once you have this you have this and you can differentiate with respect to X, with respect to T, so if you have a combination of this if you have relation between them it's called a partial differential equation of order 1, because you have only first order derivatives, if you have if you want to call this partial differential equation of second order general equation is, it involves also UXX, UTT, and UXT up to 2 derivatives, okay, so if you have all this, if you have then you say that this is, this equation is a partial differential equation of second order, so as of now so we'll use the first order so this is the general form of this PDE.

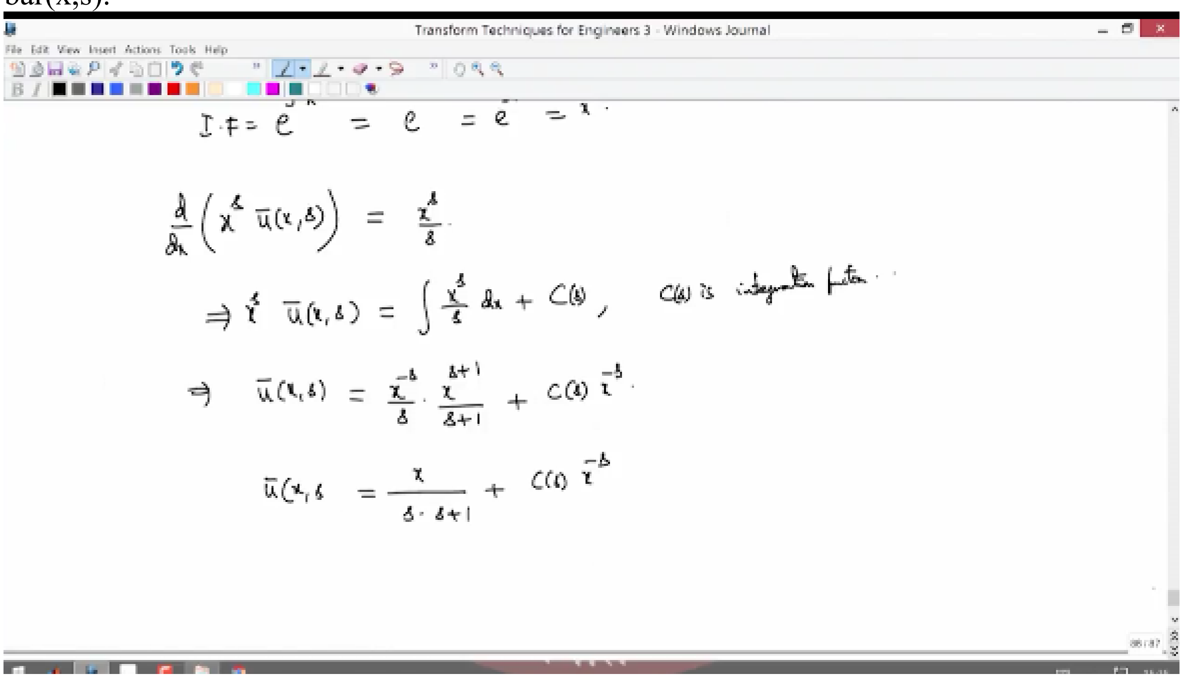
First order equation how do we solve? So what is the domain, so you want to use the Laplace transform one of your domain should be 0 to infinity, so let's do some examples how do we solve this linear equation simple partial differential equations of first order, so we'll solve some initial value problem, initial boundary value problem, okay, we will solve let me use this U is a dependent variable, UT+X times UX = X, domain is X positive, T is also positive, and what is given, so you should have initial so if you look at this as your domain, this is your domain so X and T, so T is your time variable if you think initially, that is an initial condition at T = 0,

because it's only one derivative you should provide U(XT) T = 0 you should provide some boundary condition, let me use this 0, and boundary condition means at X equal to, X is the special domain, that is X = 0 means that is this line, along this line so you have a quarter plane T positive, X positive, and you have on this line you have, on this line T = 0 you have 0, U is 0, on this line you have the boundary condition U(0) X = 0 for all times T that is also let's give it 0, T positive, so here also you have 0, U(0) here and here, so you want to find what is your X(t) here, okay, so what is this? That solving this solution of U that satisfies this differential equation, satisfying this initial condition and this boundary condition, so that is why it's called our initial boundary condition, initial boundary value problem for first order, first order PDE okay.
So how do we solve this? It's solution, we will apply let us say T positive, you can also use for X also, for X also you can use, because X is also from 0 to infinity you can use Laplace transform, but if I try to avoid X because if you use X you have to take the Laplace transform on the right hand side, if I take a Laplace transform with respect to T variable there is no T variable on the right hand side that is one, Laplace transform of 1 is simple 1/X, otherwise here 1/S square you will get, okay, so both ways you can do you can try that way.
So let's apply a Laplace transform to the equation to see that, so if you do this, this is a partial derivative with respect to T, with respect to T I'm doing with respect to the, with respect to the variable T okay, what we get is this one so dou U/dou T that is like DU/DT, what is the Laplace transform of that? That is simply S times U bar of, in the place of X as it is T variable is becoming S variable so I write like this –U(x) at T = 0, okay and that is 0 by this initial condition.
And then plus if you take the Laplace transform of X into UX that is simply X is a constant and dou U/dou X you are doing Laplace transform with respect to T variable so it's nothing to do with this dou dou X that comes out, and you only for the Laplace transform of U that is U bar(x) with respect to T we are doing so that is your S and this is equal to X times Laplace

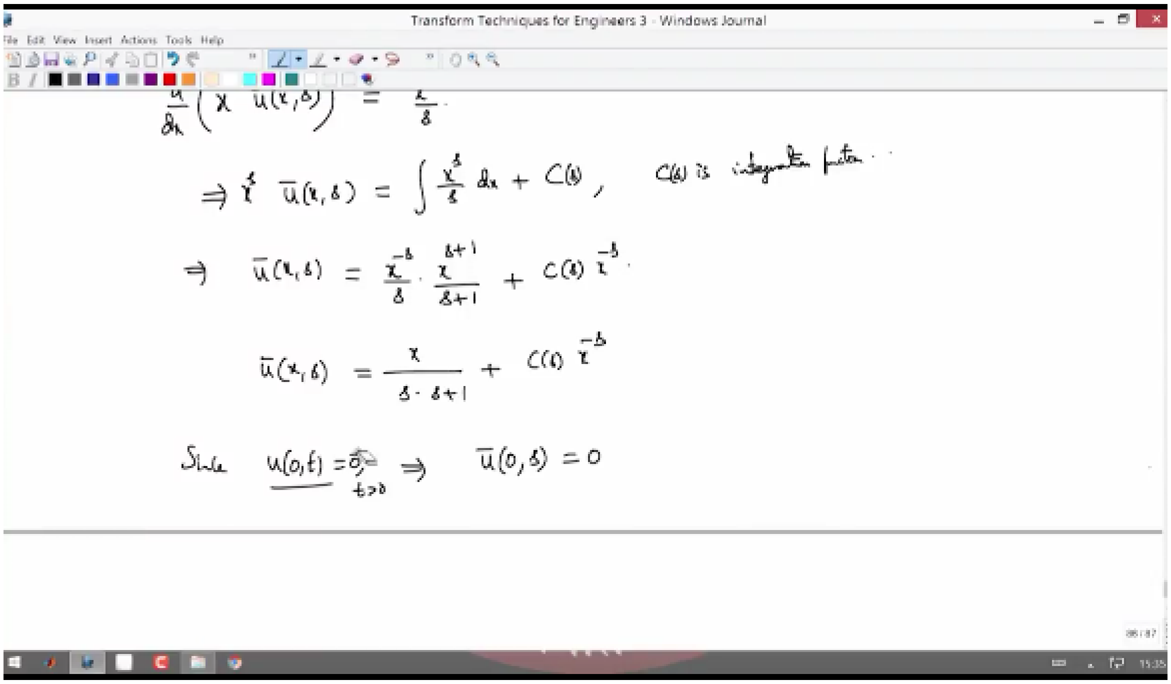


transform of one with respect to S that is 1/S, so this is what we have so far I have used this initial condition, so if you rewrite this as X times dou U bar/dou X + S times U bar = X/S because this is 0 by initial condition, by this condition this is actually 0, okay, so you end up

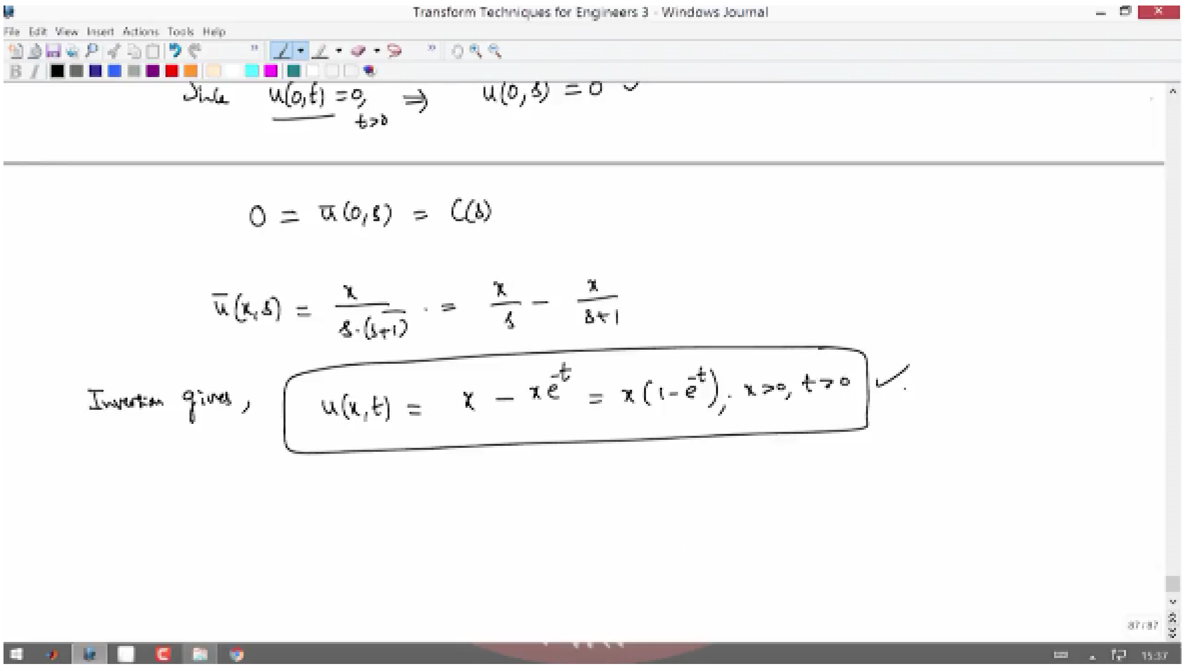
this, so if you divide S/X so this simply 1/S so you can use this is a ordinary differential equation, linear equation with respect to X if you look at it S as though, S is simply some constant, so you have a doing with respect to X, right, so X positive this is your domain of this ODE, ordinary differential equation here, so E power integrating factor is E power integral S/X DX, so you get E power S LNX that becomes X power S E power LN X power S, so you have X power S is your integrating factor, so if you multiply to the equation X power S U bar(x,s) for which if you take the derivative with respect to X, I multiplied both sides of that equation so left hand side becomes this, right hand side X power S/S so you end up getting U bar(x,s) equal to, and of course you have, so if you integrate both sides you get this one X power S.
And then you have integral X power S/S DX + arbitrary constant C, C is integration constant arbitrary integration constant, of course this C is because you're dealing with the partial differential equation there is another variable this can be a function of S okay, arbitrary integration, integrating arbitrary function okay, so it can be a function so this implies U bar(x,s) equal to, this will be X power –S/S times integral X power S derivative is, derivative of this is, what is this, derivative integral X power S DX, simply it's like X power N, so you have X power S+1 divided by S+1, and then +C(s) times this you bring it this side that will become X power –S, so you have this becomes X divided by S into S+1 +C(s) times X power -S is your U bar(x,s).



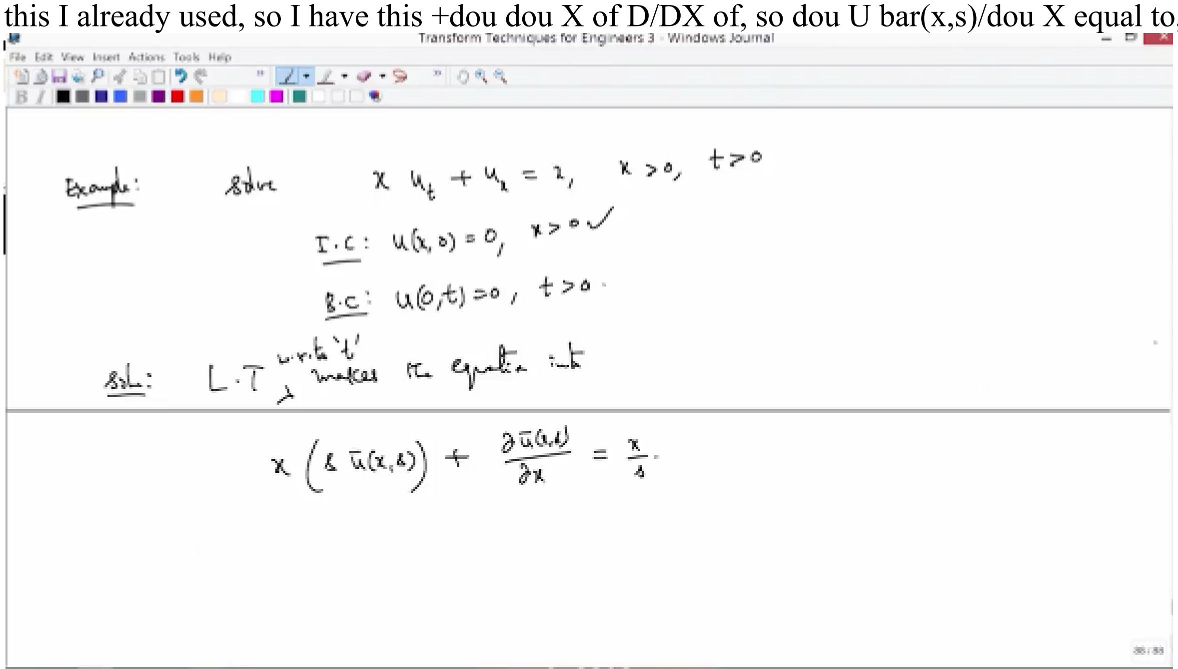
Now we can use your other boundary condition, so we have used the initial condition you know now we can use the boundary condition U(0,t), U(0,t) is your boundary condition, so let me use this boundary condition since U(0,t) is 0 this is your boundary condition, this implies you can apply your Laplace transform with this T variable for T is positive, so you can apply even here so this becomes 0, S, this is also 0, because the Laplace transform of 0 is 0.



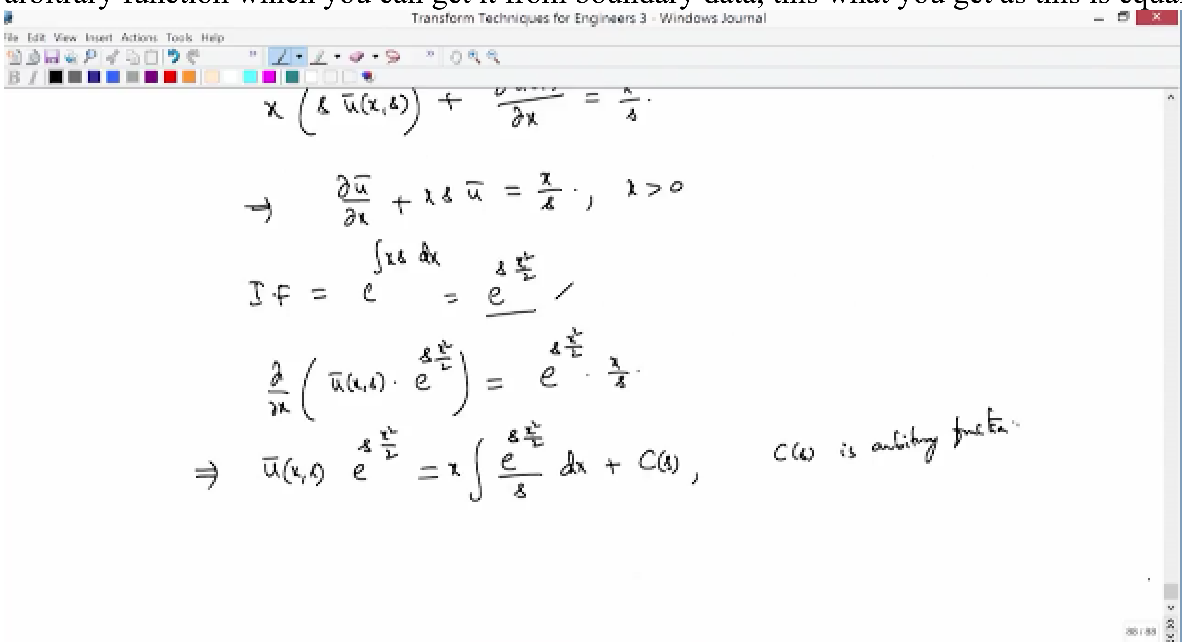
So now I use this here to see that left hand side is 0 which is equal to U bar(0,s) I'm putting X = 0 here, so this is 0 because I put X = 0 you get C(s) times X power that is 0 power –S okay, so 0 power –S you don't know exactly what it is, right, so you can put this, you can apply this as X power S+1 so you can remove this, and you have X power S, here you apply X = 0 so that you have CS is simply 0, so CS is 0 that makes it 0 so you have, what you end up is U bar(x,s) is X divided by S into S+1, so this is nothing but X/S – X/S+1, so you can now invert inversion will give, inversion gives inverse of the Laplace transform gives U(x,t) which is equal to XT - X E power –T, 1/S is actually 1 so you have X-X times E power -T so this is nothing but X times 1 -E power –T, so this is your solution, this is the required solution for X positive and T positive as your solution of your partial differential equation with that initial and boundary data, so this is how we solve this partial differential equation.



Let me use one more example how do we solve some other problem, so let me use the same, simple problems let me use, so for example let me start solve I changed a little bit this equation so I use X here, X UT + UX = X, X is positive, T is positive, and let me use the same initial data that is U(X,0) is 0 initial data, and the boundary data is also same, so let me use 0,T is also 0, for every X positive and here for every T positive, so this if you want to solve, solution is we have just use Laplace transform gives like makes the equation, makes the equation becomes X the equation into, makes into what? You have to differentiate, you have to use the Laplace transform with respect to T, with respect to T makes the equation, makes the equation into, so you have X is a constant here because and then UT is S times U bar(x,s) –U(x,0) that is 0, so this I already used, so I have this +dou dou X of D/DX of, so dou U bar(x,s)/dou X equal to, so

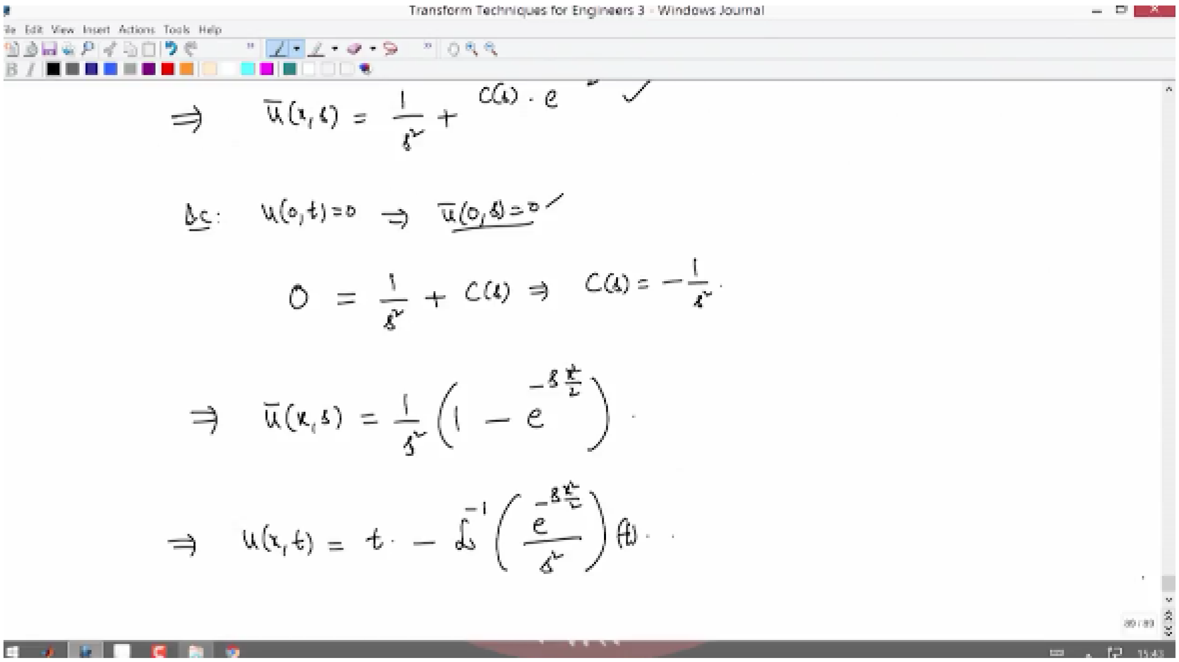


again X/S, because Laplace transform of 1 is S, so you get dou U bar/dou X + XS U bar = X/S, this is your equation, so you can use the same technique so integrating factor you can solve this by integrating factor, so X is positive domain is 0 to infinity integrating factor is E power integral XS DS, XS DX, so S is a constant so E power S times X square/2, okay, so this is your integrating factor, you multiply both sides of your equation the left hand side becomes D/DX of dou/dou X of U bar times(x,s) times E power S X square/2 is equal to E power S X square/2 into X/S, so you integrate both sides you get, you can see that U(x,s) times E power S X square/2 which is equal to integral of X comes out E power S X square/2 /S DX + some constant, arbitrary constant, so it's not an arbitrary constant it's an arbitrary function, C(s) is arbitrary function which you can get it from boundary data, this what you get as this is equal to

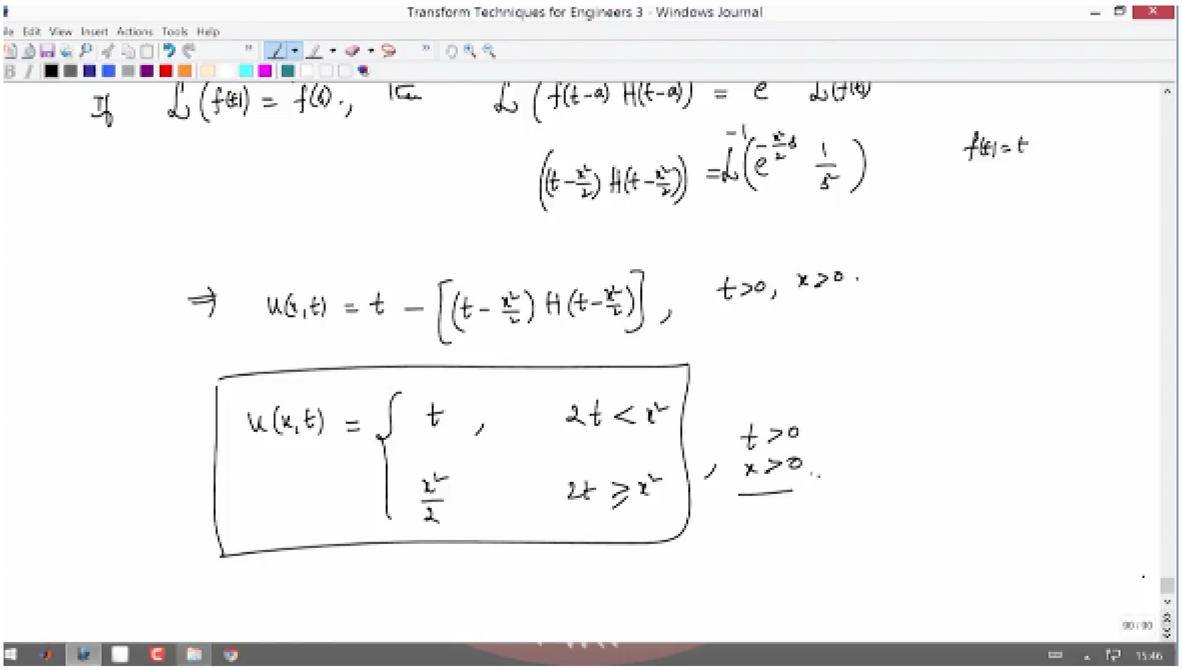


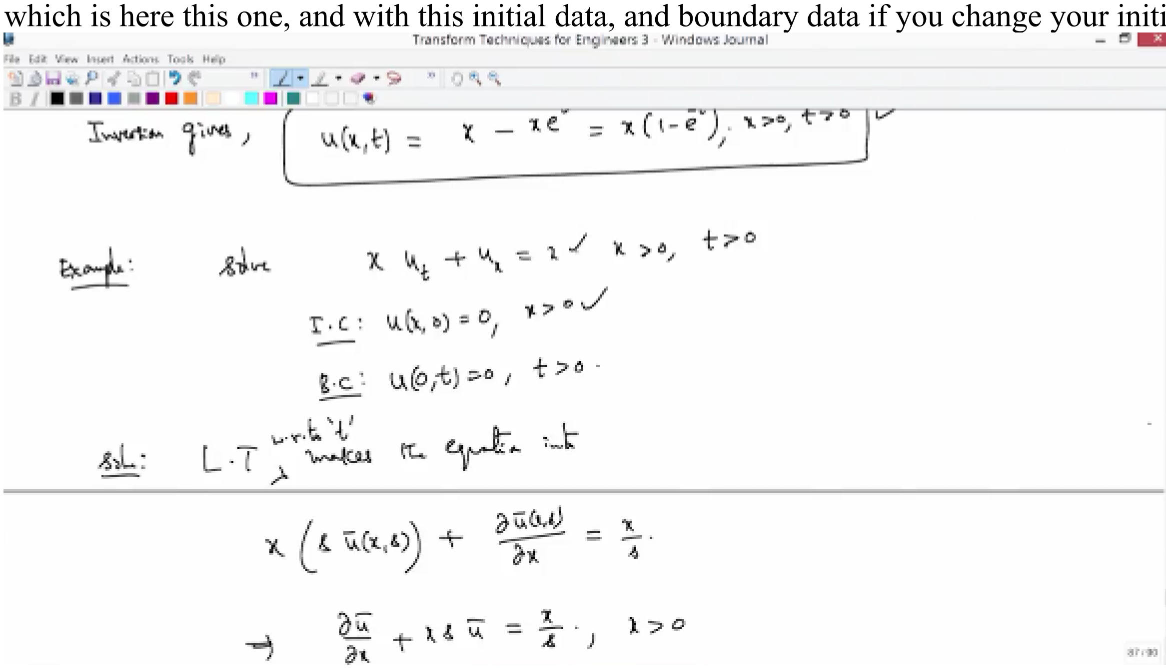
you can easily see that 2X this is simply, integration is straightforward SX square/2 S square right, so you get S square.

If you differentiate this you get with respect to X S/2 times 2X, so 2 2 goes, so you have SX, S S goes, one S so you have X/S so this is exactly what you have, so if you differentiate this you will get this X times this integrand, so this is your integration of this, and then plus you have C(s) so this gives me U bar(x,s) as 1/S square + C(s) times E power -SX square/2, so we'll use the other boundary condition that is a boundary condition is B U(0,t) is 0 so that means U bar(0,s) is 0, so if you use this condition here so you see that left hand side is 0 and you get 1/S square + C(s) put X = 0 that is 1 so this makes it C(s) as -1/S square so this implies, you have what you get is U bar(x,s) is 1/S square times that is 1, C(s) is also 1/S square that is -E power -S X square/2, so this is your Laplace transform of U, so if you invert this you can get your solution, so U(x,t) is T here, and you have a -L inverse of E power -SX square/2 /S square, so that is what is the solution which is a function of T, of course we should find.



What is this Laplace inversion? We need to know what is this Laplace inversion, so Laplace inversion of this, we use the Laplace transform of some one of the property that we have used earlier, Laplace transform of, if Laplace transform of F(t) is F bar(s) then Laplace transform of F(t-a) times heavy side function of T-A which is equal to E power -AS times Laplace transform of F(t), okay, so this is exactly we use here, so let me use so if you bring it here we know that 1/S square is the Laplace transform of T, okay, so this is Laplace transform of T which is 1/S square and this is E power A I use X square/2 into S so this is equal to Laplace transform of, here in the place of F, F of, so this is a Laplace transform 1/S square I write it as Laplace transform of T, so I can write here T-F of, so F is T, F(t) is T, okay, so F(t-a) is T-A, A is X square/2, and this is heavyside function of T-X square/2 this is what you have, you take the inversion here so this makes it 0 both sides you have to take the inversion, so this is Laplace transform of T is 1/S square again so this is exactly what we have here, okay, so if I use this you can see that U(x,t) becomes T-X square/2 times H(t-x square/2), for T positive X positive okay, so if you actually simplify it so this becomes, so this is nothing but if T is less than X square/2 that is 2T is less than X square, this is 0 so you have only T here, and if 2T is bigger than X square or equal to X square if it is 1 then you have T- T+X square/2 that goes so you simply have X square/2, so this is your required solution for T positive and X positive.



So that is a solution of your partial differential equation, first order partial differential equation which is here this one, and with this initial data, and boundary data if you change your initial


and boundary data and if you change the equation you may see the complicacy of its inversion transform maybe little bit complicated, if it's complicated you can use the general inversion where you may have to use the contour integration technique to evaluate the contour integration that integral using Bromwich contour, and then get your inversion finally get a explicit solution for your partial differential equation of first order, with of course boundary and initial data, so this is how we can solve the first order partial differential equations.

A second-order linear equations also can be solved by this Laplace transform technique, so any second-order linear, second order linear partial differential equation you can put it into one of the typical equations such as a wave equation type or heat equation type or Laplace equation type, if you already know if you have done a differential equations course you might have come at, you might have seen how to reduce any second order partial differential equation I put it into one of the canonical forms, one form is it belongs to one class that is a one typical equation that is hyperbolic type equation that is wave equation, another type is a heat equation which is parabolic type equation which is heat equation.

Other one is elliptic type, elliptic type of equations that is a typical equation looks like a Laplace equation, okay, second order term looks like a Laplace equation, so we will see one of so, at least two these typical equations with some initial and boundary data we'll be solving in the next video for certain problems will solve, and then we'll move on to, and we will move on to apply this Laplace transform to solve some integral equations, okay. Thank you so much.

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