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Transform Techniques for Engineers
Solutions of Initial or Boundary Value
Problems for ODEs'
With
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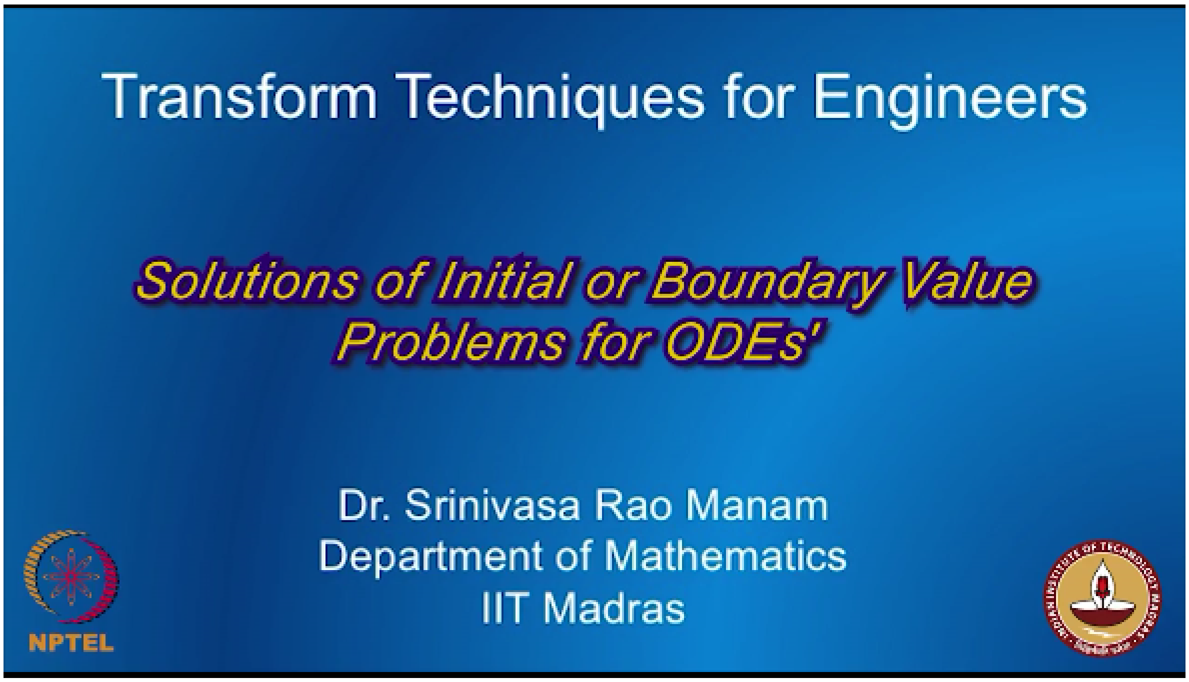
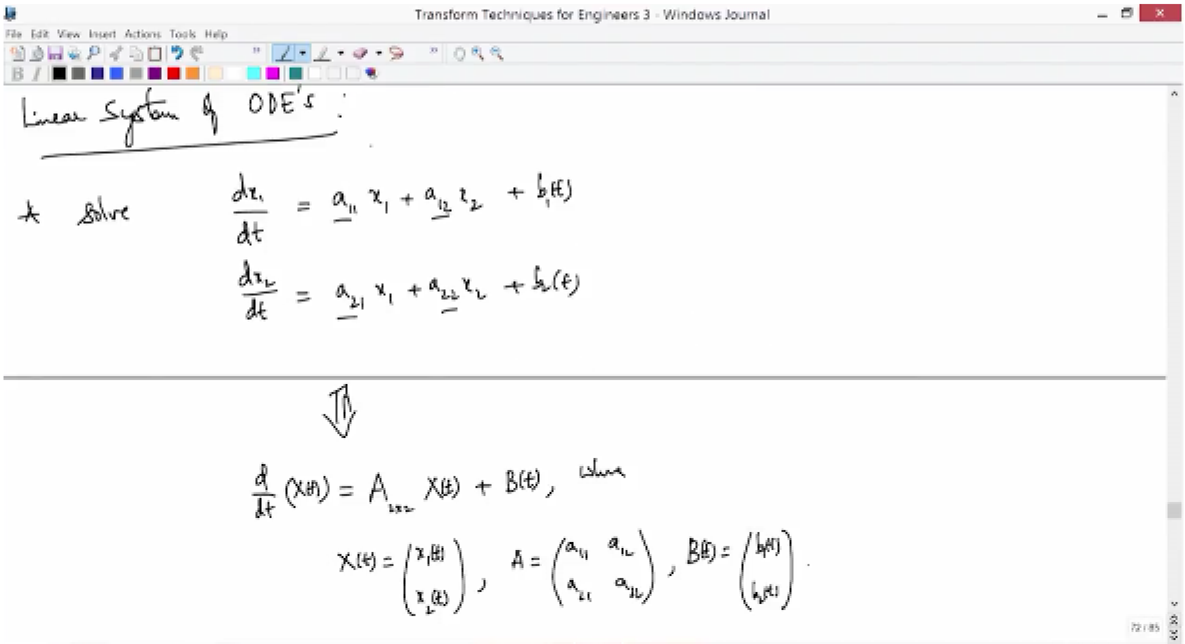
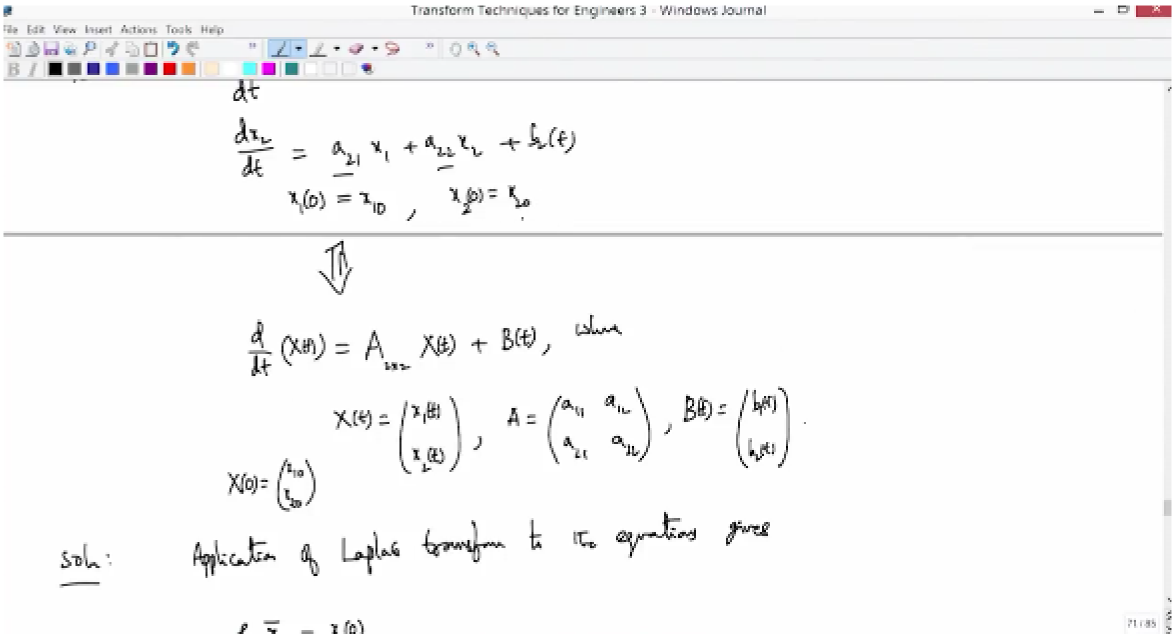


Welcome back, the last video we were discussing about the applications of Laplace transform, started with solving ordinary differential equations, first-order and second-order we have done. In this video we will start with, we will start looking at solving a system of linear equations, so let me do the general, second order we do the second-order linear system, so let me do take the general system, so how do we solve this general system, system of equations, system linear system, system of ODE's, so how do we solve this? So let me solve this two-by-two system DX1, X1 and X2 are two variables, function of T let's see A11, X1 + A12 X2 + B1(t), another system is, another equation is DX2/D2, so X1 X2 are two independent variables and T is the independent variable, so you have 2 1 X1 + 2 2 X2 + B2(t), so A1 A2 all these are constants, so we can put this in an equivalent form that is D/DT of X(t) which is equal to matrix is just, let's say A is a matrix 2/2 matrix, come with these constants and you have X(t) + B(t), so where A is, where X(t) is a vector X1(t) and X2(t) and then A is matrix, A11 A12 A21 A22, and then B(t) is a vector that is B1(t) B2(t) so this is a system of two linear equations, first order equation for vector, first order linear, ordinary differential equation for a vector.



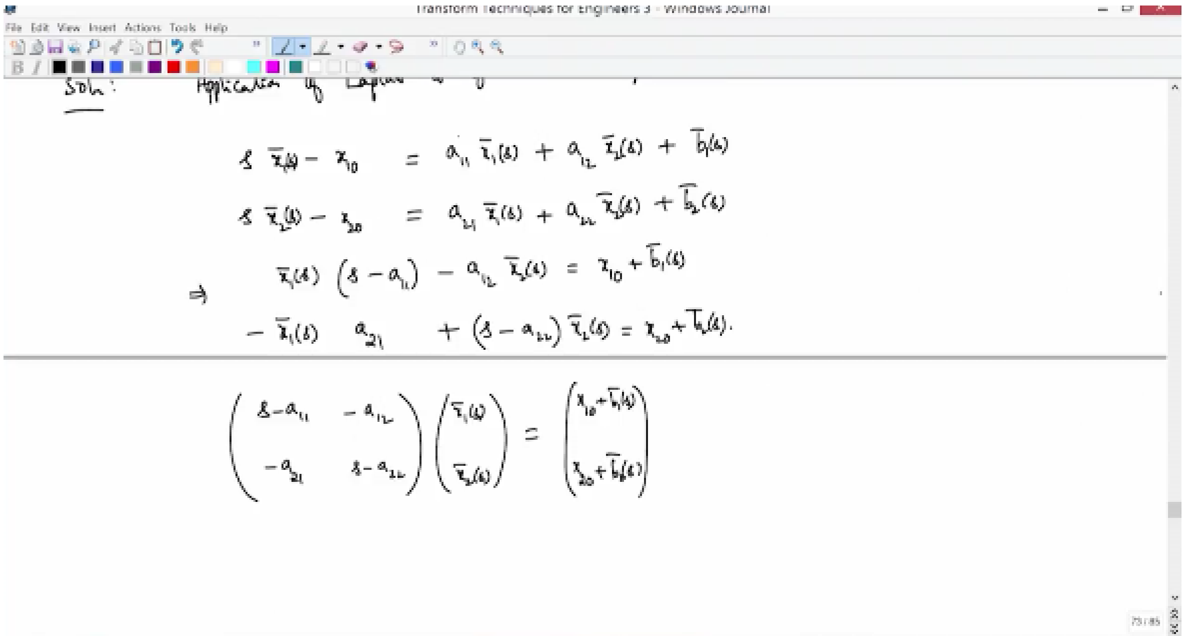
So how do we solve this? Because the T is positive you have you can apply Laplace transform, so solution procedure is to, if this is general equation, general system this is 2/2 system, so procedure is same for any order, any N/N system also, any other, nth order system if you consider that means you consider N linearly independent, and N coupled ordinary differential equations for X1 X2 XN neither procedure is same, so let me outline the procedure before we solved some problems.

Let's apply Laplace transform, application of Laplace transform to the equations, to the equations give, application gives so what are those, you apply these two equations, you apply Laplace transform, so DX1/DT that is if you apply that is S X1 bar - X1(0), so at 1 X0 it's not known, so this is the system with the initial value you have to provide that means X1(0) is let's say let me give this as X10 X2(0) is X20, so here that is so the system is this with initial, this is the equation and the same initial values if you consider X(0) is actually the vector that is X10, X20, so these are the initial values which you have here separately here as a vector, initial



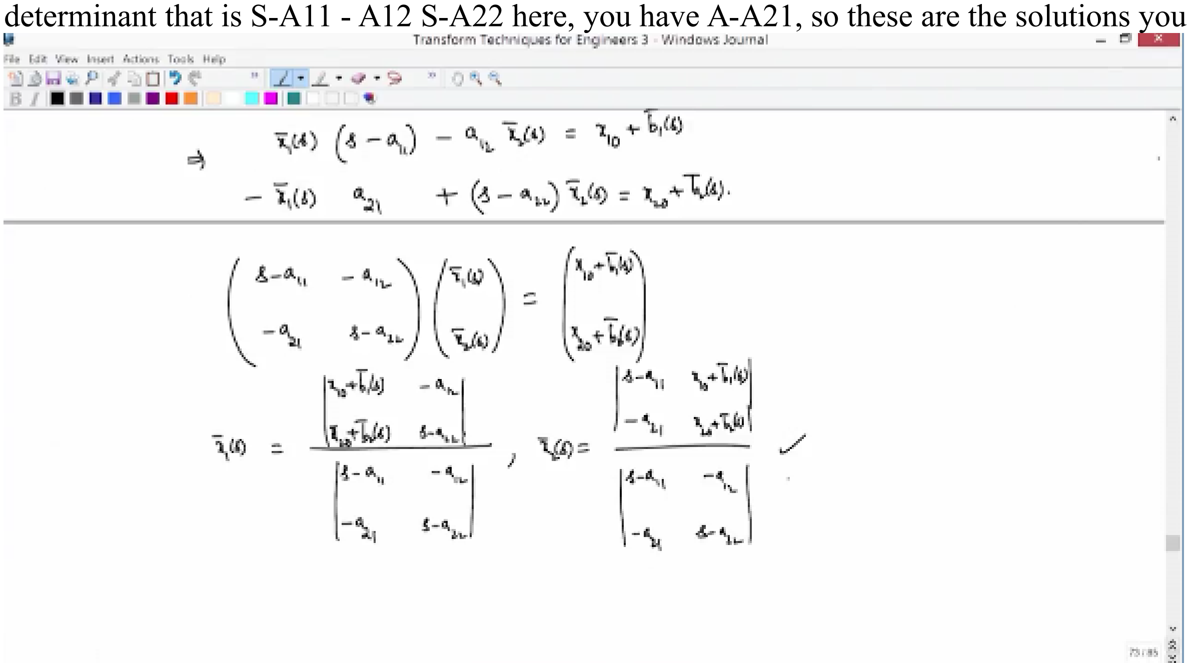
value, initial value vector, so these are the initial values, initial, okay, initial values, so those initial values you can use here when you applied for the first equation DX1/DT for which Laplace transform of this which is equal to A11 X1 bar that is a Laplace transform of this, so these are functions of S, and then +A12 you don't apply, so you apply event X2(t) you apply Laplace transform that becomes X2 bar(s) + B1 bar(s).

Similarly you apply to the other equation so you get instead of this X2 bar(s) - you have X1(0) is X10, so let me replace this as X10, if you do it here this is X20 this is equal to A21 X1 bar(s) + A22 X2 bar(s) + B2 bar(s), so this is how I applied and I used the initial values okay, so this gives me X1 bar(s) you write it as a system, so this is a system for unknowns X1 bar(s) and X2 bar(s), so if I put it together as coefficients, this coefficient of X1 bar(s) is S-A11 and you bring this on this side as X2 A12 X2 bar(s) = X10 and you have this, you bring this X10 here +B1 bar(s) that's the right hand side, other one is X1 bar(s) you bring it this side so you have A21 you bring this take this that side so that will give you minus or rather let me write this, you bring this side so you get this one minus plus here S-A22 times X2 bar(s) this is equal to X20 + B2 bar(s), so this is like a system S-A11 -A12 -A21 S-A22 this is the system matrix and you have X1 bar(s) these are the unknowns, X2 bar(s) and this is the right hand side that is known that is X10 + B1 bar(s). You know what is B1(t), so you know it's a Laplace transform, so

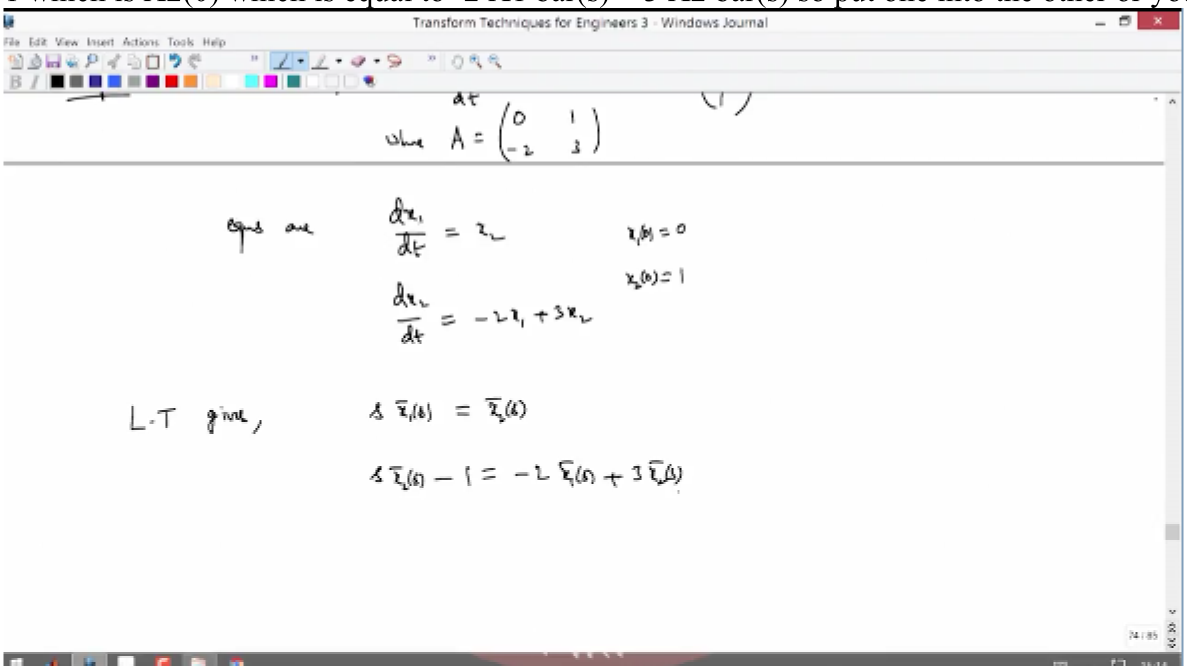


X2(0) + B2 bar(s) so this vector is known, so it's like you're looking for this so you can get the solutions easily and I write directly using this method X1 bar(s) and X2 bar(s) is I think Cramer's rule that goes like this, so this will be like, so let me write if I use that directly what you do is X1 bar I write directly this is the denominator you have determinant of this matrix that is determinant of S-A11 -A12 -A21 S-A22, and here the same determinant you consider but replace first column with right side column, so that is a first column I'm replacing with X10 + B1 bar(s), X20 + B2 bar(s), and this other column as it is, so that is M12 S-A22 so this determinants if you consider that is exactly the solution X1 bar(2).

Similarly X2 bar(s) you can get this as, first matrix is a first column as it is that is S-A11 - A21 and this will be X10 + B1 bar(s) X20 + B2 bar(s) a second one, and this determinant is a matrix determinant that is S-A11 - A12 S-A22 here, you have A-A21, so these are the solutions you

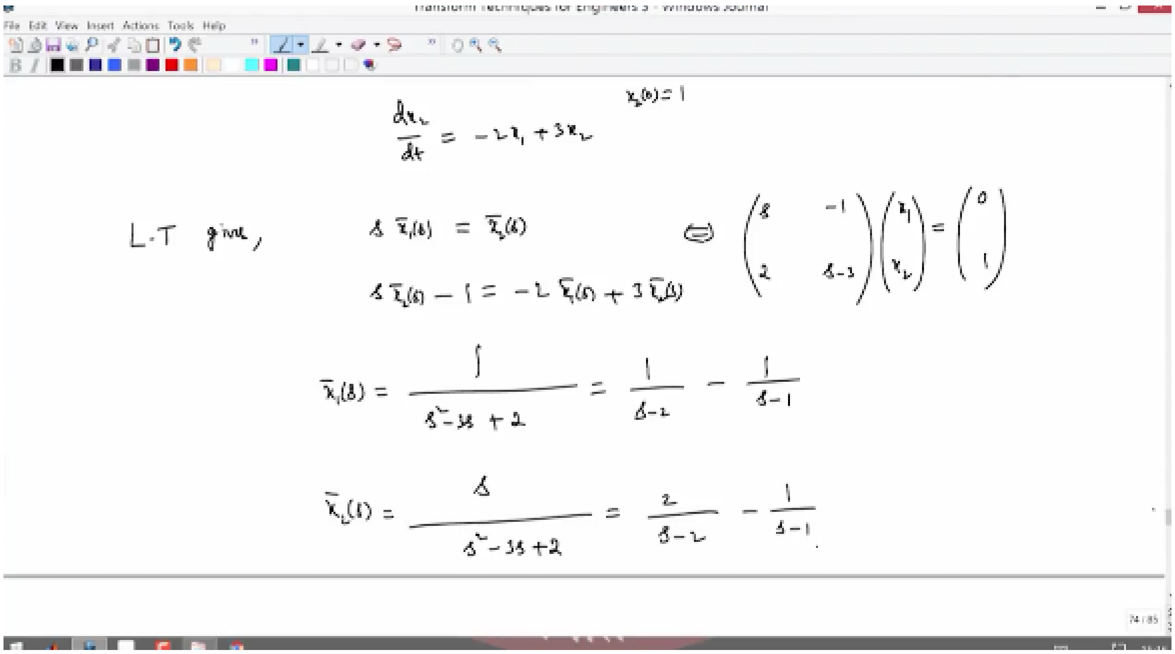


can get like this and you invert it, inversion gives X1(t) and X2(t) that is the solution you're looking for, okay, so let me do some example, example is let me solve some system of equations, solving solution of the system DX/DT which is equal to AX with X(0) = 0 1, and then where A is matrix 0 1 -2 3, so how do we solve this, let me use, you can use directly Laplace transforms if you use directly you can get first equation is you write equations, the equations are DX1/DT, so X is X(t) which is X1(t) and X2(t), so X1(t) which is equal to this is X2 and DX2/DT as -2X1 + 3X2, so these are the two equations you have, and the initial conditions are X10 is 0, and X2(0) is 1, apply the Laplace transform, Laplace transform gives S X1 bar(s) - X1(0) that is 0 so I don't use 0 which is equal to X2 bar(s), other one is S X2 bar(s) - 1 which is X2(0) which is equal to -2 X1 bar(s) + 3 X2 bar(s) so put one into the other or you

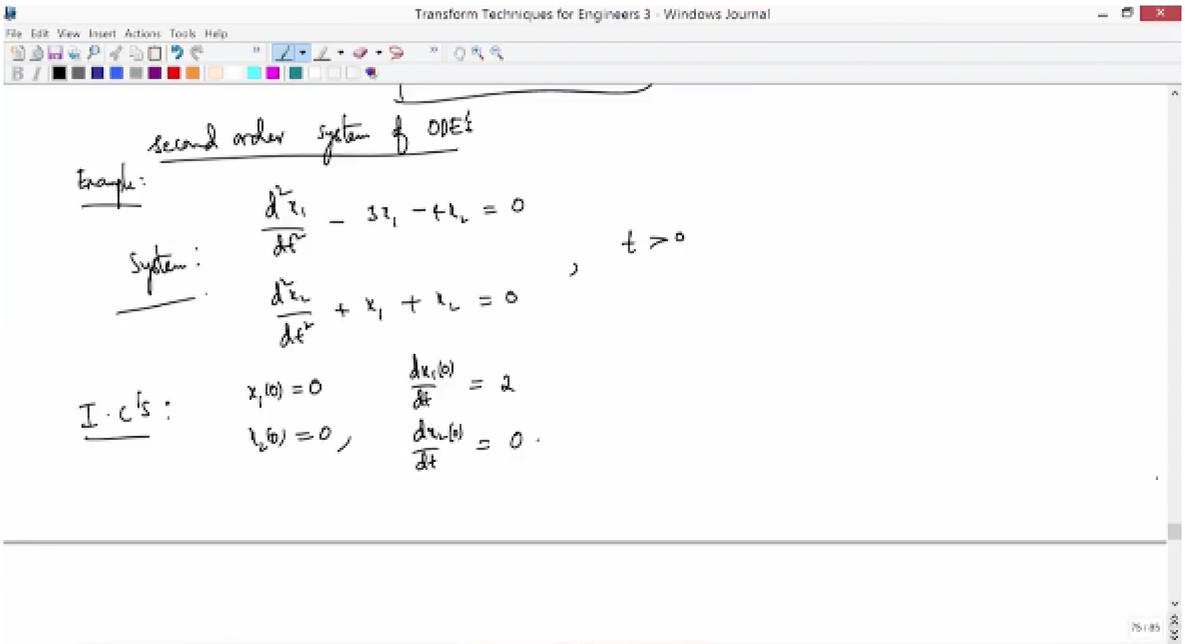


solve directly, so you can get X1 bar(s) as if you try to solve these two equations, so this is equivalent to getting S0 S-1 times here, X1 is bring it this side so we get 2, and this is S-3 of X1 X2 this is the system, so here you get 0, and here you get 1.

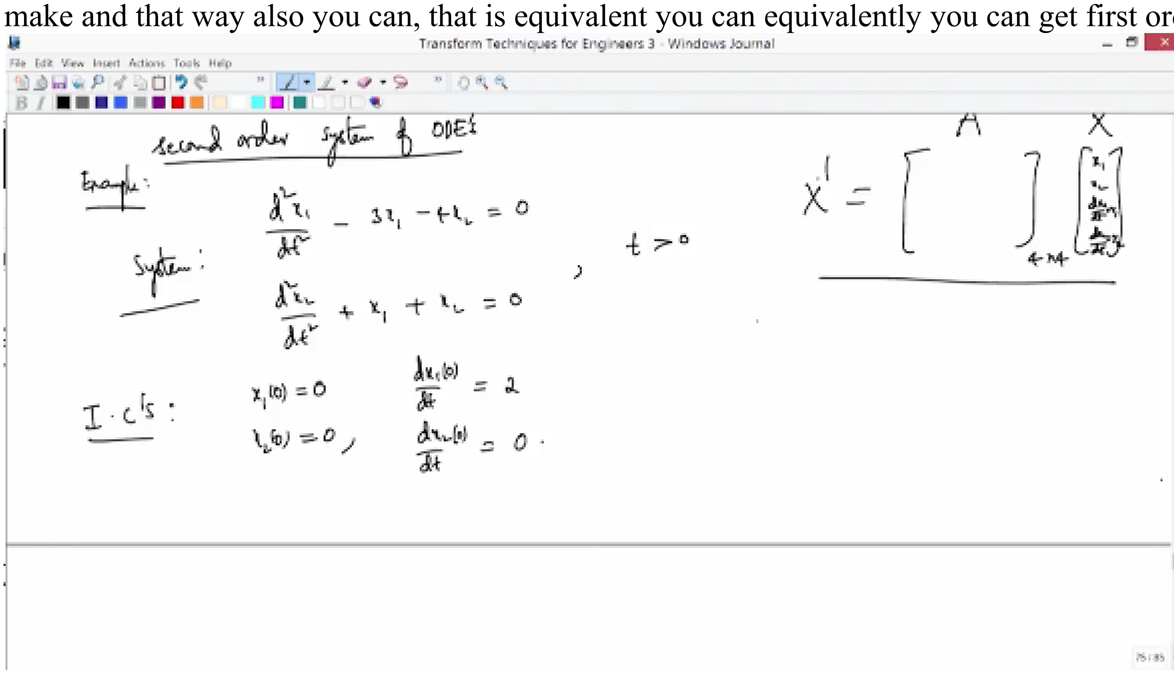
So I consider now homogeneous system here because there is no non homogeneity here, so in the general case you have the non-homogeneous term that is B1 B2 that is B(t) is now 0 here, so follow the same procedure and get this so you can get this as X1 X2 as determinant if you look at this determinant S square -3S + 2 and then here the determinant of 0 1 you put it here and -1 S-3 so that makes it 1, very simply 1 here so you get this one, and X2 bar(s) is again so denominator is same and the numerator the determinant of replace 0 1 here and this you keep as it is so you have simply S, that is simply S of that, so if you apply this use the partial fractions and write this as S-2 -1/S-1 will give me this, this one we can write 2 divided by S-2, -1 divided



by S-1 so that I get these two are same, okay, so inverse Laplace transform gives X1 of the solution, 1/S-2 is E power 2T and then -E power, and the X2(t) is 2 times E power 2t -E power T, so these are the two solutions which you can, these are your systems, so this is the required solution, solution satisfy the initial value here, okay, so this is how you solve the system, you can also follow the same procedure and try to solve system of second-order equations, for example let me do this example, one more example you can also follow the same system second order system of ODE's, follow the same procedure and you can work out, for example you have let me use D square X1/DT square - 3X1 -4X2 = 0, D square X2/DT square and then you have +X1 + X2 = 0, so this is the system, this is valid for T positive X1 and X2 are the unknowns, X1(t) X2(t) are unknowns, you need to know its initial value because it's the second order you need to provide X1(2), X2(t) at 0 so let me give this initial values at 0, these are 0 and this is 1 initial values and you also have DX1/DT at 0, which is let me give as 2, and similarly DX2/DT at T = 0 this is let me give as 0, so these are the initial values, initial conditions and these are the system.



So you can reduce this also as a system of 4 x 4, it's like 4 x 4 system you can convert with the unknowns X1 X2 X1 dash, that means DX1/DT and DX2/DT, so with these unknowns you can make a system of first order system, okay, so this has your new X, this is your A, there's a procedure to make that, okay, so assume X1 X2 and you use this as your new variable some Y1, this you knew is new variable Y2 and so that Y1 dash, Y1 dash is from the equation Y2 dash from the equation, but from X1 dash is nothing but Y1 so that gives the matrix, so X dash is equal to you can write like this way, so 4 x 4 system because is a 2 x 2, second to system of equations of second order so you have four equations, four equations of first order you can make and that way also you can, that is equivalent you can equivalently you can get first order

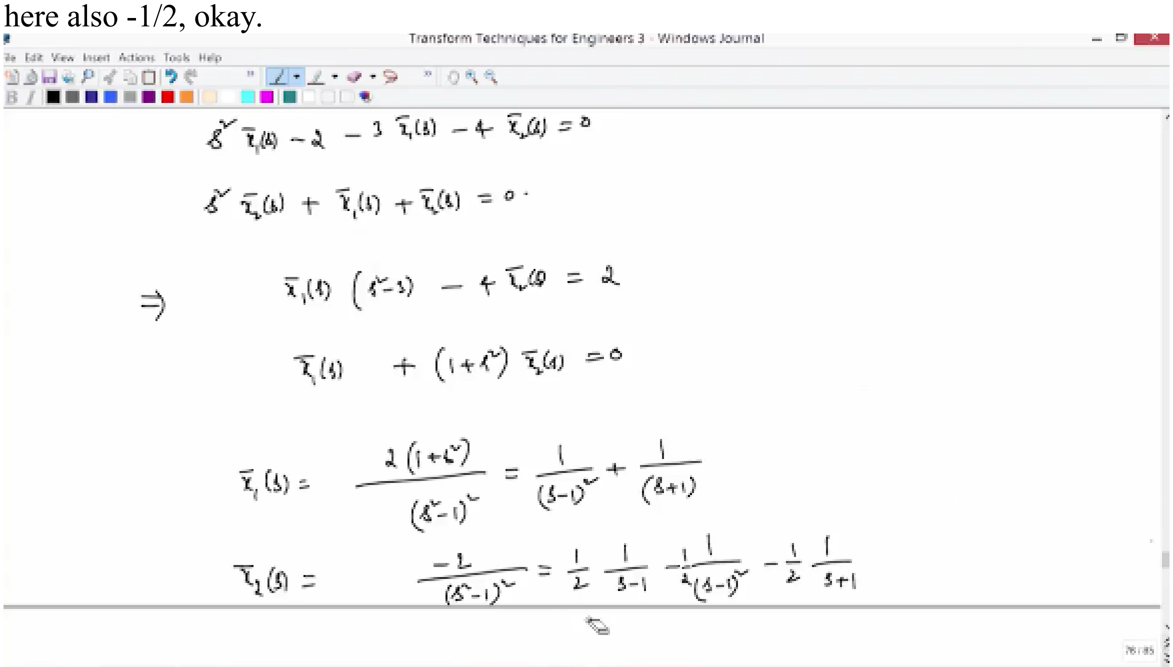


4 x 4 system, 4 x 4 coupled equations, otherwise you can directly apply anyway T positive we can apply Laplace transform, the solution is by Laplace transform apply Laplace transform to the equations, we get what we get is so you if you apply the second order derivative you get S

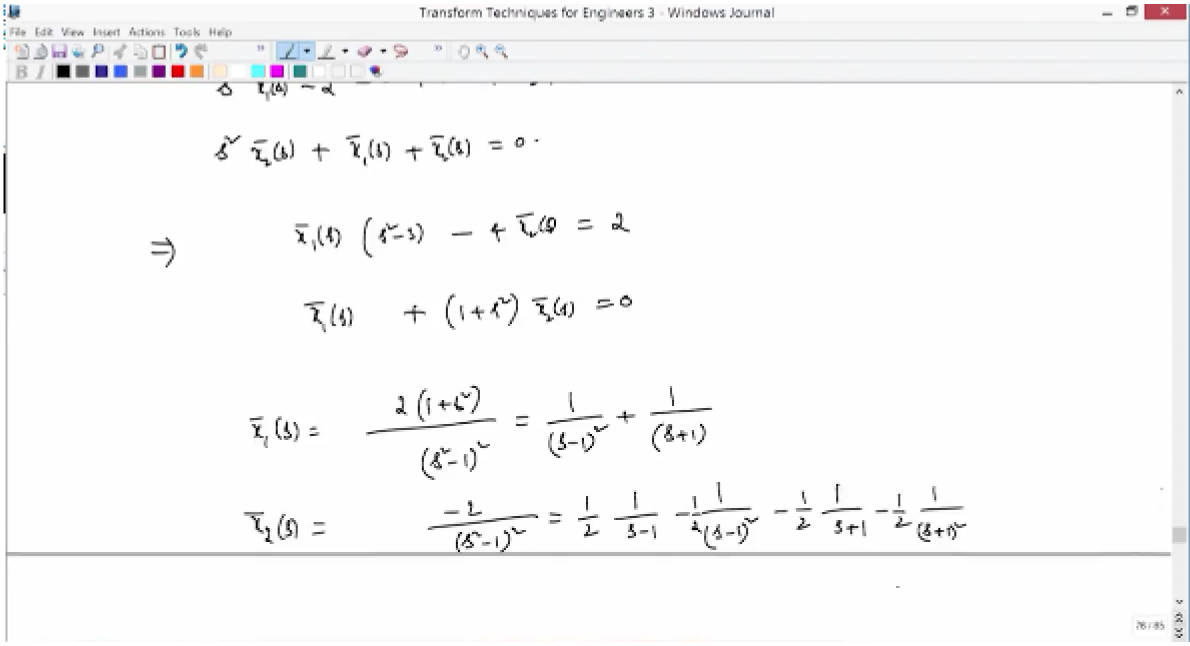
square X1 bar(s) - S times X1(0) and then -X1 dash(0) so that is DX1 by so basically I have to write DX1/DT (0) so these are the things you have to use.

So because of this X10 is 0 and DX1(0) that is 2 so I will replace this here 2 and here this is 0 so that makes it -2 here, which is A+ equal to 3X1 - 4X2, 3X1 bar(s) - 4X2 bar(s), this is equal to 0, so this is one equation other equation also you can apply the same procedure for X2 so you have S square X2 bar(s) - X2(0) S times X2 bar(0) that is also 0 so I don't write this here, and then X2 at DX2(0) DX2/DT (0) is, that is also 0 so you don't have any contribution here so you have +X1 bar(s) + X2 bar(s) which is equal to 0, so if you write them together so as unknowns X1 bar(s) is S square -3, and then X2 bar is 4 times X2 bar(s) which is equal to 2, and here X1 bar(s) + 1 + S square times X2 bar(s) which is equal to 0, so you can get your X1 bar(s) as determinant of this one which is S square -3, -4, 1, 1+S square so this is, if you actually calculate it's going to become what you get is, I write directly as S square -1 whole square at the numerator so you have 2 0 here and here - 4 1+S square, so you have 2 in 2 times 1+S square, so this is 2 times 1+S square, so this is my X1 bar(s) other one is X2 bar(s) denominator is same S square -1 whole square, and numerator this is a derivative, this is a determinant of 2 0 you put it here in the second column and you write the first column as it is that is S square -3 + 1, so you have -2 comes up, okay.

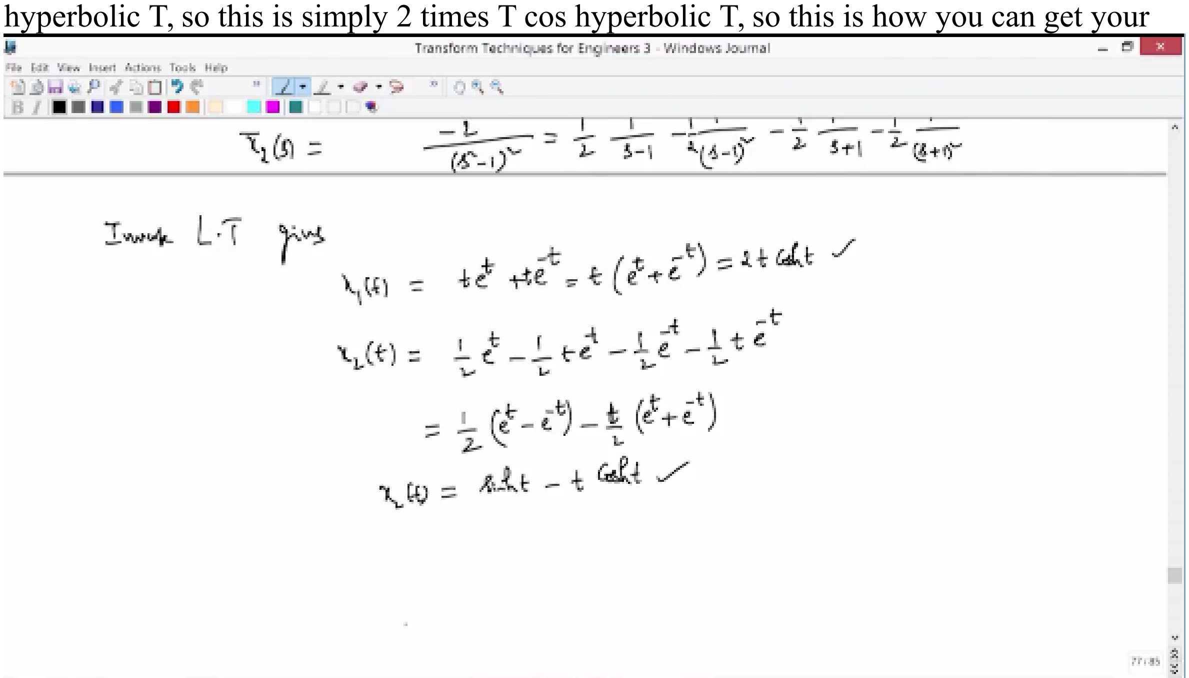
So this if you make partial fractions, write directly partial fractions, you can make a partial fractions here with S-1 whole square and S-1 S-1 whole square, and S+1 S+1 whole square, so with this if you make the partial fractions you end up getting the first one as 1/S-1 that is coefficient is 0, and square term will be there, and then similarly you can get this S+1 coefficient is 0, but 1/S+1 whole square coefficient is 1, so this sum is actually same as this and this side you will get all the terms that contribution, so 1/2 of 1/S-1 so the coefficient of 1/S-1 is 1/2, and also other one is also -1/2 is the other coefficient of 1/S-, no this is, what you get is -1 as S-1 whole square, and then -1/2 is 1/S+1 coefficient and then you have I think this is -1/2, here also -1/2, okay.



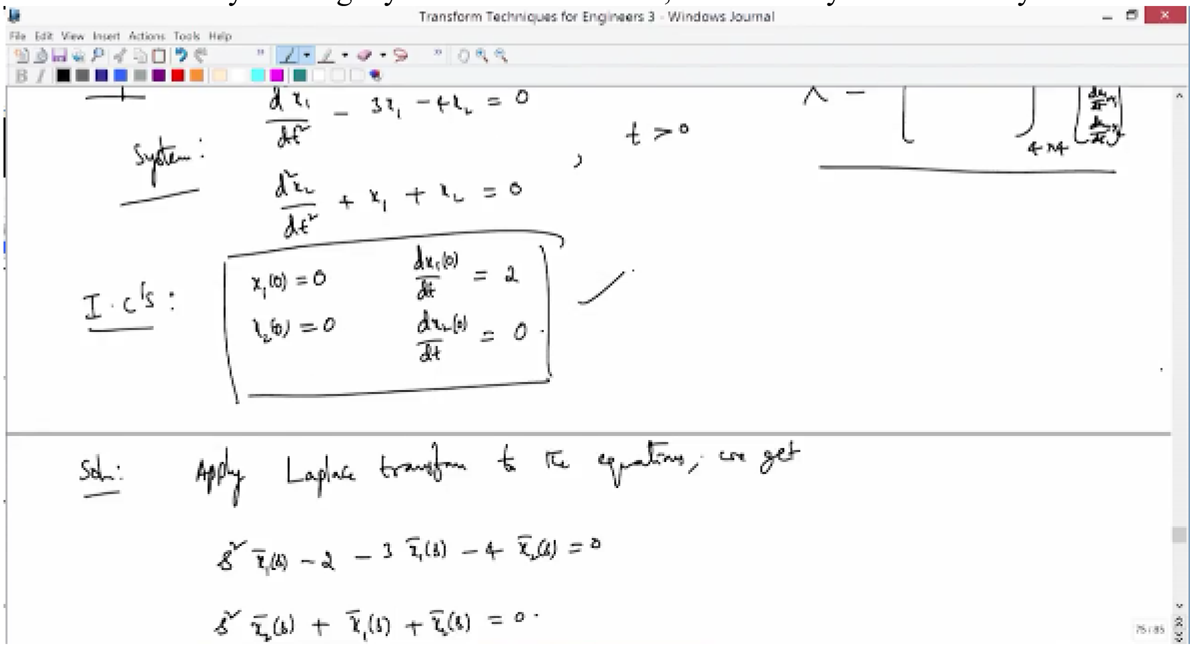
And then last one that is coefficient of 1/S+1 whole square which you will see as -1/2 here, so



this is how you factorize partial fractions you can get and the inverse Laplace transform, inverse Laplace transform gives the solution X1(t) as 1/S-1 square that is E power T because of square T into E power T, and then other one is E power -T because of S+1, this is also square so you have T times E power –T, so you can see this as E power T + E power -T as a solution, X2(t) this is also you can get 1/2 times E power T -1/2 times T power T -1/2 times the third term will give E power -T again -1/2 times T times E power –T, so if you simplify you have 1/2 is common, E power T -E power –T, and then -1/2 T/2 times E power T + E power –T, so this is exactly your solution X2(t), so this is how you can get your solutions.
So this you can also write this as sine hyperbolic T and this is cos T times -T times cos hyperbolic T, so this is simply 2 times T cos hyperbolic T, so this is how you can get your



solutions of your equation that satisfies these initial conditions all these three, okay, all the four initial conditions you can get your solution like this, this is how you solve the system of linear



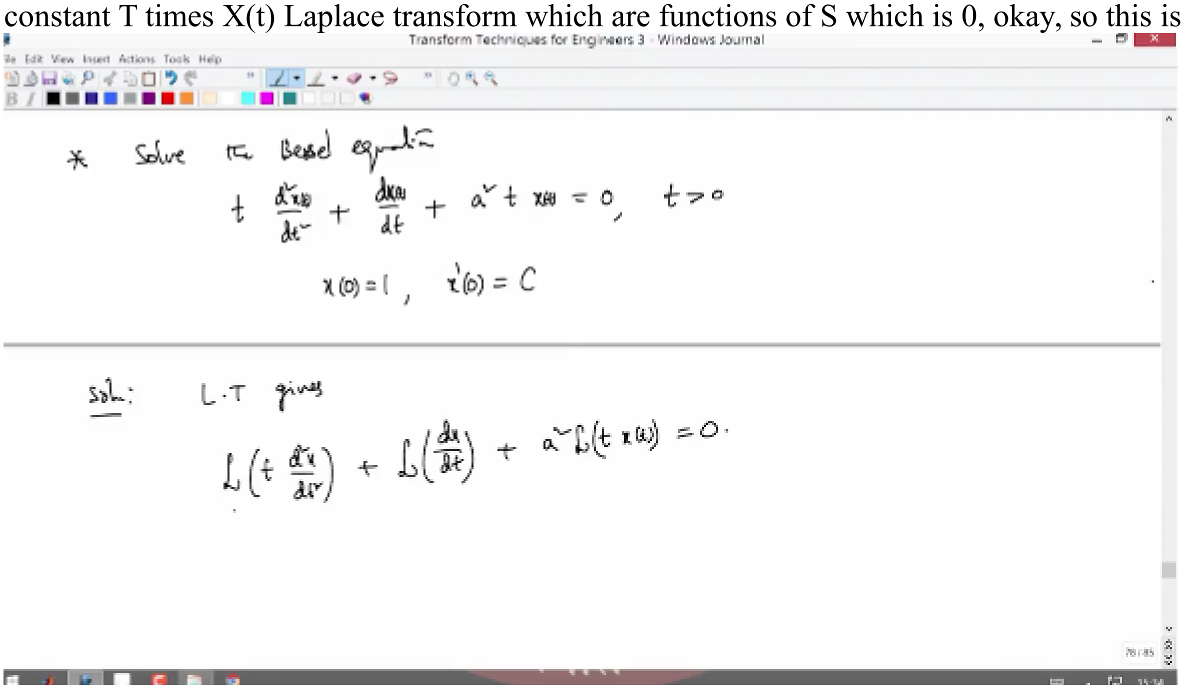
equations, linear ordinary differential equations whether it's a 2 x 2 system or any system, I mean 2 system, 2 equation system or N equation system the procedure is same though it may be lengthy to use the Laplace transform try to calculations may be lengthy but the procedure is same, also a second-order, higher-order you can follow the same procedure, because you know a derivative, any nth derivative you can apply the Laplace transform you know the procedure, use all the initial conditions and then simply solve the linear system and invert it, and get the solution.

These differential equations linear first-order and a linear the second-order equations you will come across in many applications, for example when you look at a harmonic oscillator, model for harmonic oscillator if you look at it and if you put it in a free medium there is no resistance, if you don't consider the resistance you will have a second-order equation and with resistance you will have one more term first order term you can add to the second-order equation, second order differential equation and of course initial conditions you provide and basically whatever the system, whatever equations we have solved you end up for this harmonic oscillator in a free or resistance medium so that you can, once you solve it you see that depending on the, by forcing is so if you have a forcing term in your system in your harmonic oscillator in the model, because of the forcing if it's oscillator is moving and if it's done by, the forcing is done by a harmonic oscillator initial, that's a forcing is a harmonic then you can get some kind of resonance, resonance can be overlooked by a damping term that is by including the resistance medium, so if you keep the resistance in a system you can avoid that, if you keep the resistance in the system, the harmonic oscillator system you can avoid a resonant frequency, so these things and there's basic physics you come across these kind of problems, you solve the second-order linear equations, if you are electrical student if you see current and charge in electric circuit that is you will see that model is, the model equation is a first-order linear ODE that you can solve by Laplace transform, and also not only just a circuit you can have a network of the circuit so that you have a couple system of equations, first-order equations that is where you come across this kind of differential equations, ordinary differential equations which you can solve by Laplace transform with this initial data, okay.

So these are the applications you find in these physics and electrical engineering for the ordinary differential equations, and let me solve before we close this ODE we can solve similar ordinary differential equation with the initial conditions or boundary conditions, you can solve many of these problems, you can solve by a Laplace transform technique, so let me solve some special equations for example Bessel equations also you can solve which is a second order, the procedure is same but we will see how we use initial value theorem and final value theorem in the process of solving this second-order differential equation, ordinary differential equation.

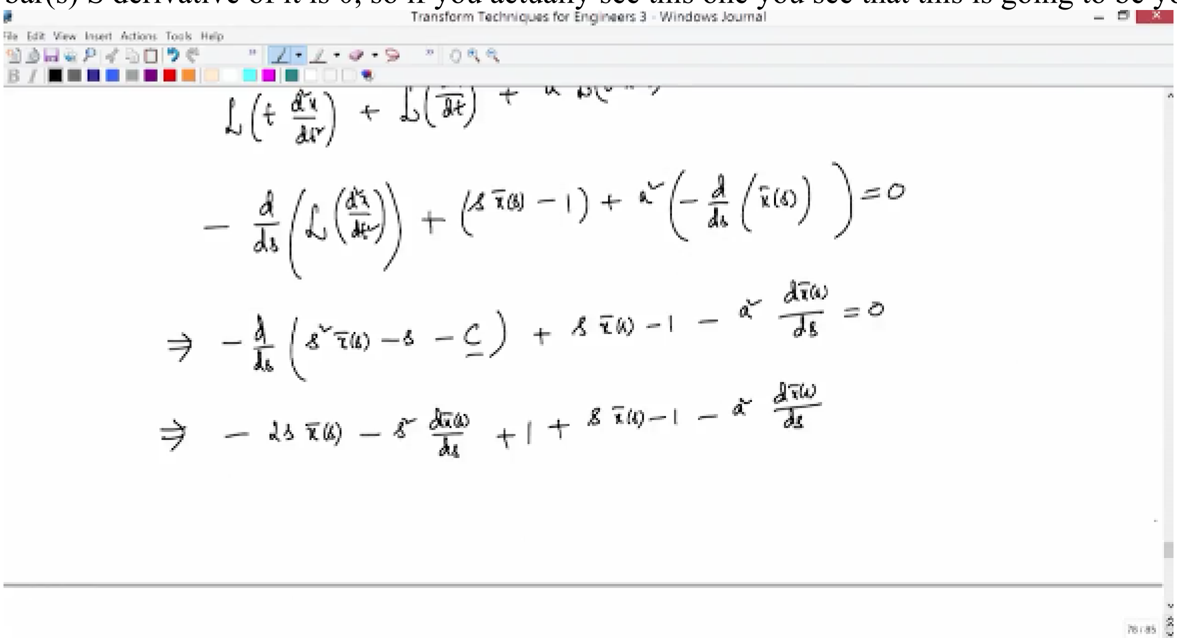
So let me just solve this, solve Bessel equation it's called Bessel equation, so let me write what it is, you have a T times D square X/DT square a second-order equation with the coefficient as being T and then you have a DX/DT and then + A square T times X, X is a function of T so it's basically function of T = 0, you have these initial conditions so I provide the initial condition here that is actually it's a second-order so you have initial condition X(0) = 1 bounded and you should also provide X dash(0), so for the time being let's choose some value, okay, for any value let us see what you get, okay.

So how do we solve this? So how do we solve this equation solution? Again so there is T is positive so you apply the Laplace transform domain is positive, Laplace transform gives application of this Laplace transform for a T variable, for X(t) if you apply so this T times you apply the Laplace transform of T times D square X/DT square and then you have a Laplace transform of DX/DT that we know already, and you have this Laplace transform A square is constant T times X(t) Laplace transform which are functions of S which is 0, okay, so this is

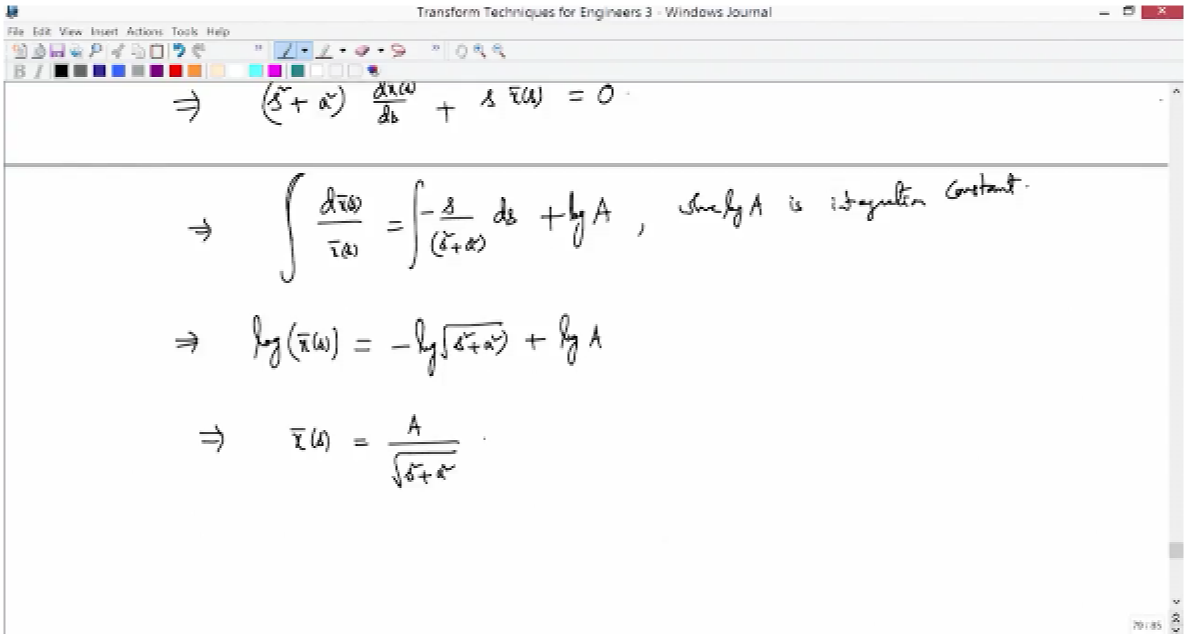


equal to and if you have seen earlier as the property of the Laplace transform we have seen this is actually –D/DS of the Laplace transform of D square X/DT square, and this is as it is so you can write directly as X bar(s) – X(0), X(0) is 1 so you have this is X(0) so that is value is 1, so I write for this + A square this is T times X(t) is again –D/DS of Laplace transform of X bar(s) okay, so that is I am applying the same technique here which is equal to 0, so if you actually do this this is equal to –D/DS of Laplace transform of the second derivative is S square X bar(s) – X(0) that is 1 -X dash(0) that is which you have taken as any constant C, okay, so that is let me

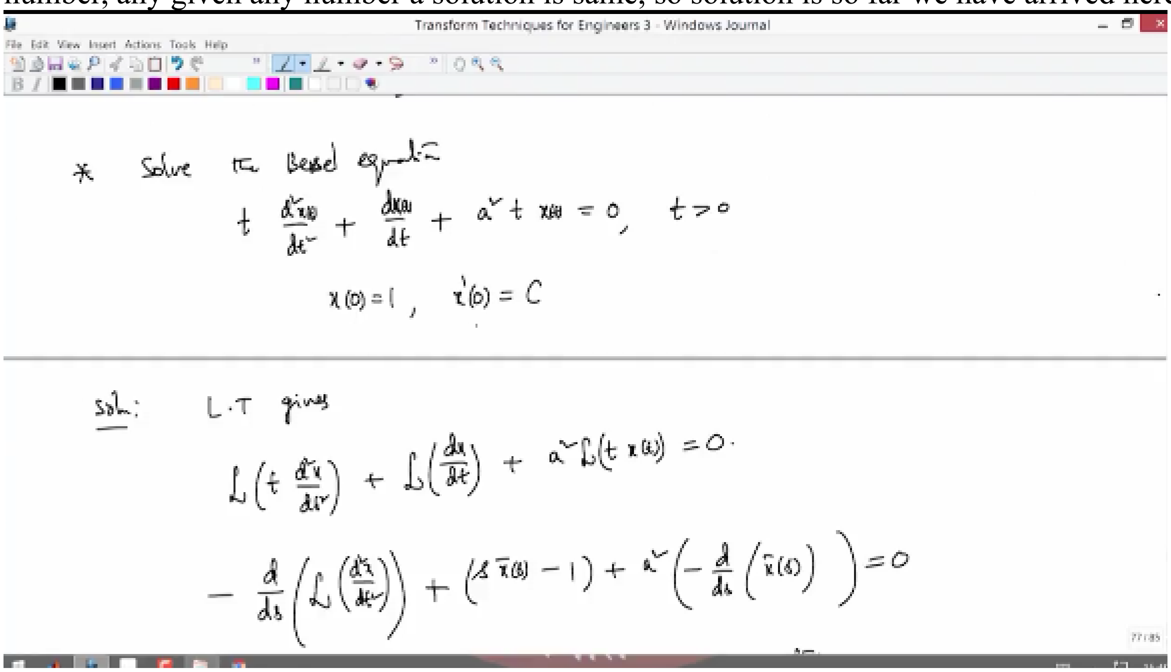
write this as C which is anyway constant, and this is S, S into X(0) that is S into 1, so that is S, this is for this term.
Other term is here X bar(s) -1 and here -A square DX bar(s)/DS = 0, so if you actually differentiate this you can see that this is going to be 2S X bar(s) - S square DX bar(s)/DS minus minus plus if you differentiate that is 1, this is a constant that's 0 + S X bar(s) -1 -S square DX bar(s) S derivative of it is 0, so if you actually see this one you see that this is going to be you



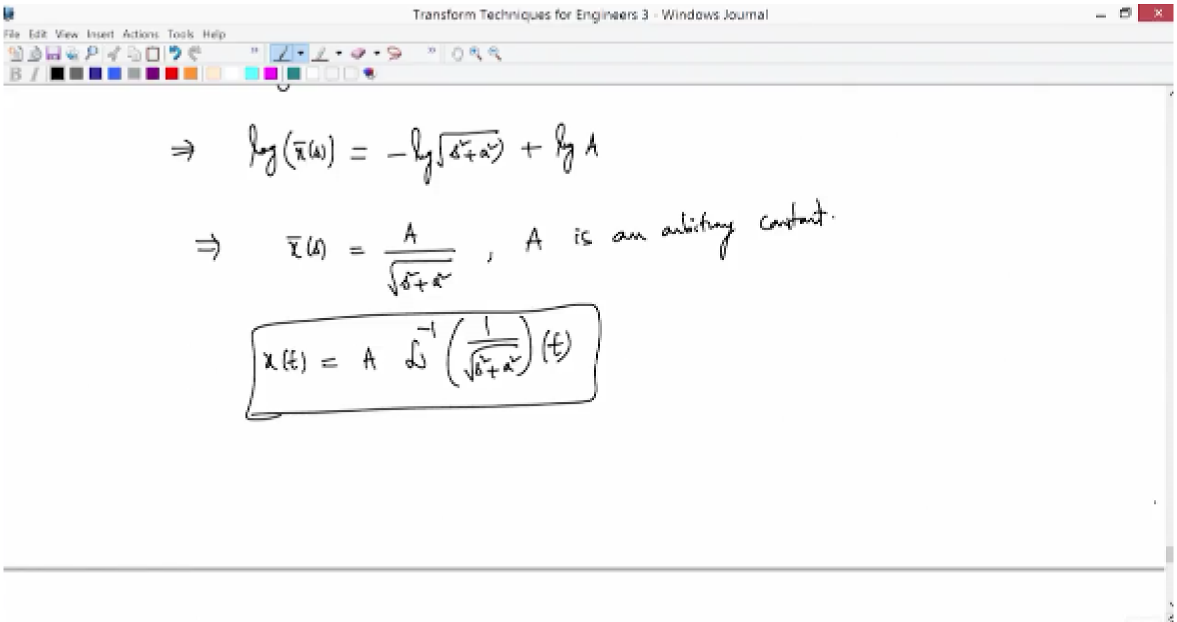
make a minus both side so you have a plus here, plus here, this is going to be minus, this is going to be minus, and this is plus, this is plus, so if you combine this S square + A square times DX bar(s) of DS and then -S this is going to be 2S -S is S X bar(s) this is going to be plus, because of 2S -S that is +S, and which is equal to that is anyway this gets cancelled so you have this is 0, so what you get is you can directly solve this because this is simply X bar(s) divided by DX bar(s) here, bring this this is separation of variable method you can solve this and you have DS so you have –S/S square + A square times DS, so you integrate both sides to see this plus some integration constant, let's call this A where A is integration constant, so what you end up is log of X bar(s) which is equal to this if you look at, if you directly write this, this is 1/square root of S square + A square, if you differentiate this so you get -1/2 and or rather I think if you simply differentiate there's an anti-derivative of this is this, okay, derivative of D/DS of this if you calculate it is actually minus of anything, so exactly so how do you calculate, how do you do this? This is again log S square + Ai square that is the anti-derivative so you have 1/S square + -2S, so you have -1/2, so that if 2 2 gets cancelled you have minus left, so this is the integration of this +E power or rather let me write this as log A okay, log A is integration constant, so you see that log if you remove both sides X bar(s) as A divided by this is going to be square root of S square + A square, this you can write this as you can put this as a square, okay, so log A-this is A divided by this, log you remove this is what you get, so A is arbitrary constant.



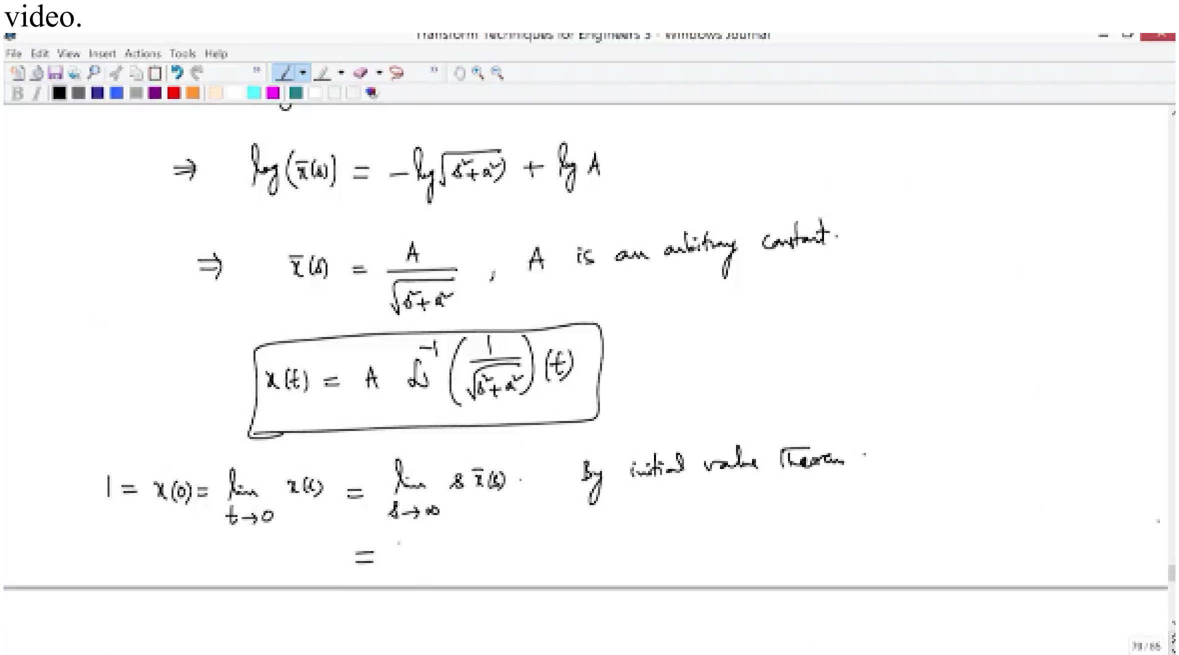
And as you see and so far although we have used this X(0) = 1, and X dash(0) which is any number, any given any number a solution is same, so solution is so far we have arrived here



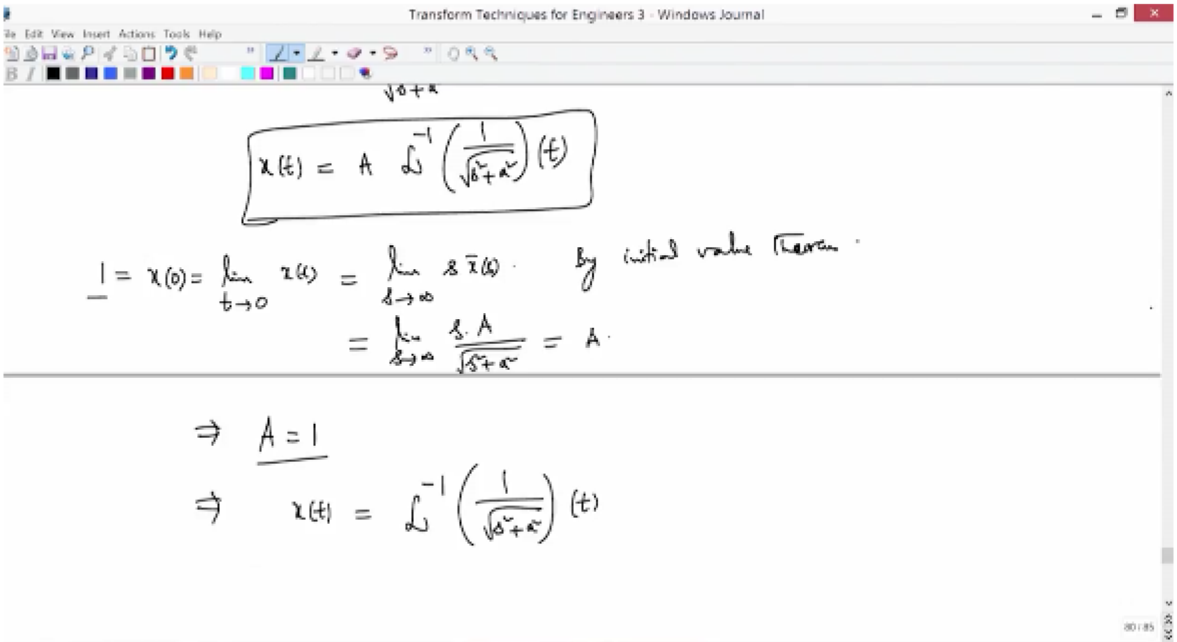
still we have an arbitrary constant okay, A is an arbitrary constant, so how do I get rid of this? If I calculate X(t) anti-derivative I mean inverse transform of this will give me something okay so A times Laplace inversion of 1/square root of A square + A square, so this is my which is a function of T so this is my solution but still I have an arbitrary constant here, so this is where I use this initial value theorem that is given that X(0) = 1, okay, so which is equal to 1 this is



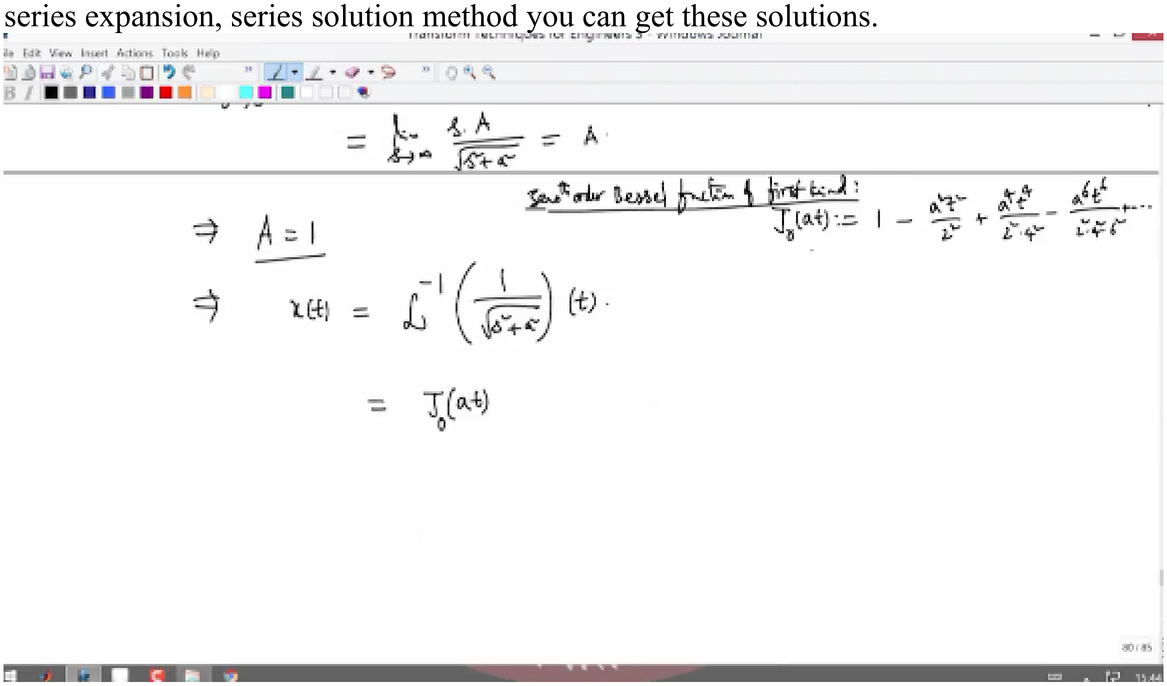
actually limit of X(t) as T goes to 0 this is exactly what we have seen is limit S goes to infinity S times X bar(s) by initial value theorem, initial value theorem that we have seen in the last video.



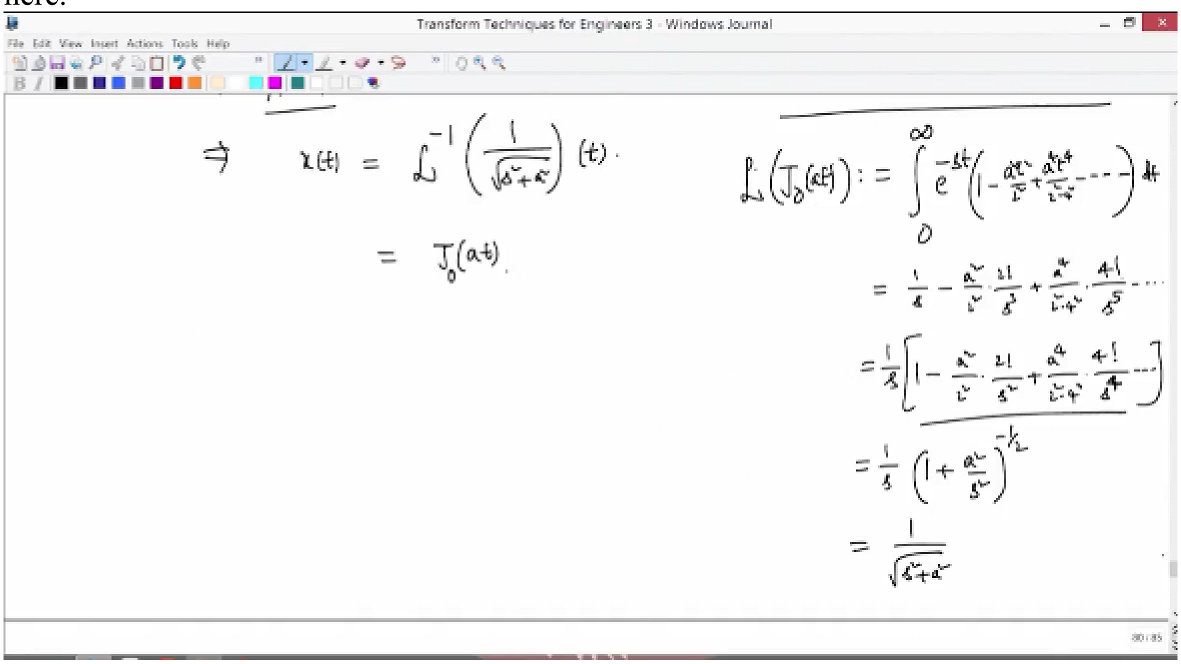
So what is this one? This is limit S goes to infinity, so you have S times X bar(s) is A divided by square root of S square + A square, so this gives me A times, and you have, this is simply A, the limit value is A because S divided by square root of S square + A square limit S goes to infinity that is going to be 1, so A is actually 1, so that means that implies A = 1, so implies X(t) equal to Laplace inversion of 1/square root of S square + A square which is a function of T.



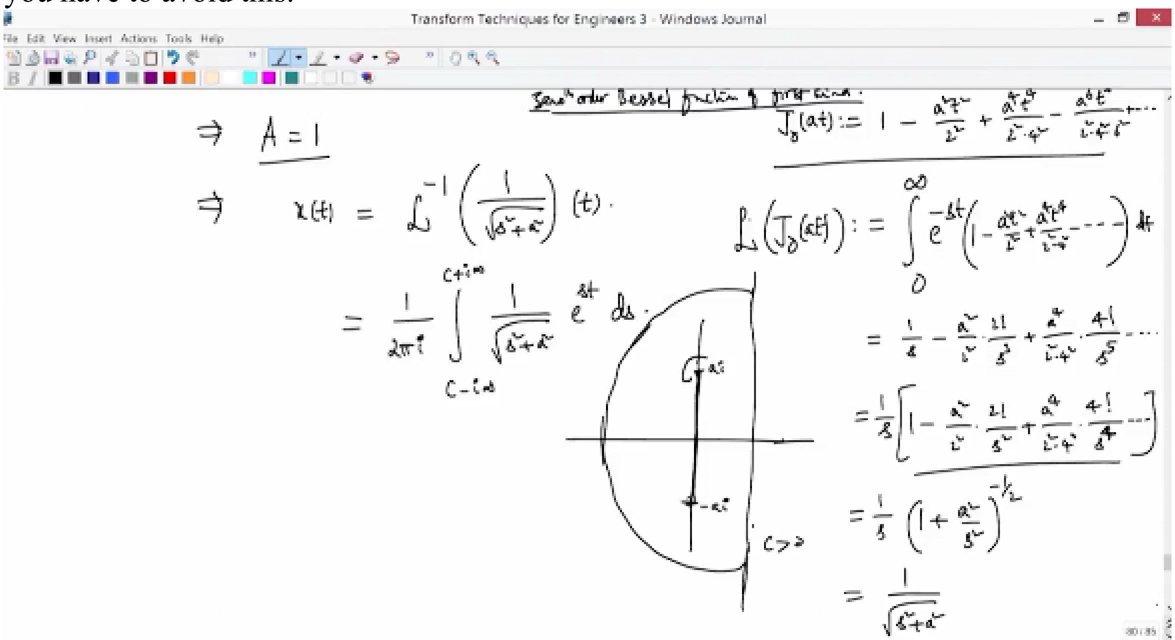
What is this inverse? To get this inverse you have to write this, this is actually special function which is J0(AT), so if this is, this you can take it as directly because J0(AT) you consider this, this is a Bessel function J0(AT) is Bessel function whose Laplace transforms if we calculate actual definition of J0(AT) is, you have this, this is a series expansion that is A square T square/2 square + A power 4, T power 4/2 square 4 square, 2 square into 4 square okay, and then -A power 6 T power 6 divided by, AT power 6 divided by 2 square 4 square 6 square and so on, so this is the definition of this special function, special Bessel function of first kind, okay, Bessel function of zeroth order, zeroth order Bessel Function, zeroth order Bessel function of first kind, if you've studied differential equations course or you understand this is a solution of this Bessel equation that we have here, you can get the solution by Frobenius method, it's a series expansion, series solution method you can get these solutions.



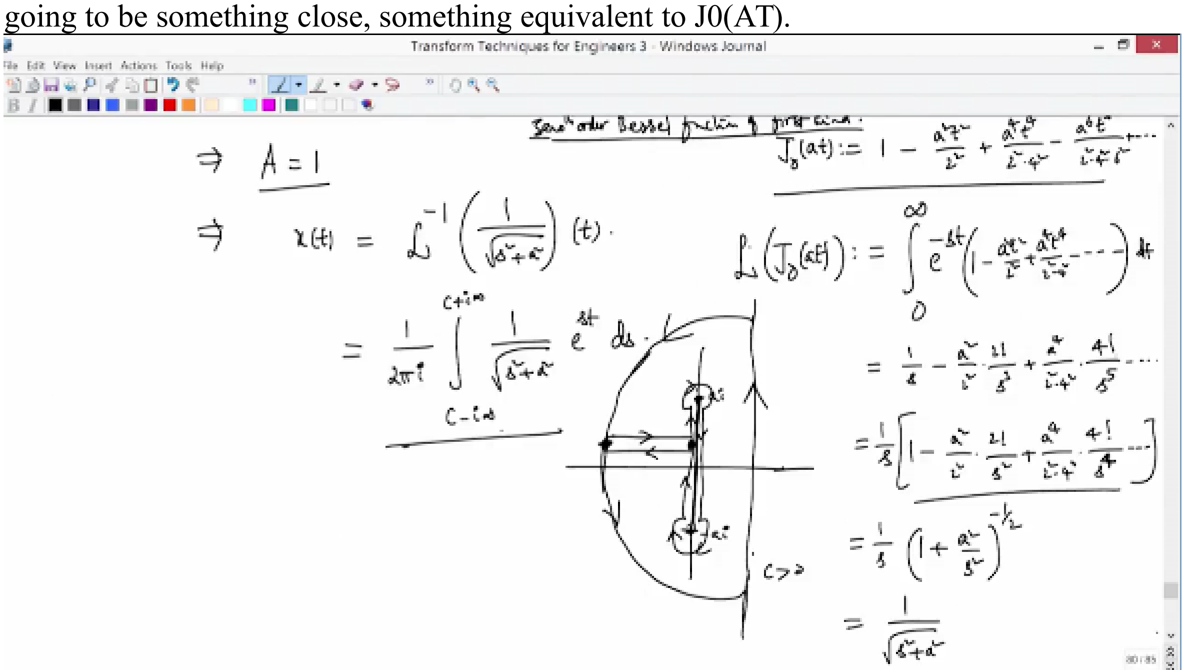
So you can see that this is going to be the value of this J0(AT), if you actually apply the Laplace transform for this J0(AT) by definition of your Laplace transform 0 to infinity E power -ST times this you put this 1 - A square T square/2 square A4, T power 4/2 square 4 square and so on, you put it and you have a DS, sorry this is DT, okay, this is the integration, so thing is you can take this integration because this is a uniform convergent series because this is J0(AT) is uniform convergence series so you can take this integral inside, so if you take that inside you see that you can apply, you can take integration term by term and you see that each term if you look at it that is going to be Laplace transform of 1 that is Laplace transform 1 is 1/S, 1/S and then you have next term is A square/2 square Laplace transform of T square is 2 factorial divided by S cube and so on like that you can go on writing like this A power 4/2 square 4 square Laplace transform of T power 4 is 5 factorial divided by S power 5, right, this is A power 4 this is going to be 4 factorial divided by, T power 4 is 4 factorial divided by S power 5 and so on, so if you actually see this one 1/S times 1 - A square/2 square times 2 factorial divided by S square + A power 4/2 square 4 square times 4 factorial divided by S power 4 and so on, if you write like this what do you see is this as 1/S times this series there is a representation for 1+A square/2 square, A square/S square power -1/2 okay, so this is how you can get this, and this is exactly what you have, so this if you simplify what you are seeing is 1/square root of S square + A square, so Laplace inversion of this is actually J naught(AT) that is what I am writing directly here.



So if you know exactly what is the solution of this one, there's another way of, see because I know that this is a solution of J0(AT) is a solution, because I know what is J0(AT) it's series I am putting into this Laplace transform and get this as a Laplace inversion, if you don't know you can also directly get, you don't need to write this expression so this you can think of like calculating this general way of finding C-I infinity to C+I infinity Laplace inversion that is 1/square root of S square + A square times E power ST DS, so this is what you have to calculate, 1/square root of S square + A square is having a branch points, so that's the different technique you have to exploit, this is advance some complex variable technique you have to use because this S square + A square is having a branch points which is at –A to A at both places, sorry this is a AI and –AI, so you have a branch points in the complex plane, S complex plane you have AI here and you have -AI so this is a cut you have to make, this is the cut you remove so that is, for which you have, then that is the if you use this as a cut in the complex plane and you can choose your C, any C positive so that all your, there is no, so you can consider this kind of contour usual Bromwich contour if you consider, so if you consider this and also somewhere you have to avoid this.



So how do I avoid this? You avoid like this, and then you connect these two somewhere here, you go here so this is how you are going, and you enter here and you come back, so you go here and you go in this direction, come here, and come here, come here and then you enter here and then you come back here and join here, so somewhere here I'm removing okay, so these two I remove and make it a single round so you have a simple closed curve you have, so if you calculate this the contributions of these branch points will give the value of this integral that is going to be something close, something equivalent to J0(AT).



There is another way of another expression, another way of finding this Laplace inversion which you see, which you may get another different expression for J naught(AT), another representation of J naught(AT) you will get if you do this way, okay, so because we know these are the solutions from the differential equations J0(AT) whose expression I know for which if you calculate the Laplace inversion you can easily see that is going to be 1/square root of S square + A square, so Laplace inversion of this directly I am writing this as J0(AT), this is the solution of this your equation, your Bessel equation with the initial condition is this and the other condition, any other whatever value, whatever may be the value of the other condition a solution is this, solution is simply J0(AT) okay, so this is how you solve this second-order special Bessel type of equation, and if you are given any second order linear ODE with variable coefficients with T or T square you can apply the Laplace transform, and use the similar procedure try to get the inverse and apply these initial conditions whatever required and somewhere you are using this initial value theorem or final value theorem, any one of them you can use to get any arbitrary constants involved in your solution.
So I will try to do one more example in this, I will try to how to use this initial value problem, initial value theorem how to use in solving the second order linear differential equation with the initial conditions, this along with other applications for example solving, we will move on to solve partial differential equations in the next video, we'll see in next video. Thank you very much.

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