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NPTEL ONLINE COURSE
Transform Techniques for Engineers
Application of Laplace transforms –ODEs'
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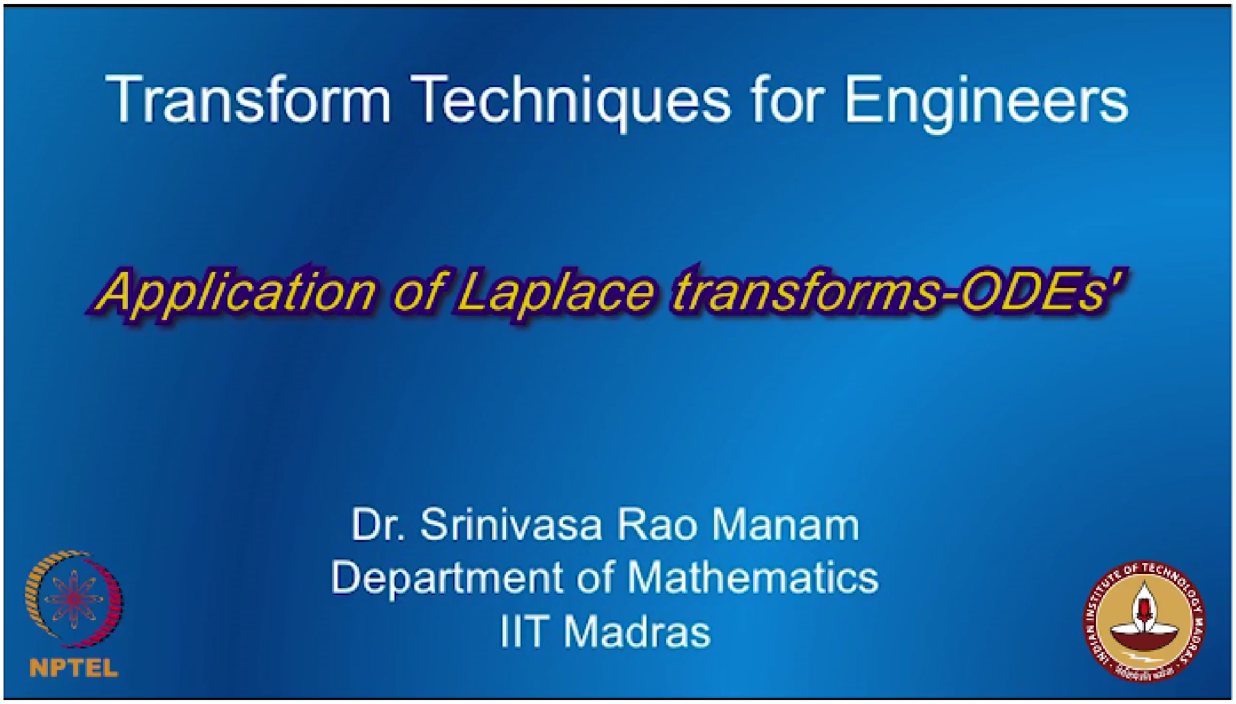

We look into applications of Laplace transform in this video, before we really get into applications, for example to solve the ordinary differential equations to start with, so what we do is we have two more results that is about the link between the initial value if you have F(x) is your function, F bar(s) your Laplace transform initial value of the function F that is F(0)and if you what is its relation F(0), what is its relation with the Laplace transform F bar(s).
So this is about the initial value theorem, and also about if you have F(x) a function and its Laplace transform F bar(s) and if you look at S goes to infinity F bar(s) with that limit, limiting value of are F at infinity if you want to look at function F at infinity if it exists, what is its relation with Laplace transform F bar(s), so these two things can be used in applications, so it's solving the differential equations, so we'll use that but before I will do these problems, so let's just quickly get these two results that initial value theorem and final value theorem.
So let me write this initial value theorem, so this tells you that if F bar(s) is Laplace transform of F(t) which is function of S, so if it exists then this limit S goes to infinity, F bar(s) = 0, so using this we can also get, also I'll just write this limit S goes to infinity, S times F bar(s) this one is nothing but a limit of F(t) as T goes to 0, that is actually F(0) okay, so if F is a continuous function at 0 so this is 0, so this is what it happens, okay in particular you can have this result, so this is what exactly is useful for us, so we need this.
Let's prove simply how we do this, so to start with we will just look at this part so that is straightforward to see, so let me choose this S goes to infinity F bar(s), so what is this value? This is actually equal to so limit S goes to infinity integral, so this is by definition 0 to infinity

F(t) times E power –ST DT, because this integral is uniformly convergent integral as a function of S so I can take this limit inside this integral so that will make it 0 to infinity limit S goes to infinity E power –ST, and anyway this is a function of T that is nothing to do with S, so this is DT, so this becomes as S goes to infinity is actually 0, okay, because this limit is 0 so you have this is 0, so this is straightforward.

<u>initial value theorem:</u>

If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then $\lim_{s\to\infty} \bar{f}(s) = 0$.

Also, $\lim_{s\to\infty} [s\,\bar{f}(s)] = \lim_{t\to 0} f(t) = f(0)$.

<u>Proof:</u>

$$\lim_{s\to\infty} \bar{f}(s) = \lim_{s\to\infty} \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \underline{\lim_{s\to\infty} e^{-st}} \cdot f(t)\, dt$$

$$= \underline{0}$$

Suppose F is differentiable function okay, so let me use this if F is differentiable function F bar(t) exists then this is also true, so this is what we'll show, okay, so second part is this, if F bar exists you look at the same way this S goes to infinity Laplace transform of F dash(t) okay, if F bar(t) exists and if this Laplace transform of F bar(t), if this Laplace transform exists this also should go to 0, okay, so this also should go to 0 by the earlier argument, this is again limit S goes to infinity, so by the same argument you can show that for F(t) power -ST DT this is limit

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If $f'(t)$ exists, then $\lim_{s\to\infty} [s\,\bar{f}(s)] = \lim_{t\to 0} f(t) = f(0)$.

<u>Proof:</u>

$$\lim_{s\to\infty} \bar{f}(s) = \lim_{s\to\infty} \int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty \underline{\lim_{s\to\infty} e^{-st}} \cdot f(t)\, dt$$

$$= \underline{0}$$

$$\lim_{s\to\infty} \mathcal{L}(f'(t)) = \lim_{s\to\infty} \int_0^\infty f'(t)\, e^{-st}\, dt = \int_0^\infty \lim_{s\to\infty} e^{-st}\, f'(t)\, dt = 0.$$

S goes to infinity E power -ST times F dash(t) DT, this is actually 0 okay, but then what does it mean, so this means we already have seen that this limit S goes to infinity Laplace transform of the derivative of the function is S F bar(s) – F(0) this has to be 0, so the place of this I wrote this, this is the definition, this is how we calculated Laplace transform the derivative of the function, so this is nothing but limit S times F bar(s) as S goes to infinity is nothing but this is a constant that is in terms of F(0) of 0 which is nothing but for a function, continuous function at 0 this is the limiting value is same as that, so this is exactly what we need to show here, okay.

initial value theorem:

If $\bar{f}(s) = \mathcal{L}(f(t))(s)$, then $\lim_{s \to \infty} \bar{f}(s) = 0$.

If $f'(t)$ exists, then $\lim_{s \to \infty} [s \bar{f}(s)] = \lim_{t \to 0} f(t) = f(0)$.

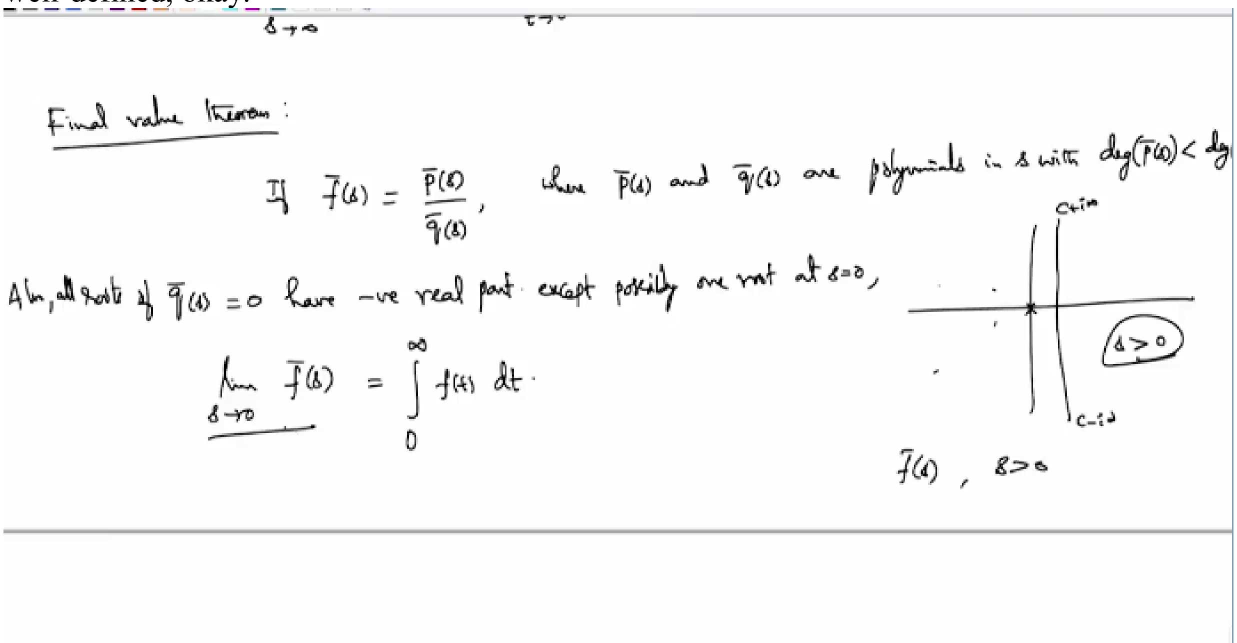
Proof:

$$\lim_{s \to \infty} \bar{f}(s) = \lim_{s \to \infty} \int_0^\infty f(t) e^{-st} dt = \int_0^\infty \lim_{s \to \infty} e^{-st} \cdot f(t)\, dt$$

$$= 0$$

$$\lim_{s \to \infty} \mathcal{L}(f'(t)) = \lim_{s \to \infty} \int_0^\infty f'(t) e^{-st} dt = \int_0^\infty \lim_{s \to \infty} e^{-st} f'(t)\, dt = 0.$$

So this is the initial value problem which is just by using as straightforward, use of definition of Laplace transform and then use uniform convergence of the integral Laplace transform that integral, definition of the integral, definition of the Laplace transform which is uniformly convergent so that you can take the limit inside the integration so that gives the result straightforward, so this is one result. And the other one is a final value theorem, final value theorem, so this may be also be used in the applications as and when it comes we'll see, so final value theorem says so if F bar(s), so let's take a simple, particular form of F bar(s), let's take this as this way P bar(s) and Q bar(s), where P and Q bar(s) are polynomials, okay, so where P bar(s) and Q bar(s) are polynomials, polynomials in S and obviously degree of P is smaller than degree of, with degree of P bar(s) is smaller than degree of Q bar(s), so assume we have such a Laplace transform, and assume also that Q bar(s) roots, all roots of this have a negative real part, that means you have poles only on the negative side of S plane, so except possibly at S = 0, okay, except possibly one root at S = 0, this can be 0 at S = 0 okay, one root can be at S = 0 if you allow, so that S is positive, so you have that C, C-I infinity to C+I infinity that inverse transform S is defined for all S positive, so that you can finally what you want is this limit F bar(s) as S goes to 0, because this F bar(s) has all the poles which are the roots of Q bar(s), which are in the negative side, so S is well-defined on the positive side, so I can take the limit S goes to 0, so F bar(s) this is nothing but so what you get is integral 0 to infinity F(t) DT, okay. If you have something like this, so which is in a simple form or any F bar(s) which is the limit, which is the Laplace transform of some function F(t) and if you have, so well defined for S positive okay, if F bar(s) this is well-defined for S positive that is possible only if, if this function this Laplace transform should have all the negative, all the roots, all the poles, all the singular points on the negative side of this plane, S plane, S complex plane, so that S positive its analytic function which is well-defined so that you can think about its limits, limits S goes to 0, so that's the only assumption so these assumptions are simply to make sure that this limit is well-defined, okay.



So finally you see that this is the value you will get, and this is one also like earlier if it's a differentiable function if F bar(t) exists for a, or rather for differentiable function okay and you can have this result that S goes to 0, S times F bar(s)this is equal to limit F(t) as T goes to infinity, this is why exactly it's called final value theorem because F(t) as T goes to infinity there is a final value that is F infinity if it exists, okay, so this is exactly this meaning.
So proof here also it's simple, you simply write what you want so this is what you want for a suitable F bar(s) for which S is positive it's well defined so that I can talk about this limit, S goes to 0, F bar(s) this limit S, S goes to 0, F bar(s) by definition E power -ST F(t) DT okay, so again this is a Laplace transform so this integral is uniformly convergent so that I can take this limit inside as a S variable so you can take this inside, limit S goes to 0 E power -ST F(t) DT, so this limit if you calculate that is 1, S goes to 0 so you have 0 to infinity F(t) DT.

Also, If $f'(t)$ exists, $\lim_{s \to 0} \left(s \bar{F}(s)\right) = \lim_{t \to \infty} f(t) = f(\infty)$. $f(s)$, $s > 0$

proof:

$$\lim_{s \to 0} \bar{F}(s) = \lim_{s \to 0} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \left[\lim_{s \to 0} e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty f(t)\, dt.$$

Now this is also, second result is also simple if F bar(t) exists what happens here? F bar(t) Laplace transform of this, as S goes to 0 is equal to integral 0 to infinity, so you simply if I apply the same result for F bar(t) you see that F bar(t) DT, so this is nothing but F(infinity) – F(0), but we already know that this is a limit S goes to 0 definition of Laplace transform for a differentiable function F bar of, F dash(t) is S F bar(s) – F(0), so that is the property, we have seen this as a property of the Laplace transform so this is your left hand side is equal to right hand side is F infinity –F(0), so this is a constants, so limit S goes to 0 so this gets cancelled both sides, so we end up getting what you want, so this limit S goes to 0 S F bar(s) = F(infinity) which is nothing but limit F(t) as T goes to 0, if it exists as a finite number this has to be a limit of F(t), so that is exactly what you need to show, so this is called the final value theorem. So these are the two results which you will keep in mind while solving the ordinary differential equations which we will see soon.

Proof:

$$\lim_{s\to 0} \bar{f}(s) = \lim_{s\to 0} \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty \left[\lim_{s\to 0} e^{-st}\right] f(t)\, dt$$

$$= \int_0^\infty f(t)\, dt$$

$$\lim_{s\to 0} \mathcal{L}(f'(t))(s) = \int_0^\infty f'(t)\, dt = f(\infty) - f(0).$$

$$\lim_{s\to 0} \left[s\bar{f}(s) - \cancel{f(0)}\right] = f(\infty) - \cancel{f(0)} \Rightarrow \lim_{s\to 0} s\bar{f}(s) = f(\infty) = \lim_{t\to\infty} f(t)$$

So when you have some arbitrary constants involved in the Laplace transform of the unknown function, and you may have to use the one of these but what is given is the unknown function I mean the solution of the differential equation initial value if it is given that means X(0) is given and you have to relate with its, the Laplace transform of X(t) that is X bar(s) for which if you look at this goes to infinity you know exactly what it means, okay, because of the initial value theorem.

Similarly you can use the final value theorem to make use of this in the applications, so keep these two results, these simple results in mind, let's do these applications of Laplace transform, applications of Laplace transform, so to start with solutions of ordinary differential equations okay, we solve solutions of ordinary differential equations, so how do we solve this? Let's start with the ordinary differential equation means you have one dependent variable and one independent variables, so you have a relation between unknown dependent variable and its derivative, if it's order first order differential equation you will have unknown dependent variable and it's one derivative so we'll have a relation between this along with independent variable, so you'll have a relation between a dependent variable and its derivative and independent variable this is called ordinary differential equation.

So we look into it some linear first-order ordinary differential equation, let me take specifically let's solve some example, so how do we solve this first order? First order case, first order ODE let me take this first order ODE how do we solve, so example let me start with DX/DT X is the dependent variable, T is the independent variable, then this is a first order this will have a derivative of the dependent variable P is a constant X is again independent variable this is equal to right hand side any function of T, so given function of it that is forcing, okay, so this is the first order, a linear ODE, so this how do we solve this? So you can use this because T is positive side if you are a differential equation if your domain is 0 to infinity you can use our Laplace transform, of course you have a initial values X(0) the boundary this is your domain, so T is positive, so T is running from 0 to infinity, so this is the boundary also initial value, so this is called the initial value time so this is X(0) is a first order, so you have to provide one initial value at T = 0, that is X(0) which you call it A constant, okay, so where A and P are constants.

Applications of Laplace transform:

Solutions of ordinary differential equations.

example: first order linear ODE.

$$\frac{dx}{dt} + px = f(t), \quad t > 0.$$

$t=0$

$$x(0) = a.$$

where a, p are constant.

Let's take this so how do we solve, so let's solution let me apply a Laplace transform to the equation, so apply Laplace transform to the equation we get, what we get is if you apply Laplace transform to this derivative so you get S times X bar, X(t) if you apply this is X, X means X(t) it's a dependent variable, so you have X bar(s) that's a Laplace transform –X(0) so this together is the Laplace transform of the derivative of X + P is a constant and you have X bar(s) and this is equal to F bar(s), assume that F(t) for which you have Laplace transform exists, so this gives me and now we have X(0) you can use here initial value that is A, so you have X bar(s) = F bar(s) + A divided by S + P okay.

So now we can take the inverse transform, taking inverse transform to get the solution X(t), left hand side if you take the inverse Laplace transform you get X(t) on the right hand side so you see A is a constant, A divided by, A is a constant comes out Laplace inversion of 1/S+P + Laplace inversion of F bar(s) times 1/S + P, so what is this one? This is actually equal to A times, this is E power –PT, okay, so Laplace transform of this is 1/S+P, so I've written this plus here this is a Laplace transform of F(t), this is the Laplace transform of E power -PT and this is F(t), so you take the convolution of this and that should be for which Laplace transform if you

$$s\,\bar{x}(s) - x(0) + p\,\bar{x}(s) = \bar{f}(s).$$

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a}{s+p}$$

Taking inverse transform, we get the solution

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{1}{s+p}\right) + \mathcal{L}^{-1}\left(\underline{\bar{f}(s)}\cdot\underline{\frac{1}{s+p}}\right)$$

$$= a\,e^{-pt} + \qquad f(t) * e^{-pt}$$

take that is what exactly you will get F bar(s) and this, so Laplace transform of this is nothing but this whole F bar(s) times this, so and you take the inverse transform Laplace goes so you end up simply getting this one, so let me write this definition of this convolution so you have 0 to T, so you can write E power -T or F(tau) E power –P, in the place of T you write T-tau D tau, so that is exactly your definition of the convolution, so this is exactly we'll get as if you apply other methods to find the solution of this linear equation, so you have this E power -PT comes out because it's nothing to do with the tau variable, so you end up getting 0 to T, F(tau) E power P tau D tau, so this is the solution of your differential equation that you started with as a first order linear ODE and with initial value, so this is how you can solve by applying Laplace transform provided the boundary, the domain of the differential equation is 0 to infinity.

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a}{s+p}$$

Taking inverse transform, we get the solution

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{1}{s+p}\right) + \mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{1}{s+p}\right)$$

$$= a\,e^{-pt} + \int_0^t f(\tau)\, e^{-p(t-\tau)}\, d\tau$$

$$\boxed{x(t) = a\,e^{-pt} + e^{-pt}\int_0^t f(\tau)\, e^{p\tau}\, d\tau.} \checkmark$$

So let's move on to solve the second order equations, second order linear ODE you can solve similarly, second order linear ODE, so of this form so let's choose some form, so for example D square X/DX DT square X is the dependent variable, T is independent variable + 2 times let me use this is just for convenience I'm considering this, P is a constant, DX/DT + QX Q times, Q is also constant X is a dependent variable and you have F(t), if T is positive and this is a second-order equation should give provide two initial conditions that is at X = 0, let's call this A and X DX/DT at T = 0, at X DX/DT that is also called X dash(0) which is nothing but some B let's choose given the constants, okay, so where P, Q, A, B are constants, if you have in this one you can solve it because you can take the Laplace transform here, Laplace transform, so if it is a Q is a function of T which possibly you may have to, it may be difficult to find the Laplace transform and get finally, so that may be difficult not all equations you can solve, some simple

Second order linear ODE.

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + qx = f(t), \quad t > 0.$$

$$x(0) = \underline{a}, \quad \frac{dx(0)}{dt} = x'(0) = \underline{b}, \quad \text{where } p, q, a, b \text{ are constants.}$$

some specific forms for example like this you can solve so if you apply, so solution is by applying Laplace transform, apply a Laplace transform to the equation we get, what we get is a derivative, second derivative of this we know that this is derivative is S square X bar(s) - S times X(0) - X dash(0), so this is the derivative, this is the Laplace transform of the second derivative + 2 constant P times S X bar(s) – X(0) this is your definition of Laplace transform of the first derivative. And then + Q is a constant, and you have Laplace transform of X which is equal to F bar(s) which you have taken Laplace transform of F(t).
Now you can use the initial values X(0), so you have A square, collect the terms coefficients of X bar so you get 2PS+Q times X bar(s) is equal to F bar(s) + you can have this S times coefficient of X(0) that is A here A here, so you have X(0) is A, you have S+2P, when you bring it to the right hand side, and the other one is X bar(0) that is B so you have B + B that's so you have this, so you have this implies X bar(s) is if F bar(s) + A times S + 2P + B divided by S

$$x(0) = a, \quad \frac{dx(0)}{dt} = x'(0) = b \quad , \text{ where } p, q, a, b \text{ are constants.}$$

Soln: Apply L.T to the equation, we get

$$\left(s^2 \bar{x}(s) - s\,x(0) - x'(0)\right) + 2p\left(s\,\bar{x}(s) - x(0)\right) + q\,\bar{x}(s) = \bar{f}(s).$$

$$\left(s^2 + 2ps + q\right)\bar{x}(s) = \bar{f}(s) + a(s+2p) + b$$

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a(s+2p) + b}{s^2 + 2ps + q}$$

square + 2PS + Q so for which if you take the Laplace transform, Laplace inversion so you can get your solution, so you take the inverse transform, inversion gives X(t) so that is simply Laplace inversion of F bar(s) times 1/S square + 2PS + Q + here this becomes A times Laplace inversion of A, A is a constant, so A inversion of Laplace inversion of S+2P divided by S square + 2PS + Q, of course this is also constant you can put it here, so this is the second order in S polynomial, this is the first order in S, so you can get this solution as this way, so one way to find this inversions are just to write this in a nice form that is a Laplace inversion of, put it together so let me write in the denominator this second-order polynomial I write like S+P whole square + if you write this 2 P's comes out P square, so you have a Q - P square so that is exactly you have to use as something which I call, if I call this as N square, okay Q - P square that is exactly let me call this some N square, so you call this N square this is on surface so because of, because we know that certain form S + P square by + 1/S + C whole square + A square so if it is like this S+C we know what is a Laplace inversion, so put it in that form let me use that form so that you have S+P here, so S+P already on here times S+2P so you have A here, so A A comes out, A times S+P + B+P that is anyway constant B+, so B B is actually B/A if you take here so you have B+PA which is a constant + F bar(s).

$$(s^2+2ps+q)\,\bar{x}(s) = \bar{f}(s) + a(s+2p) + b$$

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a(s+2p) + b}{s^2+2ps+q}.$$

Inversion gives $x(t) = L^{-1}\left(\bar{f}(s)\cdot\frac{1}{s^2+2ps+q}\right) + a\,L^{-1}\left(\frac{s+2p+b/a}{s^2+2ps+q}\right)$

$$= L^{-1}\left(\frac{a(s+p) + (b+pa) + \bar{f}(s)}{(s+p)^2+n^2}\right) \qquad q-p^2=n^2$$

So let me put it in this form so that you can write directly what is its inversion, so inversion is A comes out here, A comes out, Laplace inversion of S+P divided by S+P whole square + N square + B+PA times L inversion of 1/S+P whole square + N square, so you need this N here to get the form so you have divided by N, so I divided and multiplied so that N N goes which is second term you get.

$$\Rightarrow \bar{x}(s) = \frac{\bar{f}(s) + a(s+2p) + b}{s^2+2ps+q}.$$

Inversion gives $x(t) = L^{-1}\left(\bar{f}(s)\cdot\frac{1}{s^2+2ps+q}\right) + a\,L^{-1}\left(\frac{s+2p+b/a}{s^2+2ps+q}\right)$

$$= L^{-1}\left(\frac{a(s+p) + (b+pa) + \bar{f}(s)}{(s+p)^2+n^2}\right) \qquad q-p^2=n^2$$

$$= a\,L^{-1}\left(\frac{s+p}{(s+p)^2+n^2}\right) + \frac{(b+pa)}{n}\,L^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}L^{-1}\left(\bar{f}(s)\cdot\frac{n}{(s+p)^2+n^2}\right)$$

The other one is Laplace inversion of F bar(s) in times N divided by S+P whole square + N square, so because I multiplied one extra so comes out, so I have 1/N here, so if I write like this now you can take the inversion here depending on what is your N? N is, N square is this, so N square is say that it's positive or negative it can be 0, so in each cases you can easily see what it is, okay, if it's a positive case, so let me use if N is, if N square which is equal to Q-P square if this is positive, okay, so you have a positive + that means the + sign N square, so this is what we have seen, so in this case your X(t) that is your X(t) here, X(t) is A times Laplace inversion of this is E power because of S+P E power -PT times Laplace transform of S/S S square + N square that is cos NT okay, so this together is this one, Laplace transform of this is this one, so + B+PA/N times here this is E power -PT sine NT, okay.

$$= \mathcal{L}^{-1}\left(\frac{a(s+p) + (b+pa) + \bar{f}(s)}{(s+p)^2+n^2}\right) \qquad \left.\underline{q-p^2=n^2}\ \begin{matrix}>0\\<0\\=0\end{matrix}\right\}$$

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{s+p}{(s+p)^2+\underline{n^2}}\right) + \frac{(b+pa)}{n}\,\mathcal{L}^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}\mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{n}{(s+p)^2+n^2}\right)$$

If $n^2 = q-p^2 > 0$,

$$x(t) = a\,e^{-pt}\cos nt + \frac{b+pa}{n}\left(e^{-pt}\sin nt\right) + \frac{1}{n}\cdot$$

And then here 1/N times and this will be your convolution of F(t) that is 0 to T, F(tau) and here E power –PT, in the place of T you put T-tau and this will be sine N times T-tau D tau, so this together E power -PT sine T that is a Fourier transform of this, Laplace transform of this, this is

$$= \mathcal{L}^{-1}\left(\frac{a(s+p) + (b+pa) + \bar{f}(s)}{(s+p)^2+n^2}\right) \qquad \left.\underline{q-p^2=n^2}\ \begin{matrix}>0\\<0\\=0\end{matrix}\right\}$$

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{s+p}{(s+p)^2+\underline{n^2}}\right) + \frac{(b+pa)}{n}\,\mathcal{L}^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}\mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{n}{\underline{(s+p)^2+n^2}}\right)$$

If $n^2 = q-p^2 > 0$,

$$x(t) = a\,e^{-pt}\cos nt + \frac{b+pa}{n}\left(\underline{e^{-pt}\sin nt}\right) + \frac{1}{n}\cdot\int_0^t f(\tau)\ \underline{e^{-p(t-\tau)}\sin n(t-\tau)}$$

this one so convolution you have to, wherever T is there you have to replace T-tau, so in this place I have written T-tau so you get this one, so here you can easily see that PT is it's nothing to do with T, so you can write this outside E power -T PT so that you have together here you have only P tau, and this is exactly your solution, if N square is positive, if N square is 0 this is the case X(t) this is straightforward so you have 0 here, so A times L inverse of 1/S+P that is E power –PT, so 1/S is Laplace transform of 1, so because of S+P you have E power -PT + B+PA/N times here when N = 0 this goes to 0 anyway, you can remove this N and goes, so you don't have, so you have B+PA L inverse of 1/S+P whole square when N = 0, so that is T times, that is Fourier, 1/S square is Laplace transform of T so you multiply with E power –PT, so that is a Laplace transform of this is 1/S+P whole square + Laplace inversion of F bar(s) and 1/S+P whole square that is simply convolution F(t) into E power –PT, so T-tau times E power –P T-tau D tau, so T you can bring it out here so you have E power -PT so that you can write P tau here D tau, so this is the solution for in this case.

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{s+p}{(s+p)^2+\underline{n^2}}\right) + \frac{(b+pa)}{n}\,\mathcal{L}^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}\mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{n}{(s+p)^2+n^2}\right)$$

If $n^2 = q - p^2 > 0$,

$$x(t) = a\,e^{-pt}\cos nt + \frac{b+pa}{n}\left(e^{-pt}\sin nt\right) + \frac{e^{-pt}}{n}\cdot\int_0^t f(\tau)\,e^{p\tau}\,\sin n(t-\tau)$$

If $\underline{n^2=0}$,

$$x(t) = a\,e^{-pt} + (b+pa)\,t\,e^{-pt} + e^{-pt}\int_0^t f(\tau)\,(t-\tau)\,e^{p\tau}\,d\tau\,. \checkmark$$

Similarly for if N square is which is Q-P square if it is negative and you can write this as S+P whole square -N square, that is N square is positive with minus sign, so in this case you can have X(t) as simply cosines will become cos hyperbolic okay, so you have A times E power --PT cos hyperbolic NT because you have in this case it will be -N square, and if N square being positive, because this is negative so that is nothing but Q-P square is -N square with N square is positive but negative sign, so that is you have S+P whole square -N square in the denominator, so if you write like that so you end up getting, so you can write minus here, so -N square but I multiply only N without, only here you get a minus sign this I'm multiplying, just this I am adding actually, so once you have -N square I am only multiplying and dividing with N, so here you get a negative sign so that you have B+PA/N times you get here this is E power -PT sine hyperbolic NT, and then plus again so same way -1/N this is a convolution of F(tau) this is going to be sine hyperbolic N T-tau times E power –PT, so E power P tau D tau so that I can write E power -PT here, -PT outside, just like earlier, okay, so this is the solution finally, in this this form if you have N square is negative sign, so whatever may be the case so as such directly if you can find by some other means, if you can find this Laplace inversions you can easily write in a different form your solution, and by this technique you by known functions you bring

$$x(t) = a\,\mathcal{L}^{-1}\left(\frac{s+p}{(s+p)^2+n^2}\right) + \frac{(b+pa)}{n}\,\mathcal{L}^{-1}\left(\frac{n}{(s+p)^2+n^2}\right) + \frac{1}{n}\mathcal{L}^{-1}\left(\bar{f}(s)\cdot\frac{n}{(s+p)^2+n^2}\right)$$

If $n^2 = q - p^2 > 0$,

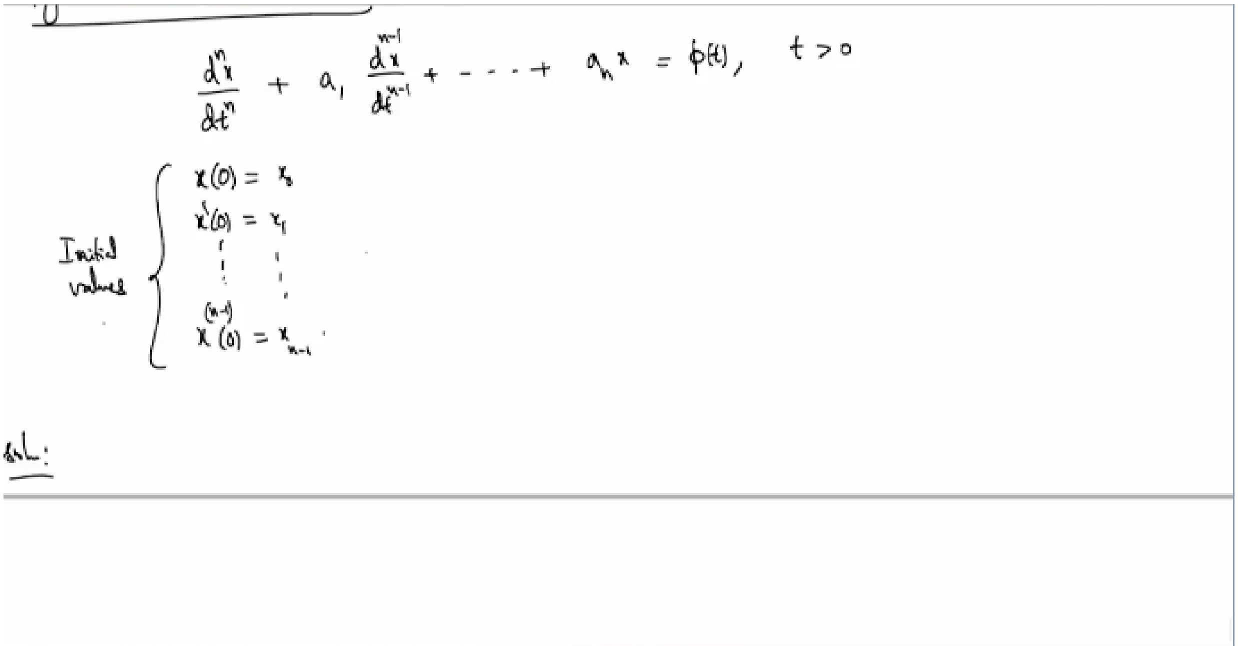
$$x(t) = a\,e^{-pt}\cos nt + \frac{b+pa}{n}\left(e^{-pt}\sin nt\right) + \frac{e^{-pt}}{n}\int_0^t f(\tau)\, e^{p\tau}\, \sin n(t-\tau)$$

If $n^2 = 0$, $\quad x(t) = a\,e^{-pt} + (b+pa)\, t e^{-pt} + e^{-pt}\int_0^t f(\tau)\,(t-\tau)\, e^{p\tau}\, d\tau$.

If $n^2 = q - p^2 < 0$, $\quad x(t) = a\,e^{-pt}\cosh nt + \frac{(b+pa)}{n}\left(e^{-pt}\sinh nt\right) + \frac{e^{-pt}}{n}\int_0^t f(\tau)\,\sinh n(t-\tau)\, e^{p\tau}\, d\tau$

in and try to recognize its Laplace inversion and write in the nice form of solution you represent in these elementary functions like sine, cosine, hyperbolic, depending on value of this N.

So the same technique you can apply for higher order equations so you can also apply the same technique for higher order equation, just me write, let me take a higher order equation, higher order a linear ODE, linear ODE, so let's take Nth order ODE if you have this, solve this linear equation with N constant coefficients let me take, so if I take our D power N X/D T power N this is nth derivative + some constant A1 times D N-1 X of D T power N-1 and so on, you end up getting AN, so that no derivative, so X X simply AN times X equal to, let me call this some phi(t) so T is positive, if T is positive you can apply your Laplace transform, and of course because its nth order differential equation you should provide some initial conditions X(0), X dash(0) and so on up to X Nth derivative N-1 derivative at 0 you have to provide, let me call them, let’s call this sum X naught, X1 up to XN-1, okay, if I call this this is your initial values, okay, this is your differential equation, so this initial value problem you can solve again by Laplace transform, so we'll see how, why is it possible, as a general case if you apply this for



the nth derivative, so let me write this apply Laplace transform, Laplace transform gives S power N X bar(s) - S power N-1 X(0) that is X(0) as X naught, I used directly X naught-S power N-2, now we have X dash(0) that is X1 and so on, finally you end up getting -S times XN-2 that is XN-, the second derivative okay, that is XN-2 – XN-1, so this is the derivative, this is the Laplace transform of nth derivative of a function X(t).

And then you multiply, so for second term again you write in a similar fashion if you write this as S power N-1, because N-1 derivative X bar(s) -S power N-2 X1 and so on, N-2 X naught here I start with X naught and so on, finally you end up getting XN-2 and so on you getting A 2 times again SN-2 times X bar(s) - S power N-3 X naught minus minus and so on finally getting XN-3 and so on you get finally AN-1, AN-1 1 derivative that is SX bar(s) - X naught + AN last thing, last term that is AN X bar(s), right hand side is phi bar(s), because phi(t), so if you have like this you collect the coefficients of all X bar(s) what you end up is, let me write you as you can see directly this coefficient, coefficient of this is S power N, so you have clearly a polynomial in for degree N, S power N here + A1 S power N-1 + A2 S power N-2, here finally you end up getting S into AN-1 + AN, so this is your nth degree polynomial as a coefficient of X bar(s) right hand side take everything to the right hand side, see that if phi bar(s) + here I collect the coefficients of X naught, X naught coefficient if you collect that is a polynomial in S with degree N-1 and here + A1, A1if you collect the coefficient of X naught, so we have SN-2 and so on, okay.

So finally you're getting a coefficient of X naught that is minus so plus when it bring it this side this is plus, so you have AN-1 okay, and up to so you do the same way and finally you get the coefficient of XN-2, a coefficient of XN-2 is this is X naught, similarly X1, X2 coefficient you can write finally XN-2 coefficient is S here, and then S and wherever else you get as your XN-2 that is going to be only this term will have, next term, so that is S+A1 and finally XN-1 so here sol: L.T gives

$$\left(s^n \bar{x}(s) - s^{n-1}x_0 - s^{n-2}x_1 - \cdots - s\,x_{n-2} - x_{n-1}\right) + a_1\left(s^{n-1}\bar{x}(s) - s^{n-2}x_0 - \cdots - x_{n-2}\right)$$

$$+ a_2\left(s^{n-2}\bar{x}(s) - s^{n-3}x_0 - \cdots - x_{n-3}\right) + \cdots + a_{n-1}\left(s\bar{x}(s) - x_0\right) + a_n\bar{x}(s) = \bar{\phi}(s)$$

$$\bar{x}(s)\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + s\,a_{n-1} + a_n\right) = \bar{\phi}(s) + x_0\left(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}\right) + \cdots + (s+a_1)\,x_{n-2} + x$$

so this is the only thing that is remain so +XN-XN-1 this is exactly you have, coefficient this is your polynomial so let me call this PN of this if you call this as a PN, so X bar(s) is now phi bar(s) + X naught times PN-1, let me call this, so polynomial in S with degree N-1 up to, this here it's polynomial of degree 1 this is simply constant, this whole thing divided by or together, so together this is the polynomial of degree S-1, so this is polynomial of degree let me call this PN-1 degree of S, this PN-1 is this whole thing is my PN-1(s) because this is a polynomial of degree N-1 you have this one is a polynomial of degree Q of, let me call this QN(s) because the degree is N so you have QN(s), of course because it’s a bar let me call this bar, puts bars okay, so let me call them as bars so that you have, taking the inverse Laplace transform gives, Laplace transform gives X(t) as your solution, because you have already used all the initial conditions, you have as a Laplace inversion of phi bar(s) times 1/QN(s) bar so this is you should know, this you can get it by 1/Q bar(s) you can write it as a partial fractions or by

$$+ a_2\left(s^{n-2}\bar{x}(s) - s^{n-3}x_0 - \cdots - x_{n-3}\right) + \cdots + a_{n-1}\left(s\bar{x}(s) - x_0\right) + a_n\bar{x}(s) = \bar{\phi}(s)$$

$$\bar{x}(s)\underbrace{\left(s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + s a_{n-1} + a_n\right)}_{q_n(s)} = \bar{\phi}(s) + \underbrace{x_0\left(s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}\right) + \cdots + (s+a_1)x_{n-2} + x}_{P_{n-1}(s)}$$

$$\bar{x}(s) = \frac{\bar{\phi}(s) + P_{n-1}(s)}{q_n(s)}$$

I.L.T: gives

$$\Rightarrow \quad x(t) = L^{-1}\left(\bar{\phi}(s)\cdot\frac{1}{q_n(s)}\right)$$

general method of finding Laplace inversion you can find and other thing also you can do the same way, so either heavyside expansion or partial fractions method or a general Laplace inversion by the contour integration technique you calculate, and you can write your Laplace inversion here that will be the solution of the higher-order equation for T positive, okay.

So this is the general procedure if you can fall, if you have a constant coefficients equation you can easily solve this, this type of equations, they come in the applications for example harmonic oscillator some equations ordinary differential equations you will come across in the harmonic oscillators, or also in the electric circuits so these are ordinary differential equations you will come across, okay, so as and when they come you can you just use this technique to solve them.

Higher order linear ODE:

$$\frac{d^n x}{dt^n} + a_1\frac{d^{n-1}x}{dt^{n-1}} + \cdots + a_n x = \phi(t), \quad t > 0$$

Initial values
$$\begin{cases} x(0) = x_0 \\ x'(0) = x_1 \\ \vdots \\ x^{(n-1)}(0) = x_{n-1} \end{cases}$$

Sol: L.T gives

$$\left(s^n\bar{x}(s) - s^{n-1}x_0 - s^{n-2}x_1 - \cdots - s x_{n-2} - x_{n-1}\right) + a_1\left(s^{n-1}\bar{x}(s) - s^{n-2}x_0 - \cdots - x_{n-2}\right)$$

We'll try to see the system of equations and also other type of, special type of ordinary differential equations we will solve in the next video, so before we wind up today let me use, let

me do some higher order equation, higher order linear equation, let me solve to show as an example here for this general technique solve this D cube X/DT cube, third order equation let me choose DX square/DT square of course with constant coefficients here only 6 times DX/DT = 0, so this is what is your equation for T positive this is important okay, only if T's belongs to full infinity you cannot do this because this is the domain is only T positive you have to apply your Laplace transform, otherwise you may have to split this equation into 2 parts, one is T positive, one is T negative side, each part you have to do it separately and get it, and then combine it together so that is more recover some, so you can instead, you can use other techniques to find the solution of that equation.
So let me so forth if this is the case initial values are except 0 is 1 given X dash(0) = 0, and X double dash(0) because the third order, up to second order you have to provide the solution initial conditions, so these are the initial conditions okay, so solution can easily work out like earlier if you write you see applying Laplace transform gives, what you get is X bar(s), so let me write this directly, so right hand side is 0 so phi(t) is 0 so you don't have that term so you have simply that PN bar, PN-1 so that is this is the third order so you have a second-order term that will come as a numerator, S square + S-1 the denominator a cube, third degree polynomial in S that is S cube + S square -6S, so S is common so you can write S S, S you remove put it outside, right like this, so this is your X bar(s), so inverse Laplace transform will give you inversion, inversion gives X(t) that is a Laplace inversion of S square + S-1 of S times S square + S-6, so this is exactly your Laplace transform, your solution of this third order differential equation, initial value problem.

eg: Solve $\frac{d^3x}{dt^3} + \frac{d^2x}{dt^2} - 6\frac{dx}{dt} = 0$, $t > 0$

I.C's $\begin{cases} x(0) = 1 \\ x'(0) = 0 \\ x''(0) = 5 \end{cases}$

Sol: Applying Laplace transform gives,

$$\bar{x}(s) = \frac{s^2+s-1}{(s^2+s-6)\,s}$$

Inversion gives, $x(t) = L^{-1}\left(\frac{s^2+s-1}{s(s^2+s-6)}\right)$; $t > 0$.

So how do we find this inversion? So this is you can use partial fractions method, you just get the partial fraction L inversion of, I write these partial fractions let me write 1/6 times you get A by 1/S + B/S+3 you have, this is S+3 and S-2, okay, so if you do like this, this product, product of these two so you have 1/S+3 B by this, the B will give me 1/3, and C by 1/S-2, C is 1/2, so you do the partial fractions this is what you will get, so that you can easily see that 1/6 times, 1/S Laplace transform is simply 1, inversion is 1 + 1/3 times E power -3T + 1/2 times E power 2T, so this is your X(t), so this is exactly your solution by Laplace transform method for this initial value problem with these 3 initial conditions for this equation if you try to solve this is how you get the solution, okay.

Inversion gives, $x(t) = \mathcal{L}^{-1}\left(\frac{\cdots}{s(s^2+s-6)}\right)$; $t > 0$.

$$= \mathcal{L}^{-1}\left(\frac{1}{6}\cdot\frac{1}{s} + \frac{1}{3}\frac{1}{s+3} + \frac{1}{2}\frac{1}{s-2}\right).$$

$$\boxed{x(t) = \frac{1}{6} + \frac{1}{3}e^{-3t} + \frac{1}{2}e^{2t}.}$$

So we will also solve, we can also solve a linear coupled, I mean coupled system of equations, system of ordinary differential equations you can solve, that we will see in the next video along with some special equations you can solve by Laplace transform technique, okay. Thank you very much.

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