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NPTEL ONLINE COURSE
Transform Techniques for Engineers
Review of Complex Function Theory
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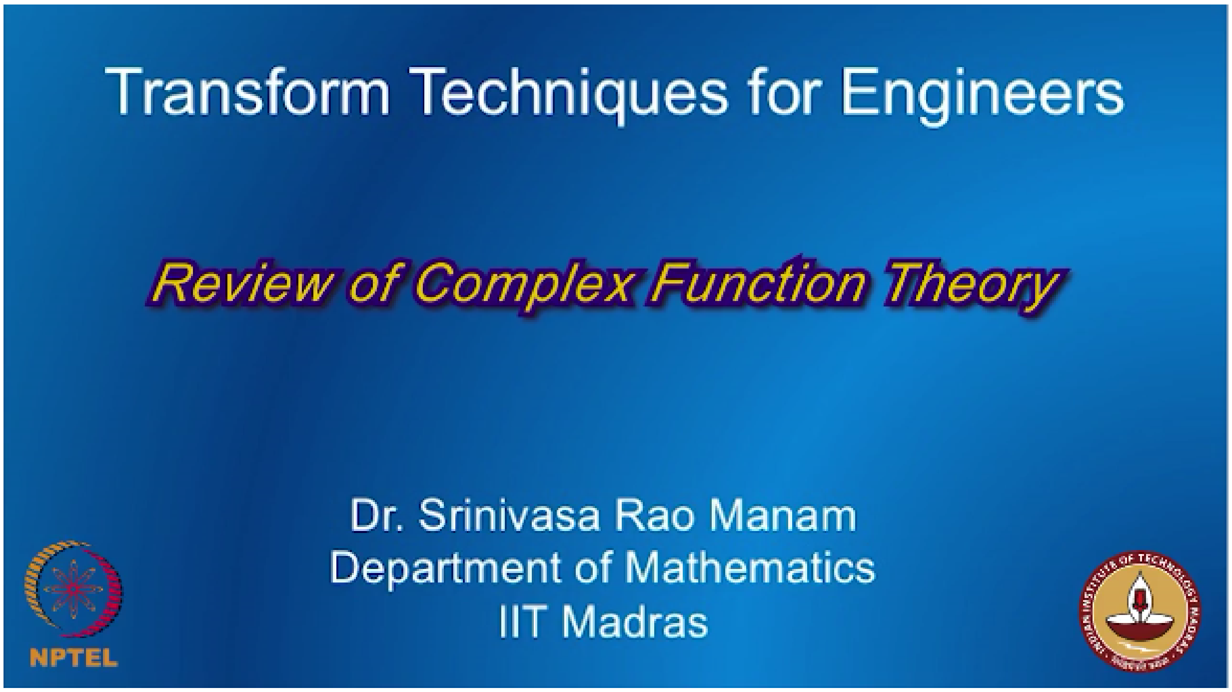


Welcome back, in the last video we have seen how to find inverse Laplace transform of elementary functions by certain methods, these are ad hoc methods so already you know because you know the Laplace transform of elementary functions and combination some simpler functions, so because of that using them and convolution some of the property for example convolution theorem which you use to find the inverse Laplace transform so far, so that is basically we have done by partial fraction method and other one is heavyside expansion theorem, so it's basically generalization of partial fraction method.

And the other one is convolution theorem which you can use, if you have a, if you look at the, if you have Laplace transform as a product of 2 Laplace transforms for which you know inverse Laplace transform so you take the convolution of that which is a property of the Laplace transform, by using that you can find the Laplace inverse.

And finally, so if you don't know any of that, if nothing works you cannot recognize, if we cannot use any of these properties to find the Laplace inverse you don't have any other choice, the general method that is finding using the definition of a Laplace in inverse that is F(t) is 1/2 pi times, so integral over a curve that is in the complex plane that is C-I infinity to C+I infinity, F bar(s) into E power ST DS, so this is a complex integration so this if you calculate so you'll end up getting residues, some of all the residues in certain special contour that is called Bromwich contour, that's what I have shown in the last video.

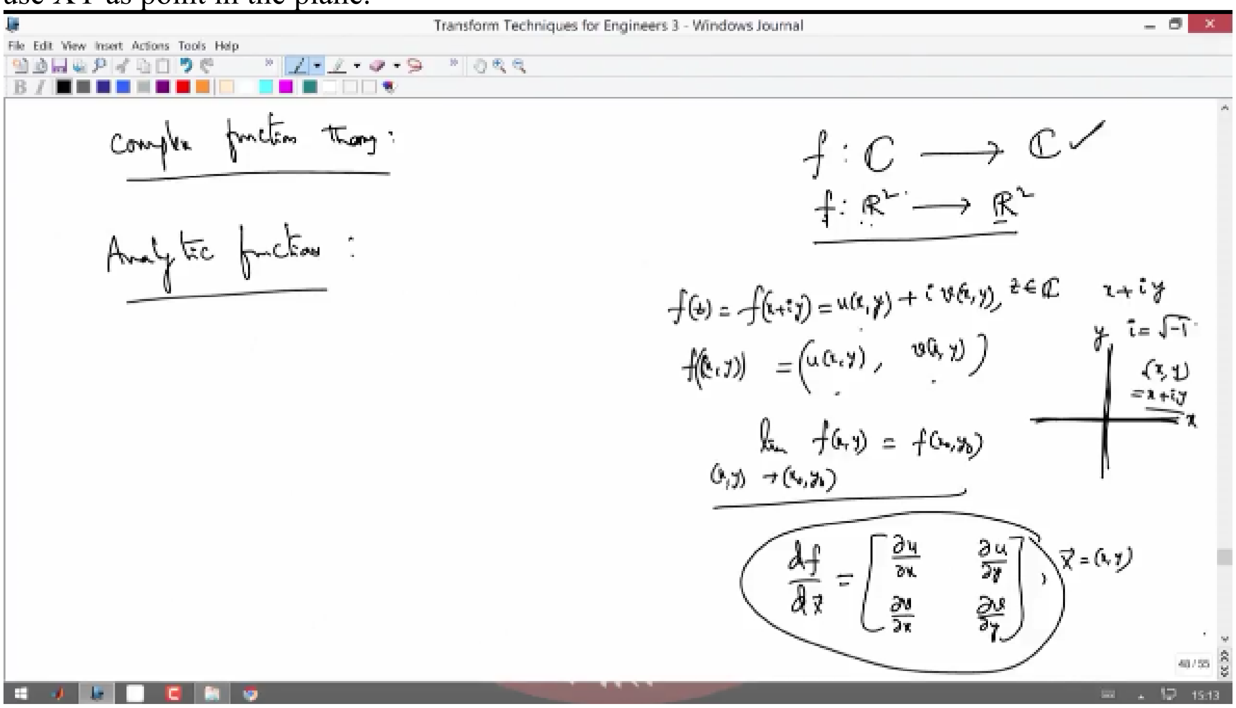
So within this contour if you actually evaluate the, you use the contour integration technique and evaluate this integral over this contour you will end up getting this integral, inversion integral so which is sum of all the residues at different values of S at those poles, poles inside this Bromwich contour that's how, so that if you know already complex function theory so you exactly know what I'm talking about , so this is basically finding all the residues within this Bromwich contour if you have all the poles, at this poles if you find the residues, contribution of these poles for the function F bar(s) into E power ST will give you the inverse transform, so that's what, that's how exactly you can get this Laplace inversion if you know complex function theory already, and if you don't know so let us review a next 2 hours let's review what is the, how we do this complex integration.

So let me start with complex function theory, so let me start with complex function theory, I'll only review this complex function theory, so start with analytic functions, so analytic functions, so what do you mean by analytic function let me explain, so if you have, first of all you should have function means, function it should have a domain that is usually you have a real line to real line, so if it is a real valued function, real valued function over the real line, so real valued function means it's a domain, the range is codomain or range is part of the real line so here and here you have a domain over which is defined, so F(x), Y = F(x) so F(x) if it's a real value, it's a real valued function of a real variable, X belongs to R, so this is how you represent a function.

So if a same way if I look at this, if I start with so let me say before I do it let me use this as a complex functions, so if you say complex valued if you put it from F is from R to C then you represent this F(x), X belongs to R as F(x) as complex valued functions, so because it's a F given any X F(x) is a complex valued function is, because it's a complex number you represent like this which is a plane, okay, so if you have like this F(x) because it's a complex valued function you call this U(x) + I times V(y), so let's you represent complex function as, complex value as X+IY, I is actually root -1, so this is how you represent this as a plane, complex plane, 1 is X, other one is Y, so that any point that is XY you represent this as X+IY, that's you identify every point in this plane as this complex number X+IY, that's how in that sense if I have a real value, complex valued function of a real variable X and this is the function, function value is real part and imaginary part with of course I is the imaginary part, so you have is called real part and this is the imaginary part of F, so this is how you have a representation for this function.

But if I remove this one here and put it as a complex value, so then what should be that case, I should simply replace here X/Z, so X/C complex plane and we don't write this as X, X when we usually we write for real variable so if I write this you put Z, so if you put Z here and what is that? It's a complex valued, F is a function of X+IY, so this is actually you write as F(z) and this is instead of U(x) you, what you get is real part as which is also depends on function of X and Y + I times V of function of X and Y, so this is how you represent this function if Z belongs to complex plane, okay, if F is from complex value to, this is called complex valued function of a complex variable C, okay, because Z is a variable, F(z) is also complex valued function because it has a real part and imaginary part, so this is how you see these functions from C to C, because C origin has a plane so you can also, you may also see that this is also from R2 to R2, you can also think of like this, F(x,y) as such you can think of this as F(x,y) which is U(x,y), V(x,y) if you think like this as your function X, Y this point is actually in this real line, this plane and then finally U, V that is also in this plane, so in that sense F is, F you can think of R2 to R2 real valued function of course it's the vector valued function you have 2 values, vector valued function of a vector, so you have a vector here, so F of this vector, if you view like this, this is a vector valued function of a vector in R2.

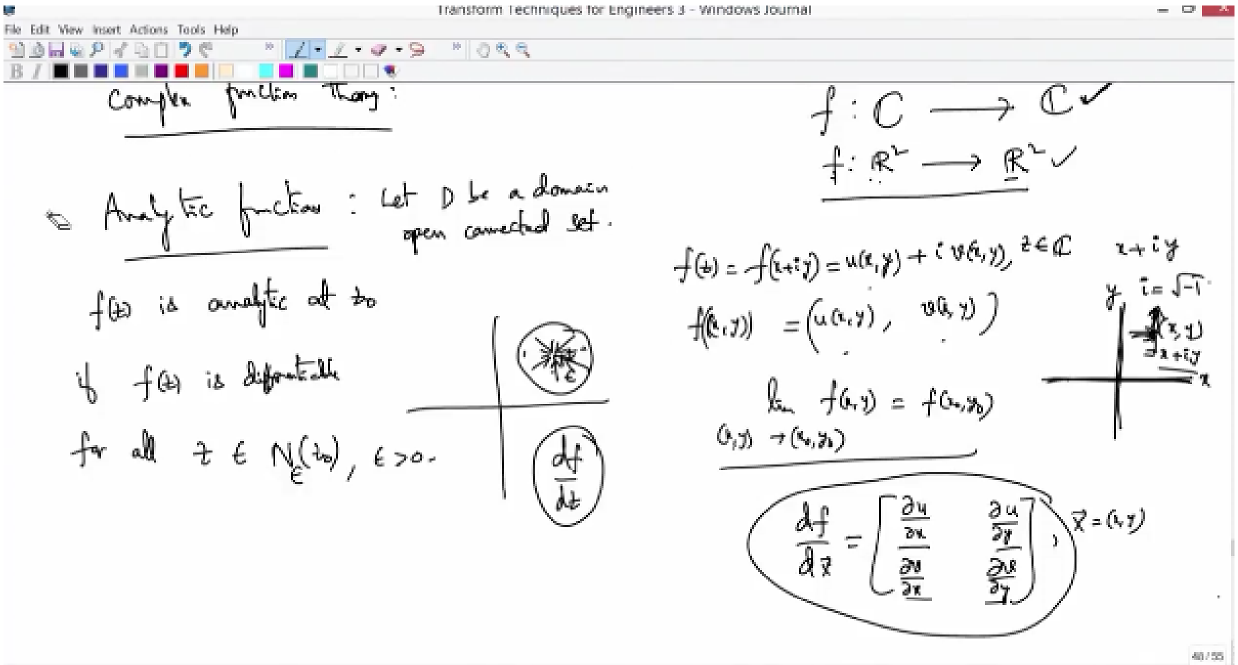
So we can see that both looks like same, but algebraically they maybe same that way but topologically they are different in that sense means, or if you see for example continuity limit, limits of this function these are all you can, usual way that you define limit of Z goes to Z naught means Z I represent as X, Y to Z naught if you call the X naught + IY naught this is going to be X naught, Y naught, so as this point goes to this point so like here so a some function of X, Y is same as F(x) so, if it is continuous this is usual R2 to R2 functions, functions of many variable, multi variable function, so in the usual way so you can, this limits continuity because they're all, continuity is nothing but limits, limits are all same, so topologically they are same so they are different, algebraically if I talk in mathematics language, so if you have, so topologically they are actually same but they are algebraically they are different, so that sense so matter that difference between these two, but even topologically only open sets, closed set they are same but when you talk about the differentiability they have a difference here, so what you have a differentiability from R2 to R2 is you have a derivative DF/DX, X is a vector, at X = X, X is here, now in R2 so this is actually from, this is matrix, that is a matrix is dou F/dou X1, dou F/dou Y1, so let's call this F1 and F2, F1, F2 is like UV, so let me use this DU/dou X1, here DU/dou Y1, here dou V/dou X1 and dou V/dou Y1. X1, Y1 or rather XY, XY if you call let me use XY as point in the plane.



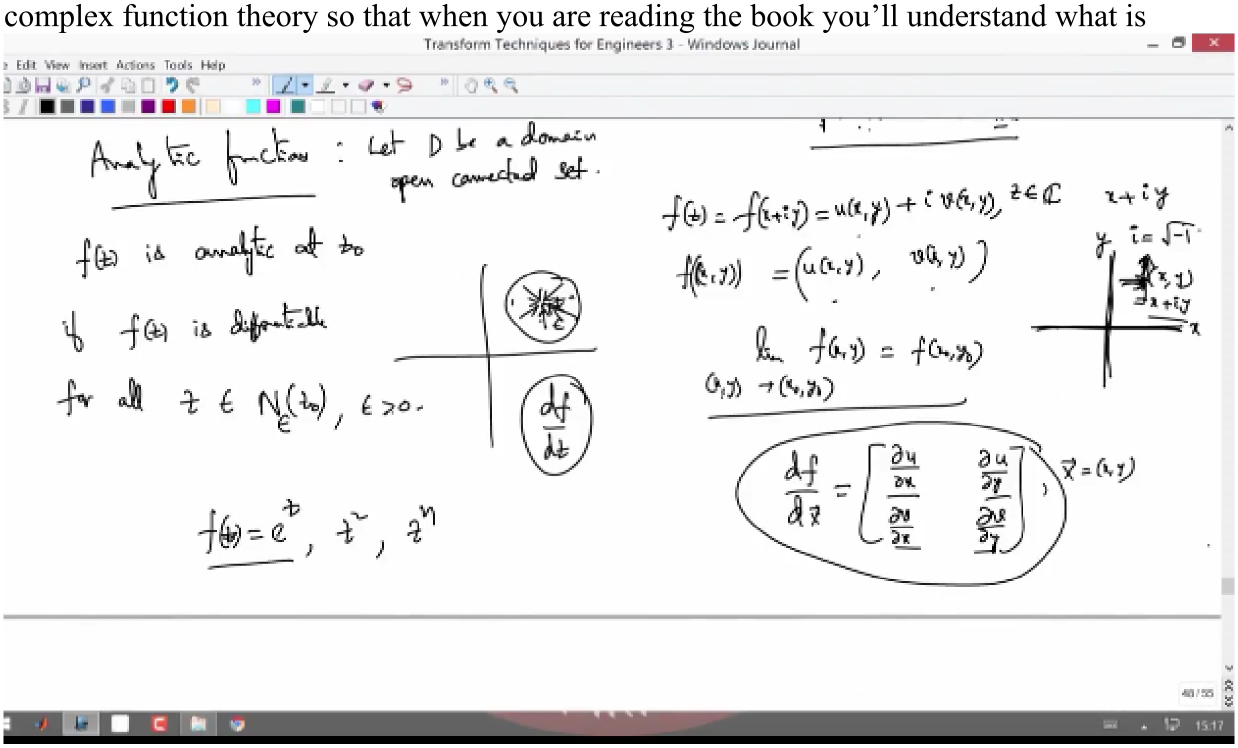
Then you have this XY is a, let me use this X bar, X bar is, where X bar is actually XY, okay, so this is how if this matrix, if this exists all the partial derivatives exists and if they are continuous, if they are exists, if they exists and as a matrix then you say that is differentiable here, okay, right from complex valued to complex valued there is a difference, it's entirely different, so if you have this partial derivative exists, partial derivative means at XY at some point you have this direction, this partial derivative for U, in this Y direction partial derivatives if you have then all this 4 derivatives exists then that is the derivative of F(x) in from R2 to R2 function, but if you look at from C to C it's entirely different, that is called analytic function, differentiability at a point means, at a point means this one, if it is differentiable so you have at a point here in the complex plane, if it is differentiable at this point and it is no other point it is not differentiable then you say it is actually same as this differentiability itself, differentiability from C to C is same as differentiability from R2 to R2, but in the analyticity so differentiability at a point and also every neighborhood of a point, if it is differentiable inside from C to C, then it's different, so how it's different from this derivative here, multivariable derivative?

If you actually see this one you view in all directions you have same amplitude and there is a small twist, amplitude so as if you view as limit, this is your Z, Z is XY, all directions it is possible, so you have only here, only 2 directions you need, but here in all directions you should have same amplitude, so the derivative exists means that is DF/DZ for example, so this exists at a point Z means in all directions you have a same kind of amplitude and only thing is it will have a twist, okay, so it's all directions, so let me I'll not confuse here, so here you need only two directions they should exists, here in all directions it should, the derivative should exists and it should have the same value, so that is where the difference between analyticity and differentiability from R to R is different, okay, so that's why we call this analytic function if you have a differentiable function F at a point in the neighborhood, every neighborhood of a point if it is a differentiable you call it a analytic function, so analytic function means F(z) is analytic, so before I do this let me let D be a domain, domain means open connected set, open set means so every point you can have a neighborhood that is entirely into it, it's a usual open sets in the plane like this one, any point inside I can always find open set, open neighborhood that is circle, circular disk which is entirely into it if it is inside.

So that's open, and you can take any point to, any point I can always connect it which is entirely, the line is entirely into the domain then you say that is connected, for example if you consider these two, this point if I take this point I cannot connect it by any curve which is entirely in this, because this part is outside, so this is not connected, okay, so open connected set means that's called domain, so if you have a domain F(z) is analytic, analytic at let's say Z naught if F(z) is differentiable, differentiable means usual derivatives, so derivatives exists at that point, not only at differentiable in every neighborhood of, every neighborhood differentiable for all Z belongs to the neighborhood of, some neighborhood of Z naught for some epsilon positive, so some epsilon neighborhood we take every point I can always differentiate around that point so as a, so in that sense one point means in the neighborhood of that, at some neighborhood at every point it is differentiable then you say that it is analytic, so this is one characterization of, this is the first definition of analytic function, and you cannot check all this every time whether the function is differentiable or not, okay.

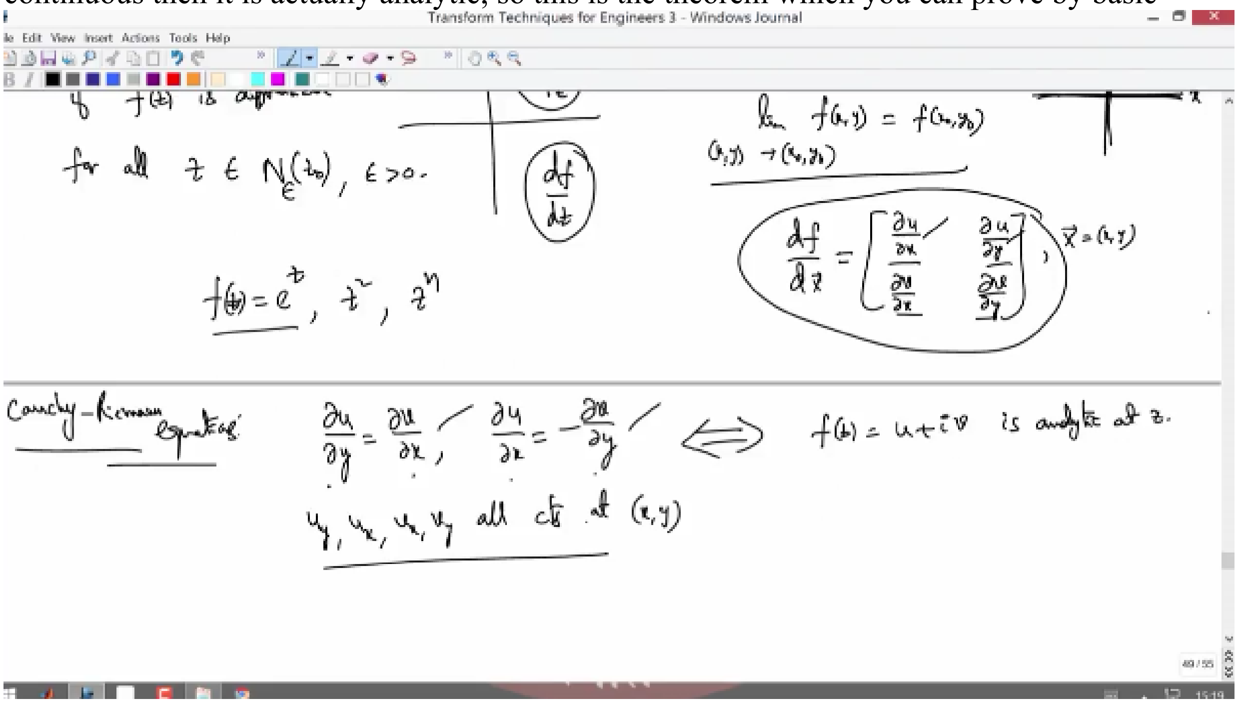


For example you consider F(z) = E power Z, okay, this is differentiable function or polynomials Z square, Z power N, okay, so like this sine Z cos Z so these are the things you have to define, I hope you can just go through some books and for example what is that book called? Churchill you can look into introduction to complex variables by or with Churchill, okay, Churchill you can read and by reading you'll understand, so I'll give you the briefly the review of all this complex function theory so that when you are reading the book you'll understand what is



actually happening, much of the technical details you can find in this text book, okay, so this is one basic definition of analyticity to check whether the given function is analytic you don't have to do this every time to this definition, this fundamental definition instead you have

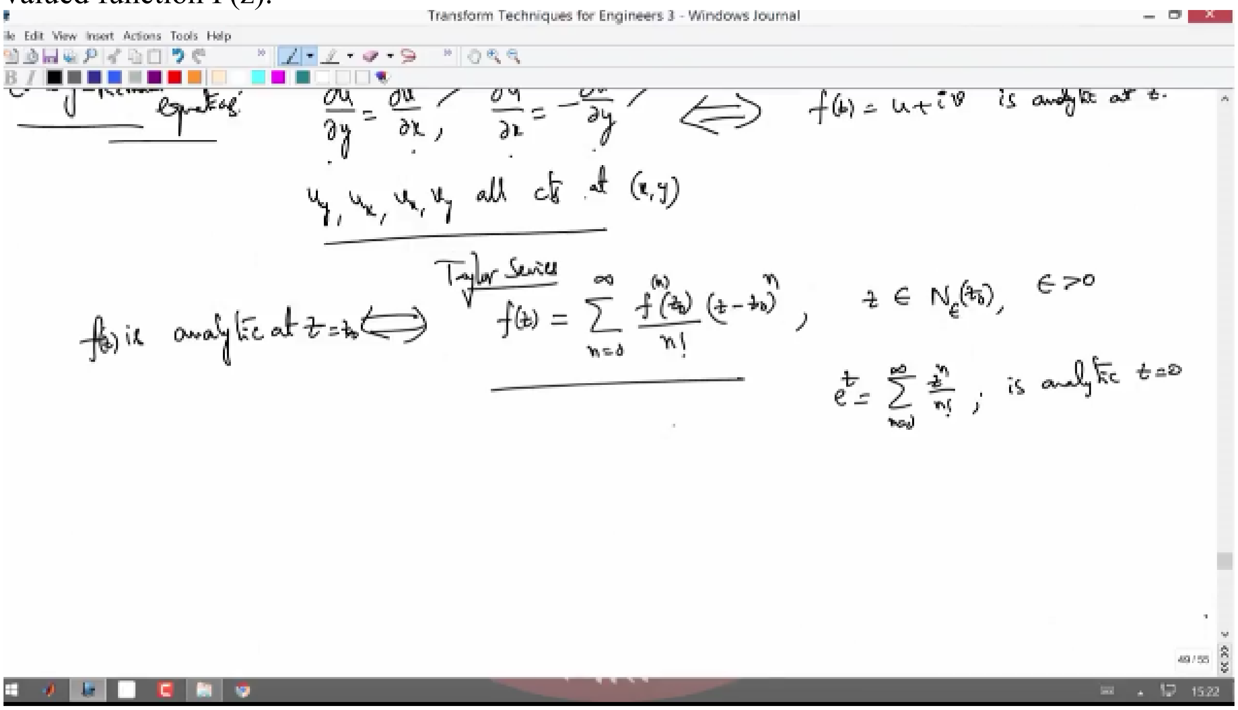
another characterization that is called CR equations, CR equations because you can always find partial derivatives, this partial derivatives, given F(z) I can find U(x,y) and V(x,y) so UV you can find the partial derivatives as a function of 2 variable X and Y, as a real variable so if you do that dou U/dou Y = dou V/dou X and dou U/dou X = -dou V/dou Y, so if these functions, this partial derivatives exist and if they are continuous and satisfying the CR equations, okay, they exists and dou U/dou X so let me call this UY, UX, VX and VY, all continuous partial derivatives exist and they satisfy this if and only if and F(z) is, which is equal to U+IV is analytic at Z, here also you have to calculate at Z which is equal to X+IY or XY at XY, okay.
At XY if you say that is same as so this is the characterization of through CR equations, this is Cauchy Riemann equations, let me write this is Cauchy Riemann equation so equations, these are the equations if you can check this and you see that this partial derivatives exists under continuous then it is actually analytic, so this is the theorem which you can prove by basic



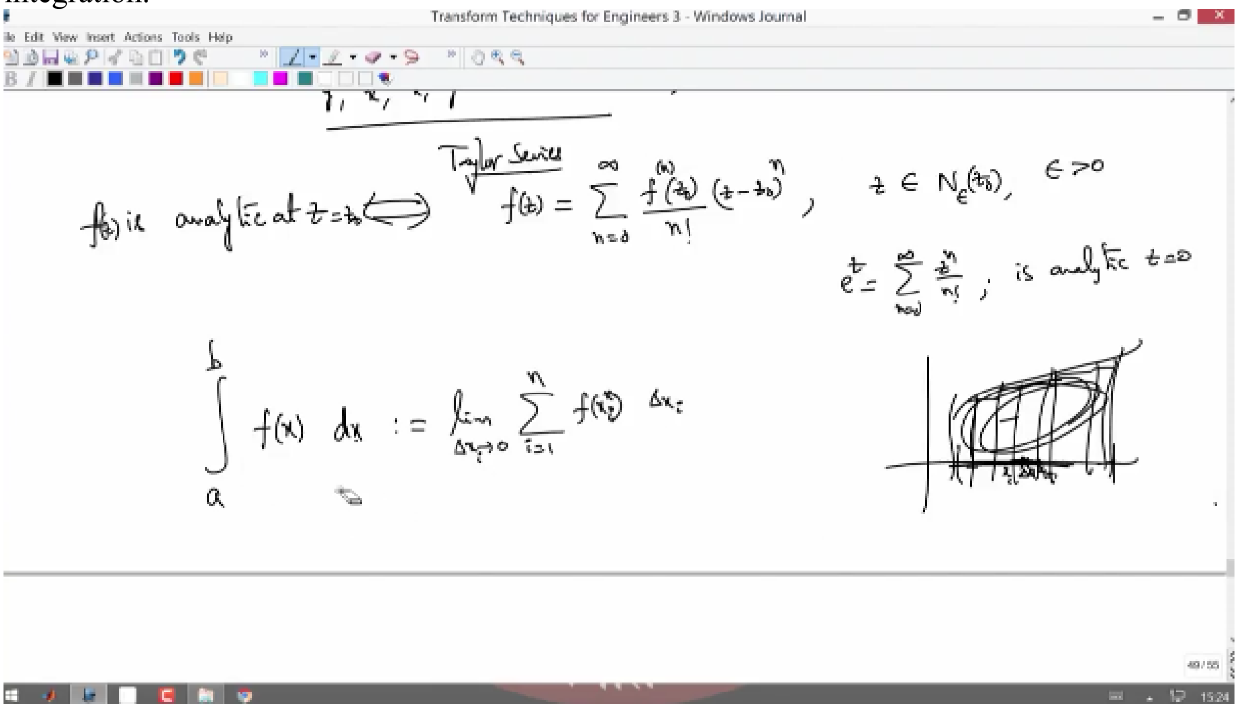
means, so if you look into this books you can understand, this is one of the characterization to check the analyticity of the complex function, any complex valued function of a complex variable F(z), so F(z) is a complex valued function of a function, complex variable Z, so this is one and what you need is for the Laplace transform, inverse Laplace transform you need to integrate, so for the integration what you, we'll start the integration what exactly the integration, so what we have is analytic functions we have 2 characterizations, one is this analytic basic definition, other one is that CR Riemann equations, there is one more characterization that I'll come to that when we need this, okay.
So maybe I'll give now itself, so another characterization is if F is analytic, if and only if at Z naught for example, okay, at Z, if you want at Z if you take if and only if F(z) is analytic at Z, if and only if, if I have F(z) has a Taylor series just like you have a Taylor series, what is the Taylor series which you can easily see N is from 0 to infinity, this is F N derivative set Z, okay, N = 0 that is F(z), so let me use at Z naught, Z = Z naught let me put then you have this Taylor series around this, so F of this is derivatives at one point divided by N factorial into Z-Z naught power N, so if you have series representation like this for every neighborhood of, some neighborhood of Z naught, if you have for some neighborhood of Z naught if you can have this

representation, this Taylor series then you say that this is also analytic, so this is another characterization, okay, if you can have for example E power Z which you can see that N is from 0 to infinity this is like Z power N divided by N factorial, so around 0, this is around Z naught, okay, F(z) = F(z naught), N = 0 when you put + F dash(z naught) divided by one factorial into Z-Z naught and so on.

So this is exponential function is in the neighborhood of 0 you have, this means you have this that means you have Taylor series around 0, Maclaurin series, so that means this is analytic, analytic at Z = 0, okay, so that means in the neighborhood of 0 it is also analytic, so that way that is how you see the three characterization of this analytic functions, where the first one is the definition, second one is the Cauchy Riemann equations, and third one is this Taylor series, so you have a Taylor series, okay, Taylor series for the Laplace, for the function, complex valued function F(z).



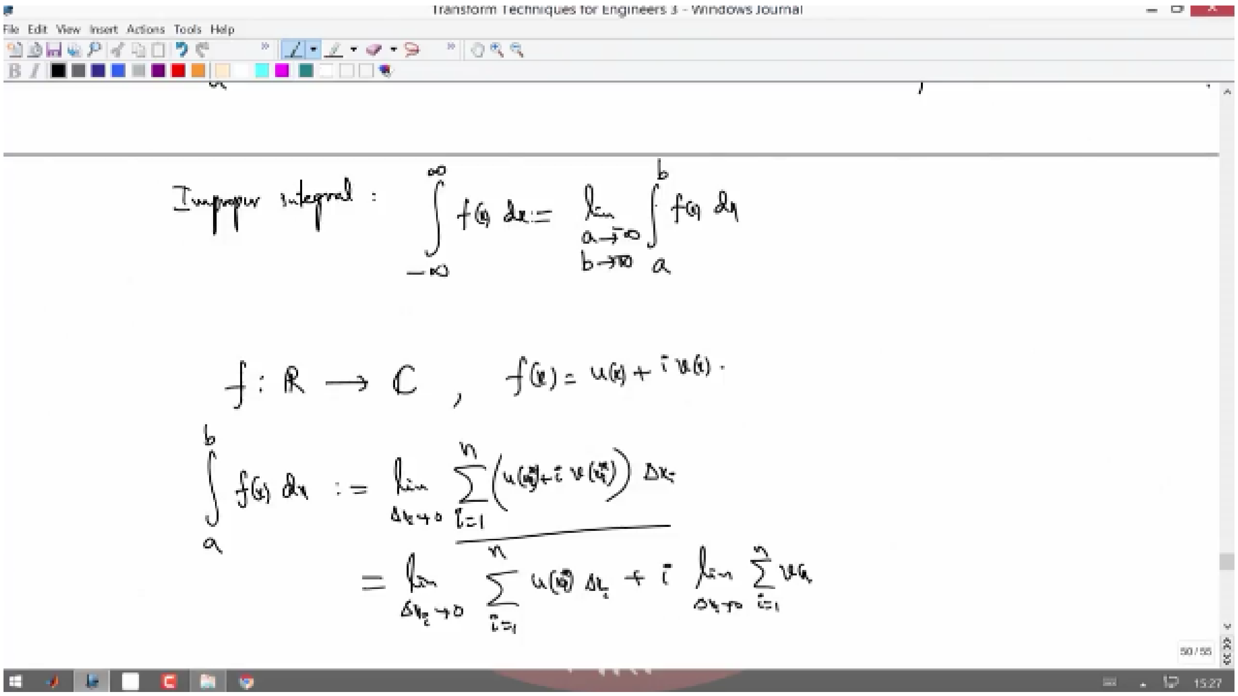
Once you have this three characterizations, let me use what is that integration, integration of, so what you mean by integration, what you know integration, integration what you know is if you start with the real valued function F(x), if you have F(x) let us say it define on A to B, A and B X belongs to AB, so that means it’s a real valued, X belongs to AB, so it’s a real valued function, so real valued function or it can be complex valued function of a real variable, F(x) is it can be, we don’t know so first of all you know that is a real valued function on a real variable, so this is by definition if you actually see what do you mean by this is, area under the curve, so this is the curve you can represent this as a curve between AB, you break this into, so equal parts as a rectangles and finally you take this epsilon, delta X goes to 0 you sum it it’s like area under this curve is exactly you see this as a limit, delta X goes to 0 and what you have is this summation F(XI) okay, F(XI) times XI – XI-1 so that is exactly your delta XI, okay, so this XI star is one of them, so this is between this here, somewhere here, if this is XI, this is XI+1 these two points in between you can take any point or XI itself, any point you can take on this that is this, and I is running from 1 to N, so like that you finitely make and finally allow this delta XI, biggest one, biggest one of the maximum of them, maximum of all the R, anyone of them and all of this each part has to go to 0, if we choose this you see that this is going to be the definition of this area under the curve that is exactly you will end in the calculus this is your integration.



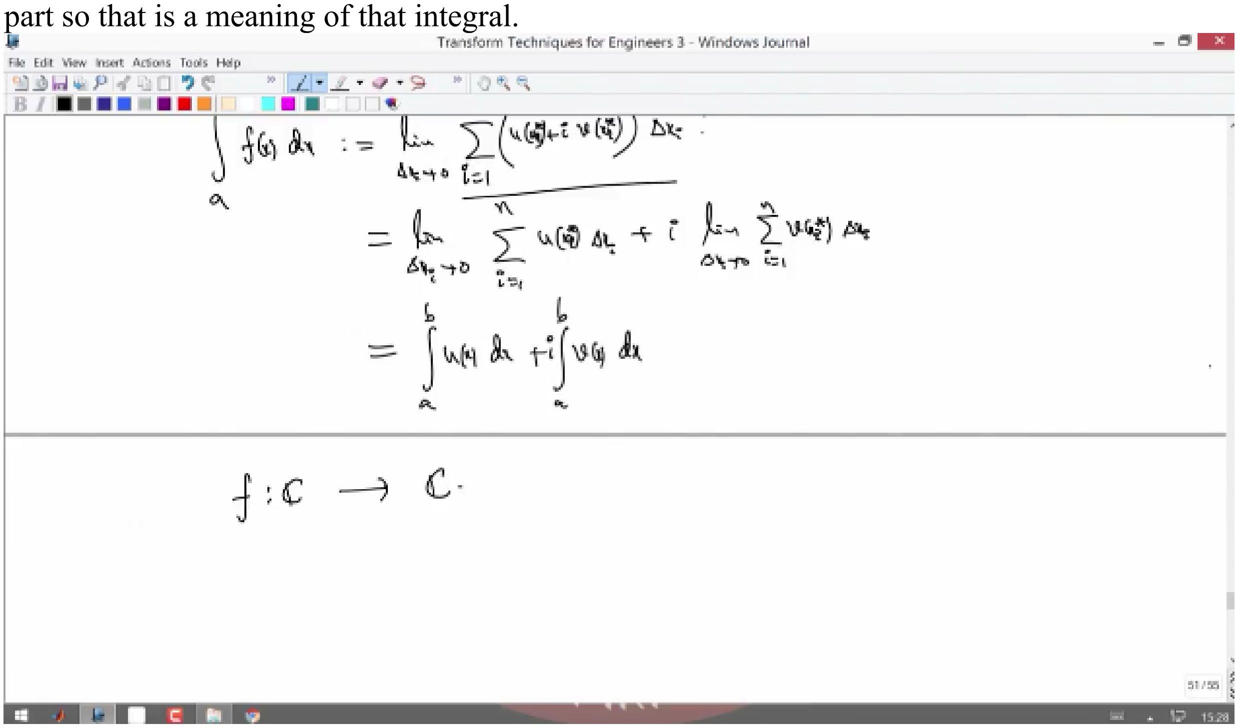
So if I have, if once you have this, this is the Riemann integral this is the finite integral so there is something called improper integral, improper integral is you also know that what is improper integral, improper integral means it’s going to be from -infinity to infinity or 0 to infinity, finite infinite or –infinity infinity, you view this F(x) DX as a limit of a finite sum, so for example 0 to A F(x) DX, this meaning I know from this definition, usual integration definition of a real function of a real variable.

Now what I do is this means by definition this is actually as A goes to infinity this is the meaning, so a same way if you have –infinity infinity you allow B to A or A to B you put it here and you allow A goes to infinity and B goes to, A goes to –infinity and B goes to infinity, okay, so like that if you can view this is well defined, now finally you look at this limit and what you see is this is the improper integral, so this is what you know from calculus, this integral.

Now if I have function from real variable to, but it is a complex valued function so that means F(x) is U(x) + I times V(x), so what is the meaning of this integral? F(x) DX of A to B, this is again same way you can do but by definition this is actually, you split this into A to B U(x) DX + I times A to B V(x) DX or you put it in the same definition because of the finite sum you can always do, so let me, so by definition you do the same things, as a limit delta XI goes to infinity, this summation I is, K is, whatever I used, I is from 1 to N, if I use delta XI, here F(x) I star for example, if I use this F(x) star if I use this U so this delta XI here and U(XI star) + I times V(XI star) if I have this, this is a finite sum you can split into two parts, okay, so this is limit delta XI goes to 0 this summation I is from 1 to N, U(XI star) into delta XI + I times, again limit delta XI goes to 0, so this is legitimate, and we only have a twos, you can split this sum, okay as V(XI star) delta XI, so what you have is finally this is actual usual definition

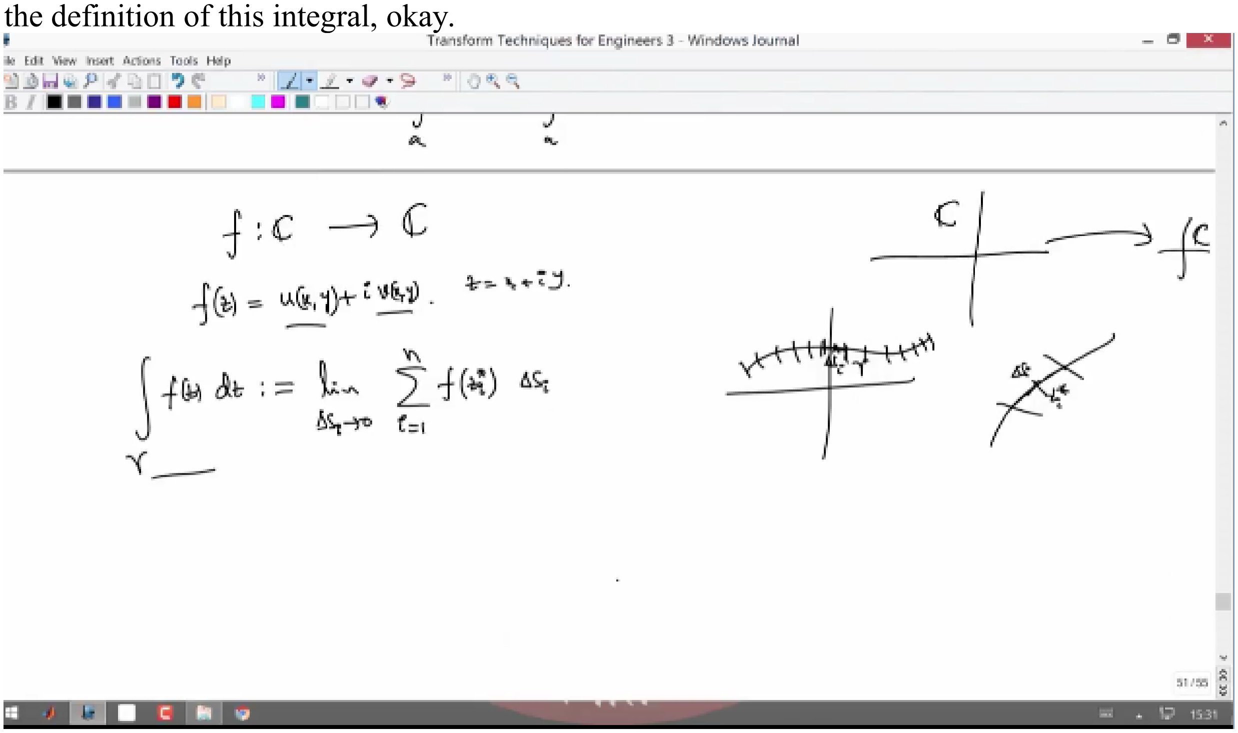


which you can generalize for a complex valued function, but finally you end up getting integral A to B U(x) DX + integral A to B I times integral A to B V(x) DX, so you can take this as your definition directly, so it is actually true so what you see is if you take this usual way, usual definition way if you do with this definition if you start you end up getting this integral, so this definition, this means complex valued function you simply integrate of a real part, imaginary part so that is a meaning of that integral.

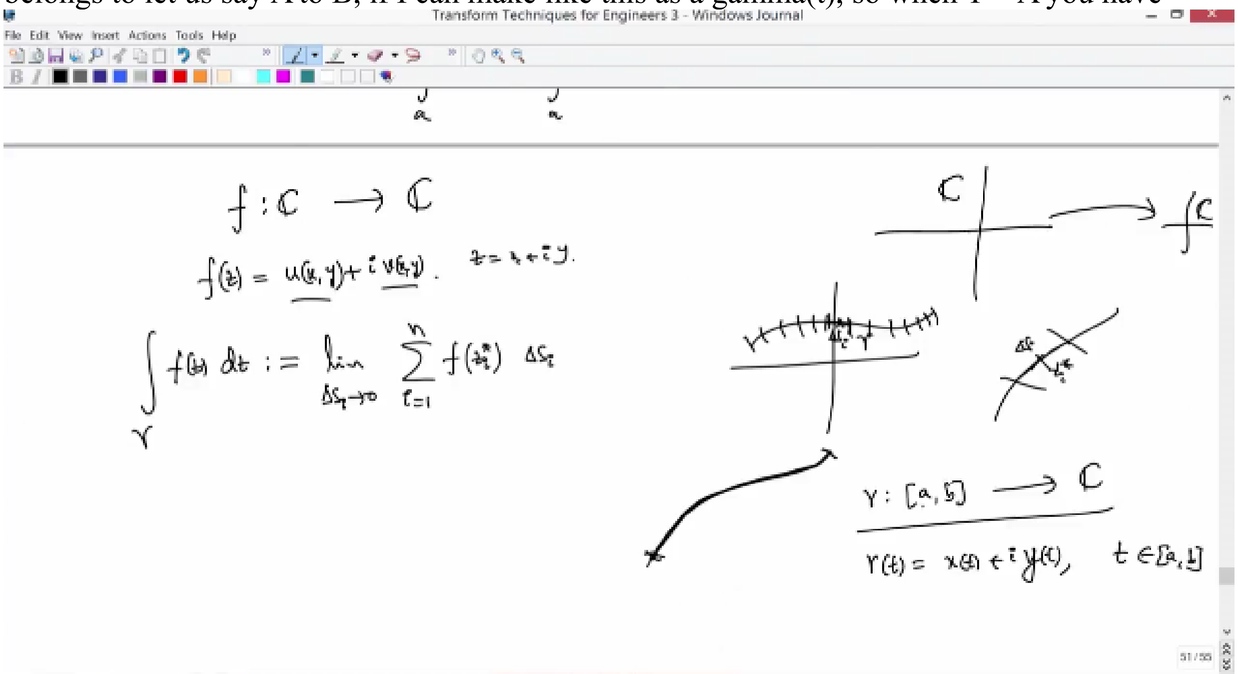


Now if I have integral of, integral over rather F is from C to C, so what you do here? What is the meaning of C to C means what? C to C means or rather let me do it from C to R first, C to R, so what is the meaning? So you have a function defined over this and it goes to real line R,

okay, this is complex, okay, complex to real valued, so that means you have F(z) which is equal to simply some function of U(x,y), that's it, Z is basically X+IY so you take this X and Y put real valued as U(x,y), imaginary part is 0, so you can also include C so that this is I times V(x,y), so you have a complex valued to complex valued if you have this is what you have, okay, this is how you have a function.
Now what is the meaning of F(z) DZ over there is no meaning of real, because it's not a real line, I don't say it is from A to B, so this is over you have a plane, the complex plane let me use some curve, this is your curve, this is the curve let us say C or let us say gamma, so gamma is the curve in the complex plan, so over gamma so you can define this as, so how do I define this? You need to parameterize this as gamma, so first of all you can represent this as a definition because is a real and imaginary part you first write like this U(x,y) DZ and this is over gamma + I times over gamma V(x,y) DZ, okay, so this is the definition, this is you can do the same way, so how do I do this? Okay, let me do not the definition I give directly, so like here I start with some finite sum so this is the usual way, you break this as a sum and then finally you take the limit that is going to be, like this same definition you can apply here.
So what do I do? You break this gamma as equal parts for example, if you break this into equal parts by definition this limit let me choose this delta SI, so delta SI, SI is arc length for example, delta SI goes to 0, so this is the starting definition, fundamental definition and you break this into finite pieces, so over each of that finite pieces I, let us say I is from 1 to N, so if you make this delta SI and then you have F at some ZI star, so ZI belongs to here somewhere in this piece, on this piece, if you have a curve which you have a piece here at this point I'm calling this ZI star and this length is delta SI, so again when you choose this delta SI goes to 0, so like that you break pieces into pieces, pieces, pieces and finally you add it up that is exactly the definition of this integral, okay.

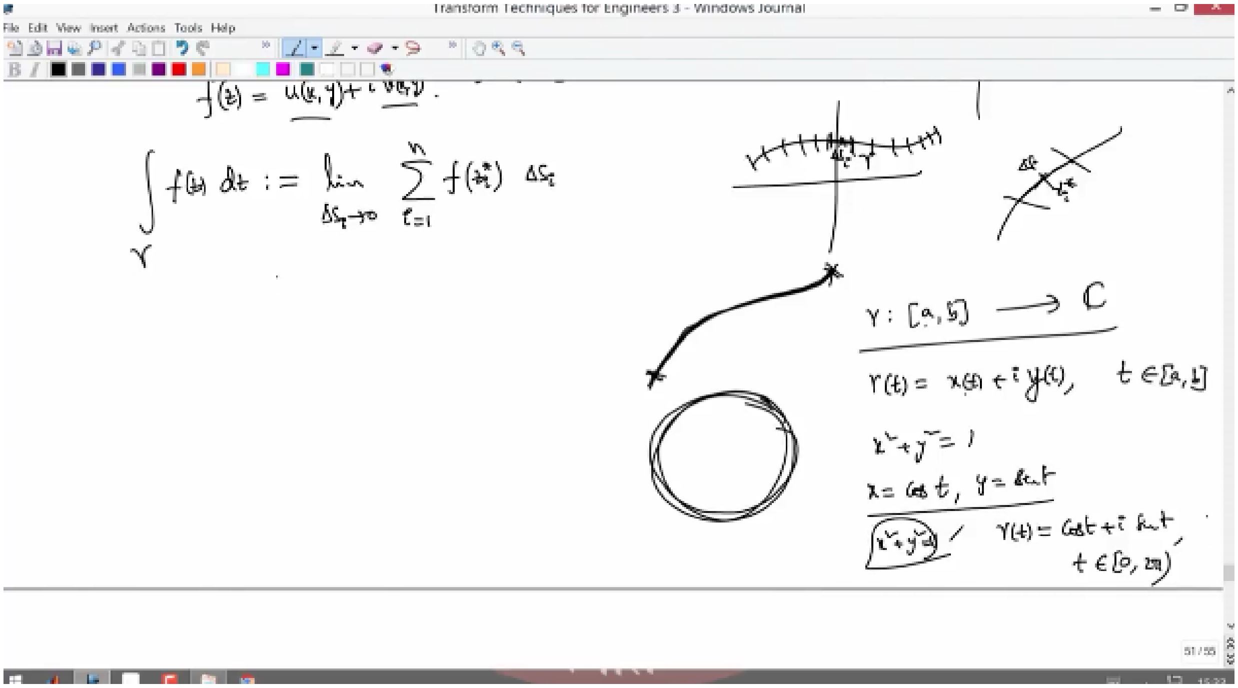


So by manipulation what you end up is if you actually use the parameterization this curve, so any curve C you can parameterize this is actually Z of, this is Z right, this is the curve gamma, gamma is from, so gamma is a curve in this plane, okay, in this complex plane so parameterization means I try to make a mapping from some AB to the complex plane, so at A this is the point, at B it’s a real valued, complex valued function of a real variable that is exactly the curve, so as you move from A to B I move on from, on this curve from this point to this point, okay, so if you can make like this gamma, so gamma T which is X(t) + I times Y(t), so T belongs to let us say A to B, if I can make like this as a gamma(t), so when T = A you have



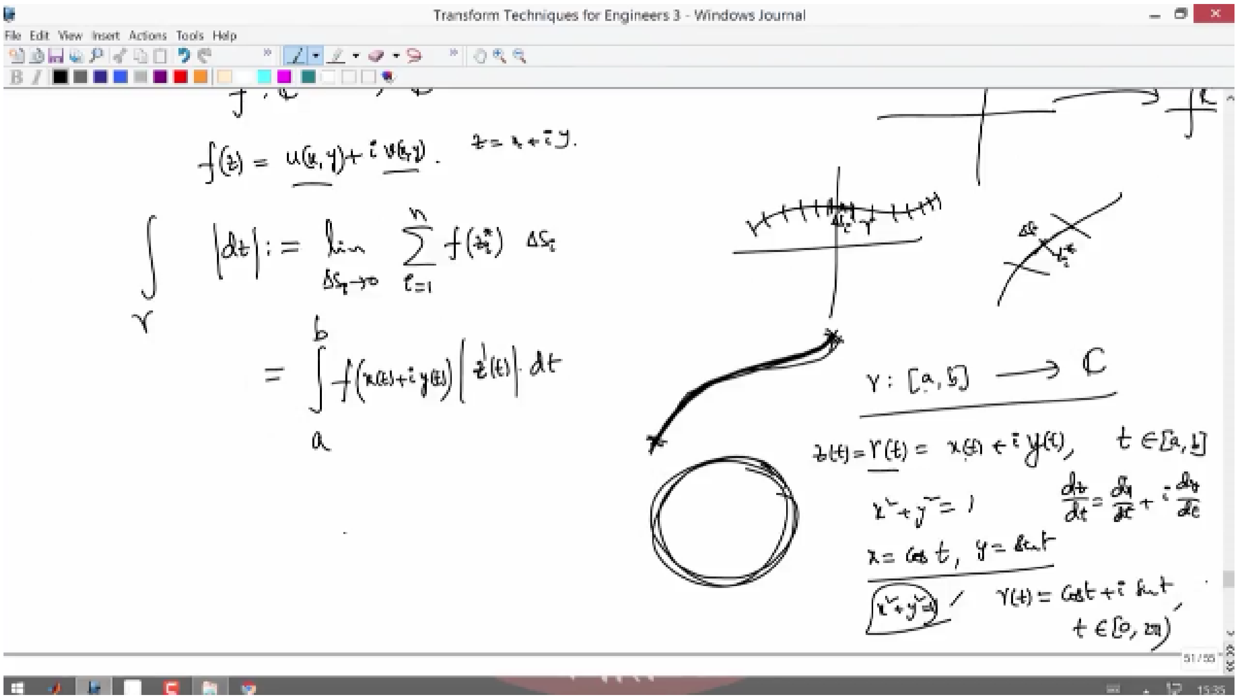
X(a) Y(a) that is this point, when T = B, X(b) Y(b) that is this N points, so as you move from A to B this X(t) Y(t) moves along this curve.

For example you take the circle X square + Y square = 1 is the circle, unit circle, so if I can write like this as a cos theta Y equal to let me use cos T, Y = sine T, then what is this one? This is actually your representation of the curve, so you have with this representation clearly X square + Y square = 1 so this is your curve, and so you have a parameterization is gamma(t) which is a unit circle which is cos T + I times sine T, so T belongs to 0 to 2 pi, so this is your parametric representation of this circle, okay, the complex plane.



So the same if you do eventually I'll not look into the details, so you look into the text book that is Churchill, I suggest you to look into the Churchill you finally see that this is actually equal to, once you have the parameterization, so you see that now the gamma is from this, from the complex plane is your domain, now you bring it it's over AB, so you'll go to AB parameterization F, where Z is there you put this parameterization, okay, Z is the gamma, gamma means Z belongs to gamma, gamma is basically Z, so Z(t) is X(t) and Y(t) so you write X(t) + I times Y(t) in the place of, what is DZ? Because Z is this, Z(t) and you have DZ by, DZ you need DZ/DT is same as X dash(t). DX/DT + I times DY/DT, okay, so DZ is simply Z dash(t) that is DZ/DT into DT, if I multiply with the DT that is exactly your DZ, in the place of DZ, okay, so this is actually one can show that this two are same, okay, so let me not show that so you can look into the thing, books, and get it.

So this is exactly, and this is the complex number, okay, so this is Z dash(t) DT is your DZ, and if you have this modulus here what you do is you put this modulus, Z dash(t) into DT with modulus you can replace, so if you want to have curve length you take a curve length, then what you have is this mod DZ is a curve line that is actually same as, you remove this F, mod Z dash(t) DT and whatever is, T is parametric value of the curve, okay, so let me not use that this

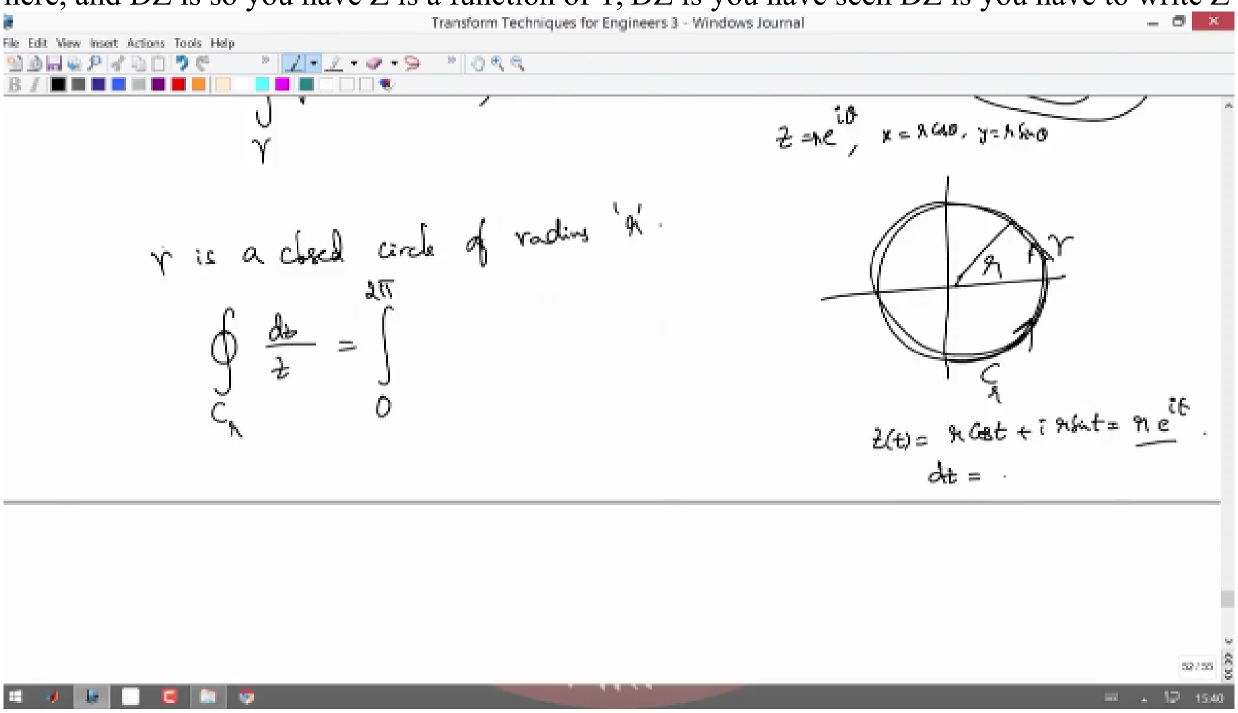


is anyway F(z) DZ, this integral this is the meaning so this is the definition, this equivalence you can easily see just by basic definition you can come finally here, okay, because of this parameterization.

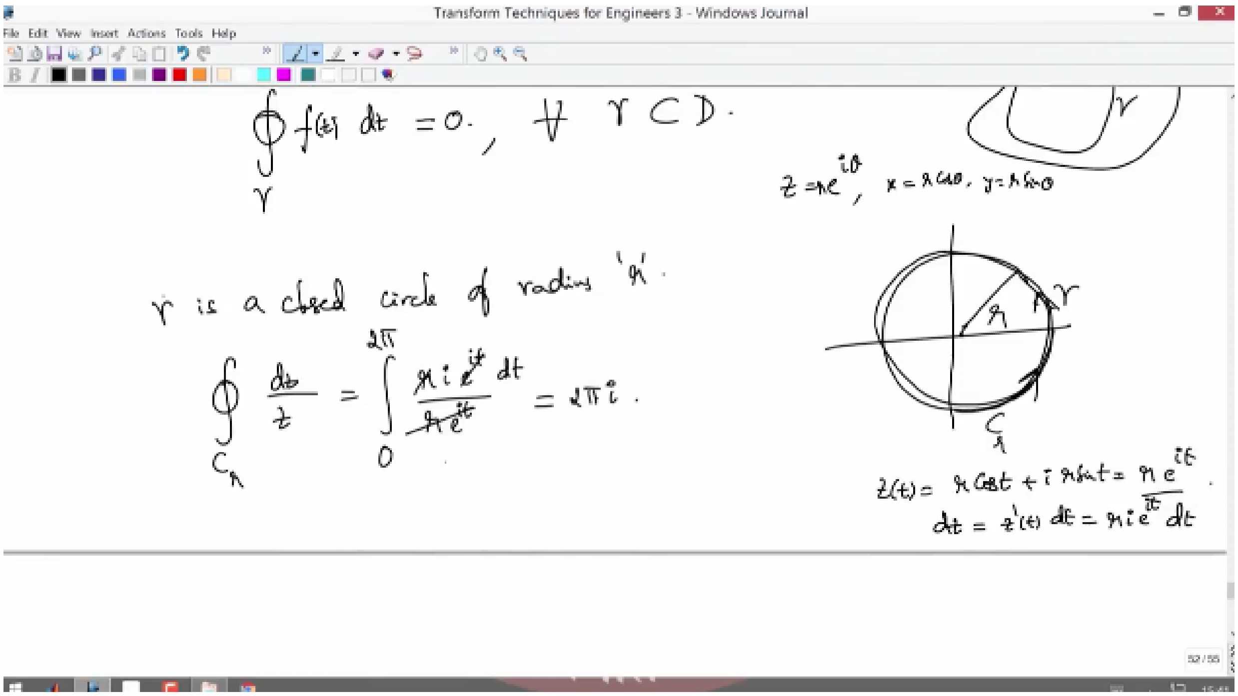
So if you have a complex valued function and a complex variable and so that is over integral means, integral means over a curve F(z) DZ meaning is the integral over its parameterization from A to B and this is what you have, so like this you can once you have this meaning for the integration of, integration over a curve, so this curve maybe either open curve like this or closed curve like this any curve, but we assume that is smooth, because we are differentiating, it is differentiable, continuously differentiable parameterization if you have that is, so that you can use here in the place of DZ, I'm using Z dash, if you are assuming that Z dash(t) exists and it is continuous so that this integral is, that is inside this integral, if you know that is the continuous function you know that integration is guaranteed, so it's well defined, so for that reason we assume that, we assume this curve is continuously differentiable curve that is also called smooth curve, okay.

So if you take such a smooth curve gamma we have this meaning for this integral, so once you have this meaning for this integrals, so what we have is, let me give some results of complex function theory, basic important thing is Cauchy theorem, that theorem tells you that if analytic function, what happens to this integral, okay, so F is analytic function so F(z) is analytic, analytic in a domain D, in D, okay, in a domain D let us say, so if you have a domain means open connected, and gamma B a closed curve in D, so you take any closed curve inside this gamma, okay, then integral of F(z) DZ and this is over gamma because it is a closed curve I give a notation like this, I put a round on the integral sign this is actually 0 for every gamma inside D, so this is the important theorem this is called Cauchy theorem, so this is the basic thing, basic fundamental result in the complex function theory, so using this everything you can derive, okay, so using from this you can derive, so once you have this, once you mean, once you get this meaningful definition of this integral you have this Cauchy theorem, and what you can calculate, for example if your gamma is let us say if gamma is closed circle, circle of radius R, let us say R okay, if this is the circle of radius R if you have with center at 0, okay, radius R, what is this? This is your gamma, gamma is this and another convention we use is if it is a closed curve we always choose this orientation you go from this side to this side, this is anticlockwise that is a positive orientation because as you move along the curve domain that's inside the closed domain which is on your left hand side, okay, so this is the positive orientation, if you come from the negative side that is a negative orientation, so if you come from this side it's going to be –gamma, gamma is a positive orientation thing.

So and then one of the property of the integral is –gamma F(z) DZ is actually equal to minus of F(z) DZ over gamma, so if you go over negative direction, positive direction the only sign change so that is what, that is a result you can easily see, so if gamma is such a close curve can I find this DZ/Z-Z naught, certain functions, if F(z) is 1/Z over a closed curve let us say CR, CR is my closed curve, what is this value? Okay, so let me find this value because I know that is parameterization this closed curve is, circle is Z(t) is R cos T because radius is R, R cos T + IR sine T, okay, so if I do this it's going from 0 to 2 pi, DZ is, what is DZ? This is also called R times E power IT, so this is another representation for this complex number so that is from Euler, Euler representation Z you can always represent E power I theta, where X = R times E power I theta, so X = R cos theta , Y = R sine theta, so because of this I have this representation here, and DZ is so you have Z is a function of T, DZ is you have seen DZ is you have to write Z

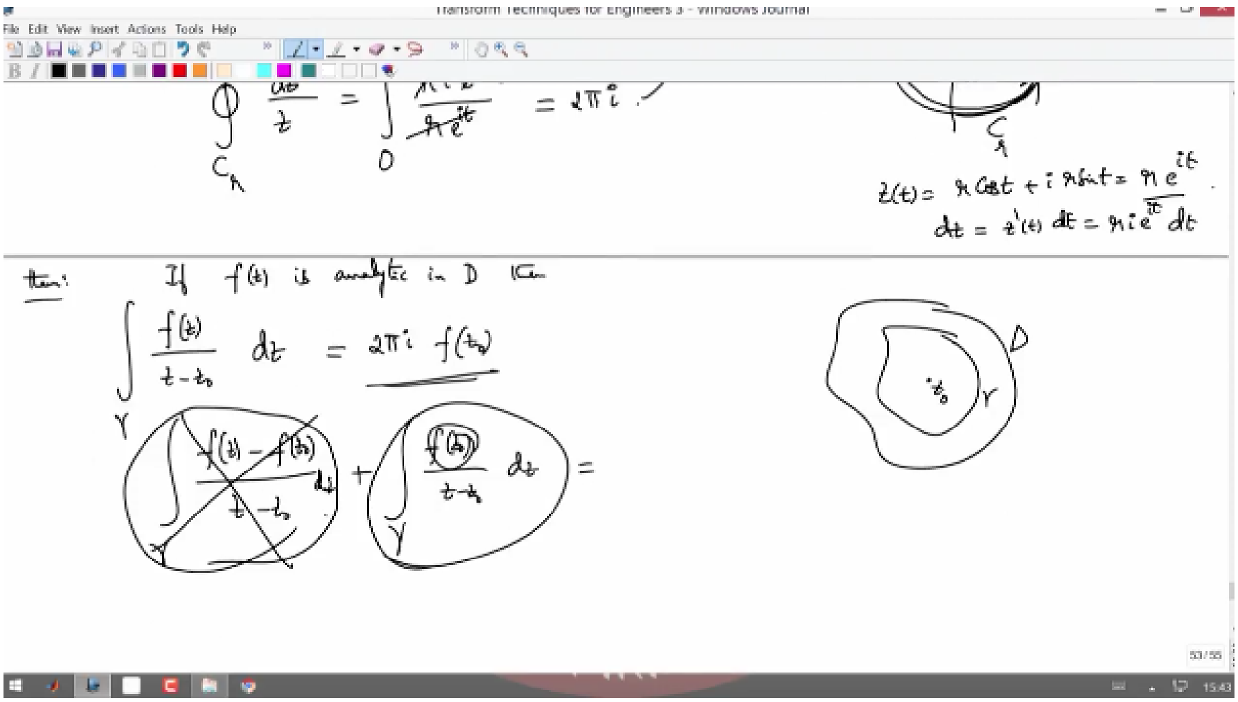


dash(t) into DT, right, so this is Z dash(t) is R times I times E power IT into DT, so you write this in the place of D, RI E power IT DT divided by, R is, R into E power IT, so this gets cancel, what you end up is I times into 2 pi I, so 2 pi times I, so I comes out integral 0 to 2 pi DT is 2 pi, so finally with 2 pi I, okay.



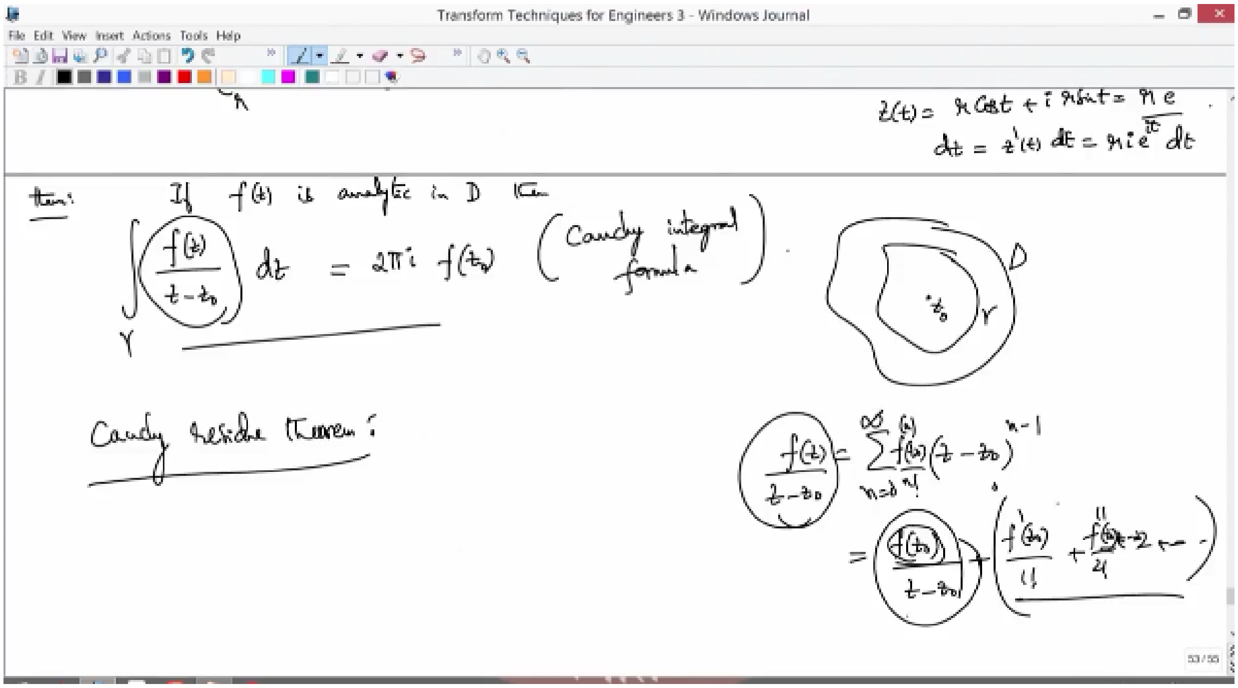
So you get like this, so can over because of this, because of this easily you can evaluate over a circles some integrals, okay, because of this integral we can have, if F is analytic in a domain D okay, and gamma is a curve in this D is domain inside, gamma is this curve and suppose it is not differentiable, it's not analytic at some Z naught, okay, suppose it is not analytic at Z naught, right or so F(z) is analytic so let me use this F(z) as analytic everywhere inside D, okay, and I simply put as integrand Z-Z naught where Z naught is here, so that means if this is, if you look at this integrand this is analytic everywhere except at Z naught because of 1/Z – Z naught, so 1/Z-Z naught is analytic everywhere except at Z naught, okay, so this is 0, so this is over gamma DZ, if you want to find evaluate this where if F(z) is analytic, analytic in D, okay, then this is the result, so this is the theorem of complex function theory based on the Cauchy theorem you can and this simple result we can show that this gamma is actually to 2 pi I times F(z naught), okay, so this is easily you can show by rewriting left hand side here, F(z) I rewrite F(z) – F(z naught) divided by Z- Z naught + F(z naught) divided by Z-Z naught.

So now I integrate over this gamma and gamma DZ, DZ, I will show that it's actually this goes to 0 and this is the value what you end up is actually this one 2 pi I times which I use here, 2 pi I times F(z), F(z naught) is a constant that comes out you use this, so only thing here remains in this theorem to prove is this integral has to, you can make as small as possible, if it's a closed curve, okay, so with that idea we can actually get this result, this is the Cauchy integral formula



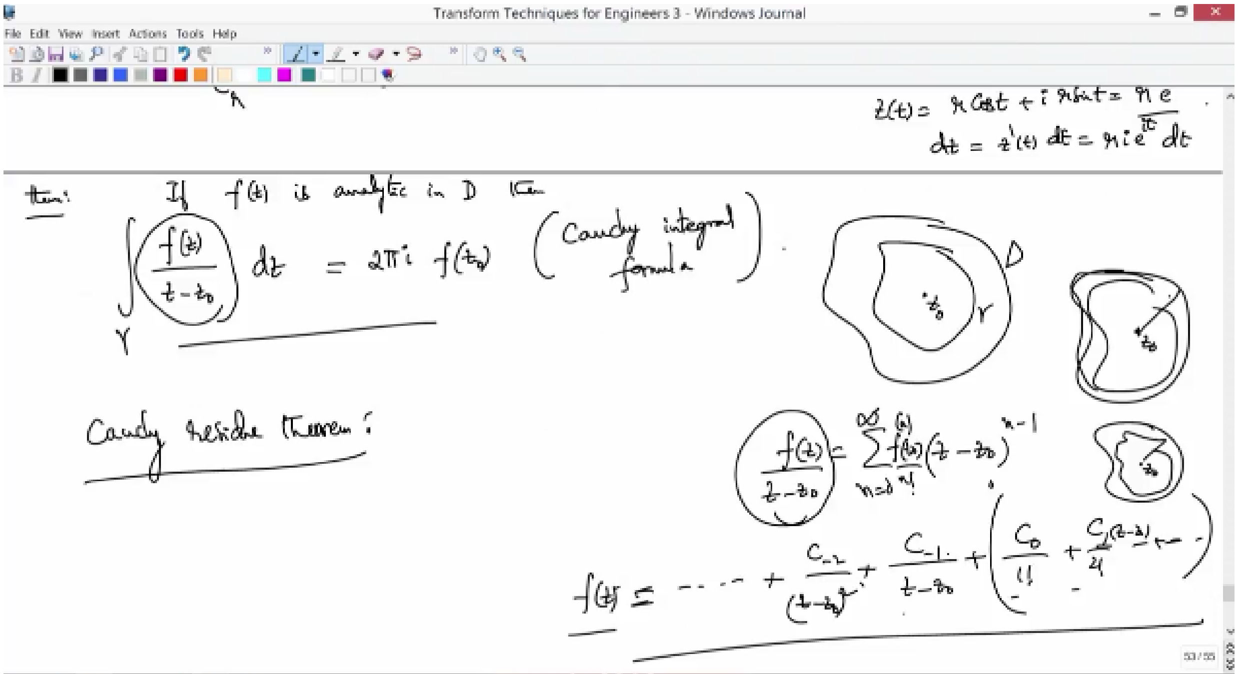
so this is Cauchy integral formula, so this is another result which you can use, which you have in the complex function theory and also from this you can easily see there is a Cauchy residue theorem, okay, so let me write directly now Cauchy residue theorem so there is a Cauchy residue theorem, what do you mean by residue? Residue means and if your function F(z) is analytic you have a characterization, 3 characterizations you have, one of the characterization is having a Taylor series, if F(z) is analytic you have a Taylor series, okay, you have a Taylor series and it doesn't have, none of the terms is singularity, okay, so everything is like F(z), F dash, Z-Z naught, okay, you have Z-Z naught, if it is around Z naught if you look at the analyticity around Z naught you have Z-Z naught power N you have a sigma, N is running from 0 to infinity, okay, some constant of course you have that is F, N derivatives of at Z naught divided by N factorial, this is what is your Taylor series.

Sometimes if you have, you can have a representation, not only so if F is analytic, residue means coefficient of 1/Z-Z naught in your Taylor series, if you write a Taylor series and for example this one, if F is analytic I can have this Taylor series and if you look at this integrand, this integrand divided by Z-Z naught if you put I have Z-Z naught, okay, so one cancels what you have is N-1, okay, so if you have N-1 what is the coefficient of N, when you put N = 0, coefficient that is simply 1, right, N = 0 that is simply F(z naught)/0 factorial which is 1 that is the coefficient by Z-Z naught + F dash(Z naught)/1 factorial into Z-Z naught power 0 that is 1, like that you go on, okay, this is actually nice of, this doesn't have any singular point, because 1/Z-Z naught is all powers of F double dash(z naught)/2 factorial Z-Z naught, all remaining are Z-Z naught square, cubes and so on, so this is the term, so this coefficient of 1/Z-Z naught is actually residue, Cauchy residues that is called residue of this function F(z), okay, this function this whole function okay, F(z)/Z-Z naught, so if F is not analytic you may have, if F is analytic you have this Taylor series, if F is not analytic you may end up having like this, this terms, if it

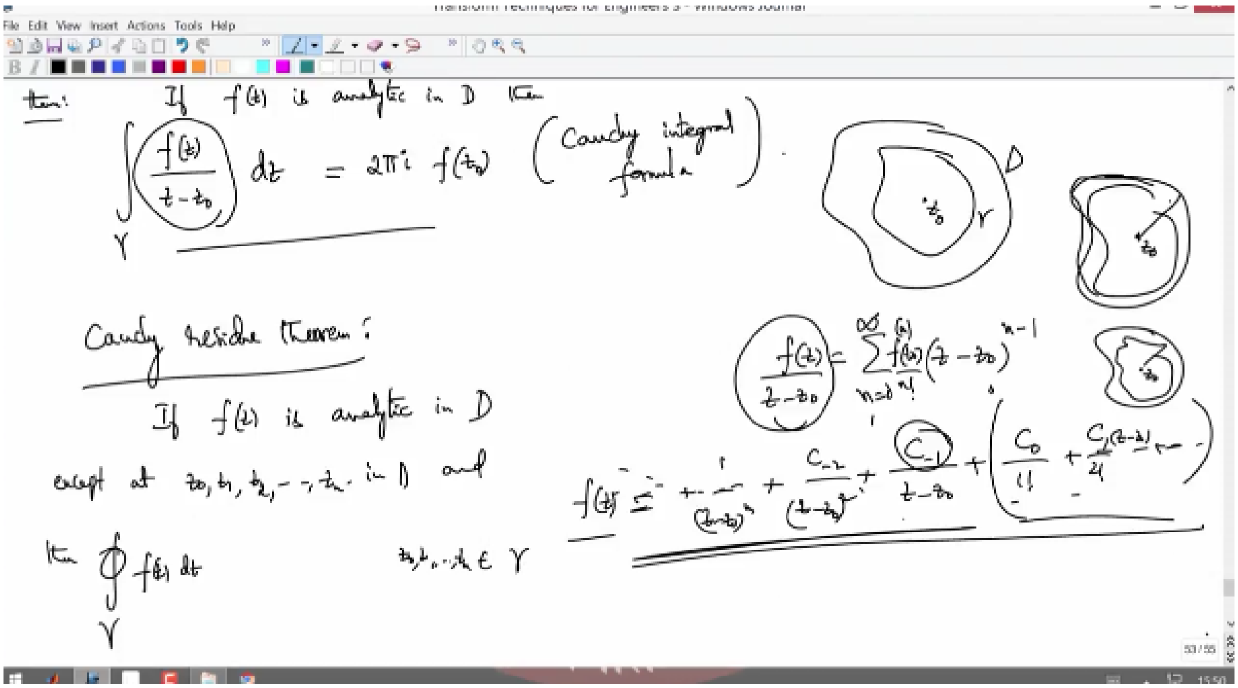


is like this it’s a simple pole, if you have if F is not analytic but also have, let us say some other coefficient, let me use C’s, okay, so C-2 divided by Z-Z naught square, then you say that if F(z) is like this and if I use this as let us say C-1, C0, some coefficients and you have like this, if you have like this Z-Z naught and so on, these are nice like a Taylor series.

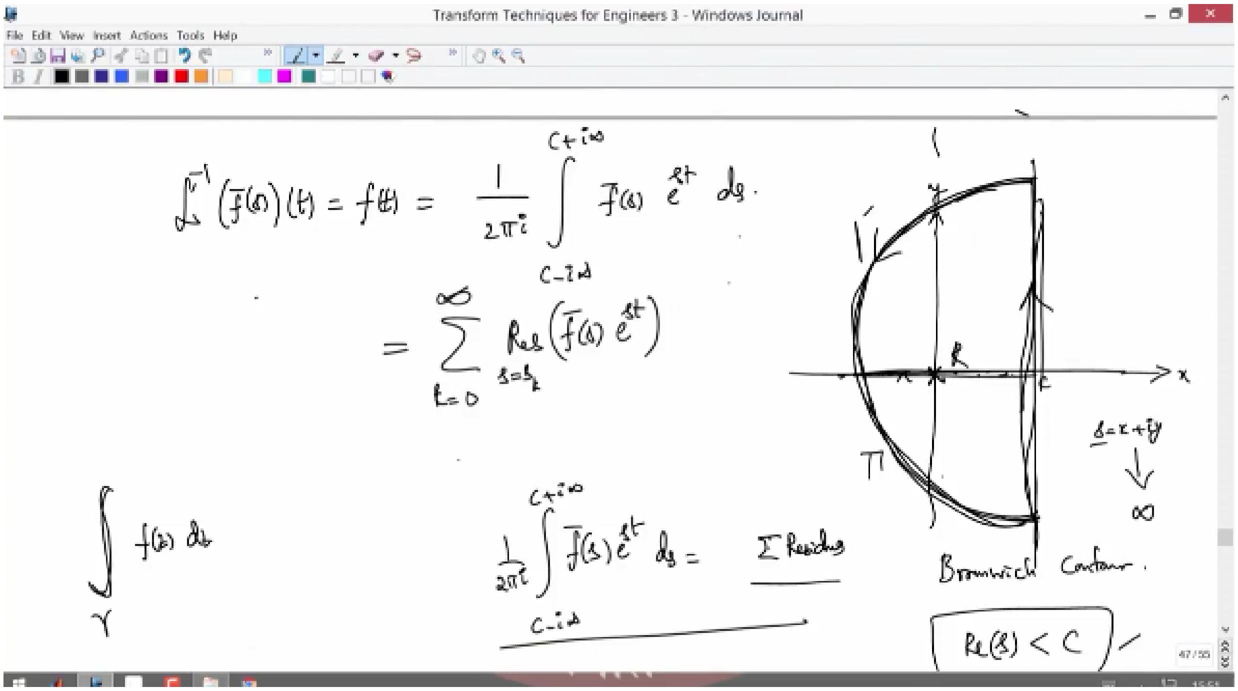
And left hand side you have poles, so if it is like this it is a pole of order 2, okay, so still you see that this C-1 that is the residue of this function F(z), and if you have if it is not analytic at all, if it is still not analytic some more poles if you have, if you have infinite series like this then you still that is called Laurent series for the function F(z), so only thing is you have a series, you have like this, you have Z naught within this except this point and within this, this Laurent series is valid which is convergent, so this series is convergent at every point inside this domain D except at Z naught inside this, okay, inside this so such a thing is called Laurent series.



In the Laurent series coefficient of 1/Z-Z naught is your residue, so if you can get this Laurent series, now you can read the text book and you can, if you get the, if it is analytic function you can get a Taylor series so that all this negative powers will be 0, so that your residues actually 0 because there is no negative powers, okay, and if you have this negative, finitely many negative terms then it is called pole of order some finite order, its goes from up to 1/Z-Z naught power M and there is no other negative terms this side, you say that it is the pole of order M, if you still have negative terms this side that is pole of infinite order that is called essential singularity, so they have at different kinds of singularity you can think, if it is a point where it is not analytic it is called the singular point, and first of all first singular point is it is pole which is only simple singular point you have at, you have at Laurent series expansion you can, if you expand that function you'll get like this up to here, so pole of order M means you'll get expression like this, expansion up to negative powers M, if you can get all the negative powers, the negative left hand side, negative powers, infinite negative powers you say that is essential singularity, okay, so these are three different singular points and in the residue theorem tells you that if F(z) is analytic, analytic in D open connected at D except at Z naught, Z1, Z2 up to ZN, some finite points inside D, in D, okay, then integral over F(z) DZ over this gamma, so gamma contains, okay, so let me use gamma inside the R, except at this points in D and this gamma contains this is closed curve, this is the closed curve, gamma is the closed curve, this Z naught, Z1, Z2, ZN belongs to this closed gamma, so they are inside gamma, okay, or inside gamma closed curve.



If you have that this is actually equal to 2 pi I times sum of all residues of F(z) at Z equal to this singular points, these are singular points because this is not analytic except this points, so these are the singular points Z = ZI residue, here itself we write Z = ZI, so I is from 1 to N, so 0 to N so you have N+1 singular points, so whatever maybe the singular point you can find the residue different kinds of singular points or most of the examples which you see, you may have only a pole, simple poles so that you can easily calculate the residues, so that is how you can find the residues, then you have this expression, so integral over a closed curve this complex integration is actually 2 pi I times residue of that function, so you can calculate this residue and that is exactly what we were talking about in the last video, let me briefly go there and explain you, so if I use this Cauchy residue theorem what you have is if you consider this, if you want to evaluate this integral you have to consider this integral over this closed curve like this, this is



called Bromwich contour, so that as R goes to infinity it has all the negative roots inside, because this F bar(s) E power ST is analytic this side, so all wherever it is not analytic it will be only this side, left hand side of this line, so it will have all the poles if F(s) has poles or singular points all are the left hand side, as R goes to infinity this Bromwich contour will have all its singularities inside this curve.

So by the Cauchy residue theorem integral over this closed curve this is equal to 2 pi I times, 2 pi I times all the residues, all the residues of this function at all the singular points inside, okay but what is the left hand side? Left hand side is this total integral but one can show that because of the behavior of F bar(s) over this piece, this circular piece we can actually show that this integral is actually going to 0, so the contribution is only over this line that means what you end up is only this integral, left hand side will be only this integral so that is the reason 1/2 pi if you divide so this is equal to sum of all the residues, okay, so if you can find all the residues, residues where F(z) this function has singular points E power ST is analytic function, so F(s) or given F(s) is only having, wherever in the you can easily find what are the, what points it is not analytic functions, so those are singular points, those singular points you find the residue and that will be this inverse transform of, inverse Laplace transform of the function F bar(s), so anyway we will see that, we'll try to give you some more examples of how to find this residues, okay, Cauchy residue theorem is this.

In the next video I'll give you, in the next video I'll try to present how do you find for simple functions how do you find the residues, and what are the different ways to evaluate some real integrals so that, so you need only this Bromwich contour integral, but in the next video I'll try to explain you how to do this contour integration technique for, when you have different types of integration, okay, some real type of integrals, how do you evaluate using this contour integration technique or Cauchy residue theorem, how do you make use to evaluate some real integrals, okay, so this is what we'll see in the next video and then we will get back to this finding inverse Laplace transform of, general Laplace transform that is F bar(s) using this Bromwich contour, okay, so this is what we will see in the next video and later videos. Thank you so much.

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