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Transform Techniques for Engineers

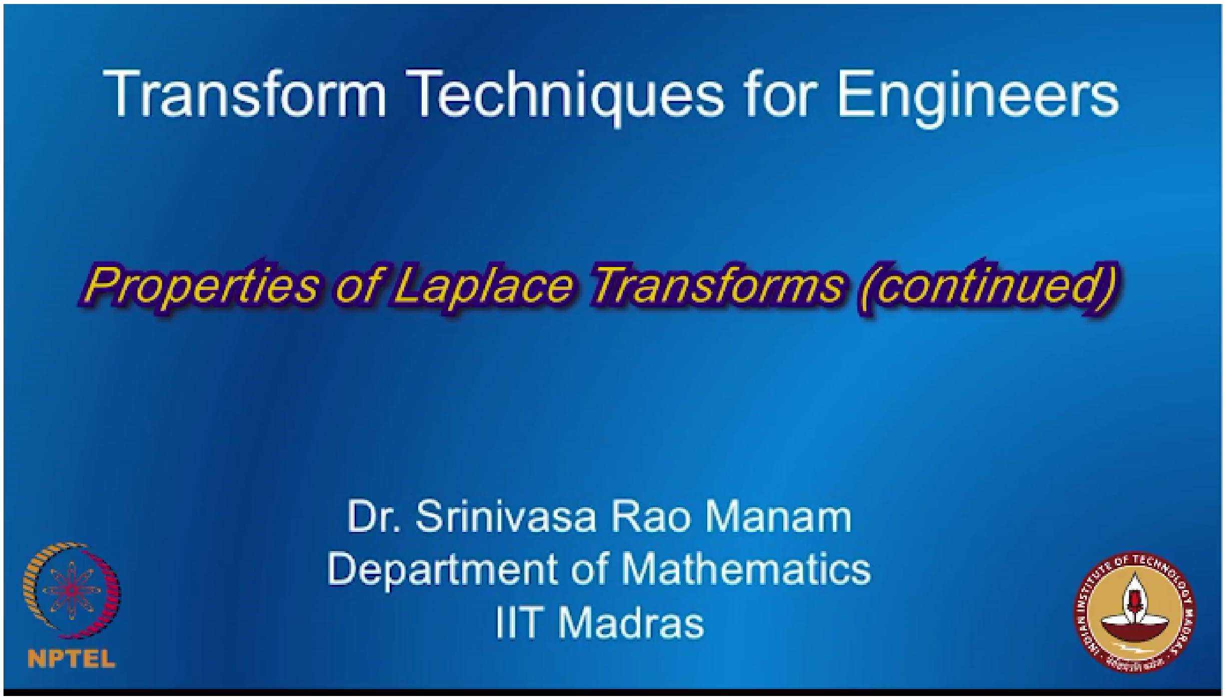
Properties of Laplace Transforms (continued)

With

Dr. Srinivasa Rao Manam

Department of Mathematics

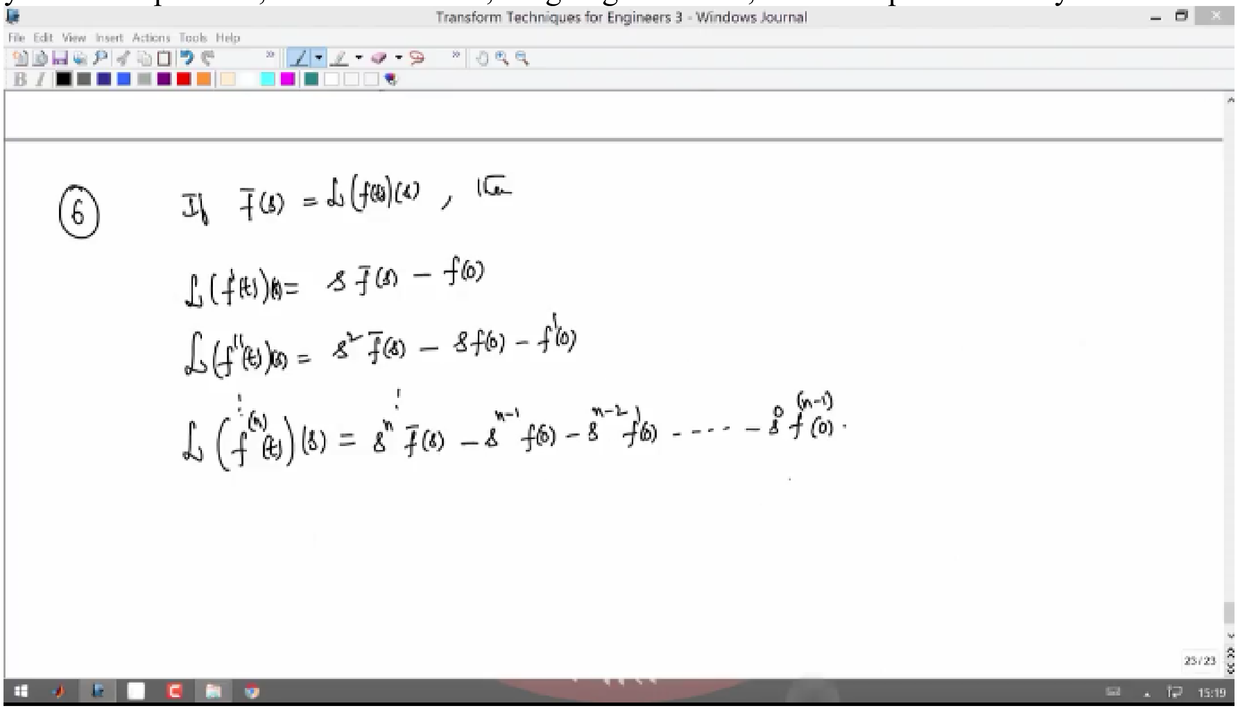
IIT Madras



Welcome back in the last video we have seen certain properties of Laplace transform, such as shifting properties, scaling properties, Laplace transform of a periodic function, how to calculate very easily if it's a periodic function which is defined between 0 to infinity, and we have actually seen what is the property of a Laplace transform that integral, Laplace transform integral is uniformly convergent so that means you can take the derivative when you differentiate the Laplace integral you can take the derivative inside the integral, and you integrate the Laplace integral you can take the integration inside the Laplace integral of the Laplace transform, so that's what we have seen so far.

So let's do some properties of derivatives of the Laplace transform, so let me do this property, property number 6, so what we had if F bar(s) is a Laplace transform of F(t) which is a function of S, then and suppose F is a differentiable function and what is the Laplace transform of F dash(t) and F double dash(t) and so on, so what is this one and Laplace transform of F double dash(t) that means 2 derivatives of Laplace transform is a function of S, what is it, so these are the questions we have.

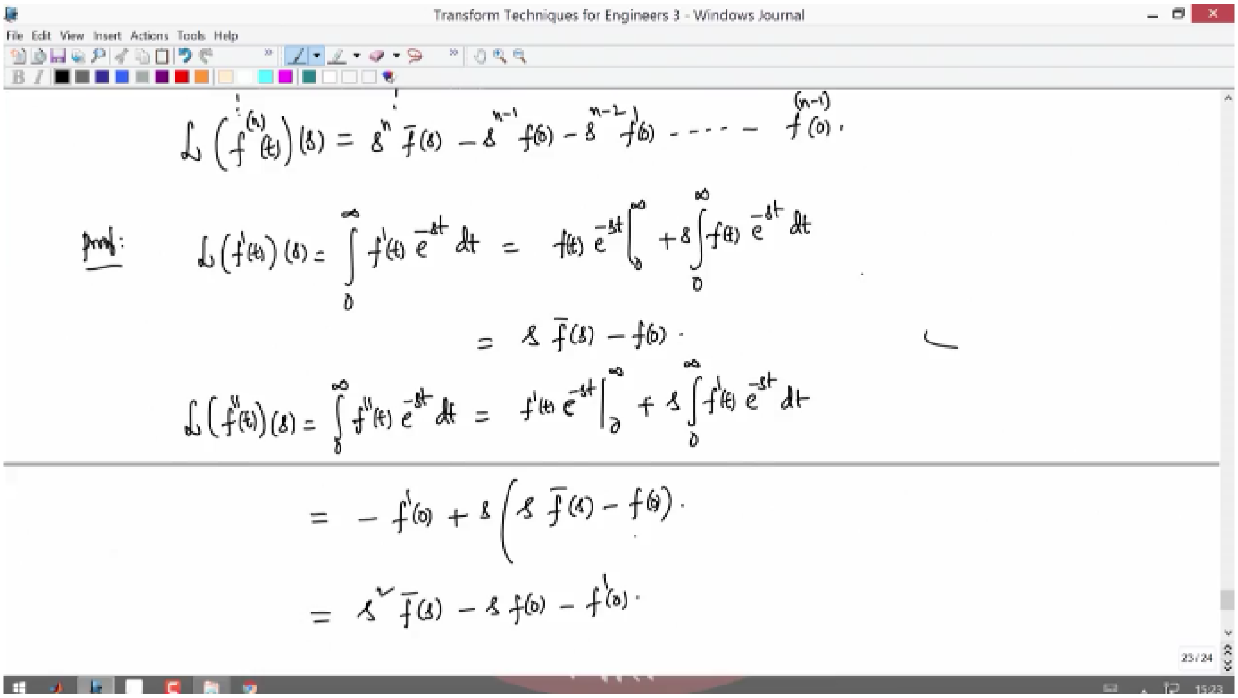
In general what is the Laplace transform of N derivatives of function of T, which is a function of S at the end, so what is this one? So let me write this, so what we will have is S times F bar(s) which is a Laplace transform -F at 0, the value of function at 0 and here you have S square F bar(s) and then –SF(0) and then -F dash(0), and what you can easily similarly can guess it is S power N times Laplace transform of F- S power N-1 times F(0) -S power N - 2 times F dash(0) and so on, you end up finally S constant times minus this one S power 0, and you have S power 0, so N-2 +1 this is, it's going to be N-1, so here S power 0 and you have N-1



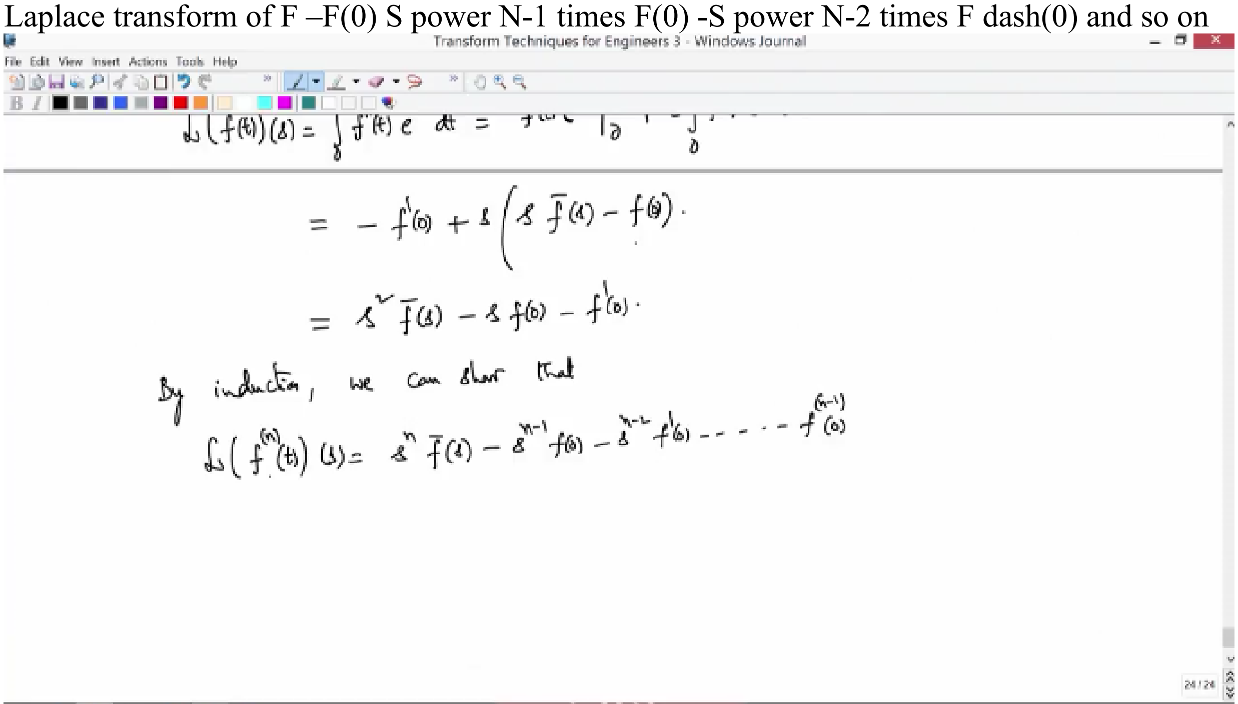
derivative at 0 that's what you will have, okay, so if I write functions in the superscript, if I write in the brackets that's the meaning of derivative, if I don't use dashes that's the meaning, S power 0 is 1, so you can need not right, so this is your formula for Laplace transform of N derivatives.

So you can easily see this one just by integration by parts, so what is the Laplace transform of F dash(t) as a function of S is by definition F dash(t) E power -ST DT do the integration by parts F(t) times E power -ST from 0 to infinity, -integral 0 to infinity F(t) times you differentiate this you get so that you get +S E power -ST DT, so minus minus plus and you differentiate this with respect to T that is S comes out, -S comes out, that minus makes it this plus, so this is exactly equal to F at infinity, F is a continuous function which is exponential function but because S is bigger than order of the exponential function, exponential order, so S is bigger than order of the exponential of this function F, so at infinity this quantity is going to 0 so what you have is -F naught F(0) and this is going to be +S times, this is F bar(s), so if you rewrite from this side so you have S, F bar(s) – F(0) so that is first one.

And you can do the similar way, you can calculate the same way so you can calculate the other thing also, so if you do what is a Laplace transform of F double dash(t) which is a function of S, what you have is 0 to infinity F double dash(t) E power –ST DT, twice if you do the derivative integration by parts here so you get F dash(t) E power -ST from 0 to infinity, now minus minus plus S times 0 to infinity F dash(t) E power -ST DT, so this is clearly you can see that this is going to be 0 again because this exponential order is S is bigger than this derivative function okay, so that's 0 and you had it 0 - you have a negative sign -F dash(0) this function is 1, and here +S times you repeat the same thing, you end up getting what we had yesterday, what we had earlier so this is this one, so you can substitute this, so you see write, if you write that F bar(s) – F(0), so this is nothing but if you put it together so you see that S square F bar(s) - F dash(0) – F(0), or rather so S times F(0) – F dash(0).

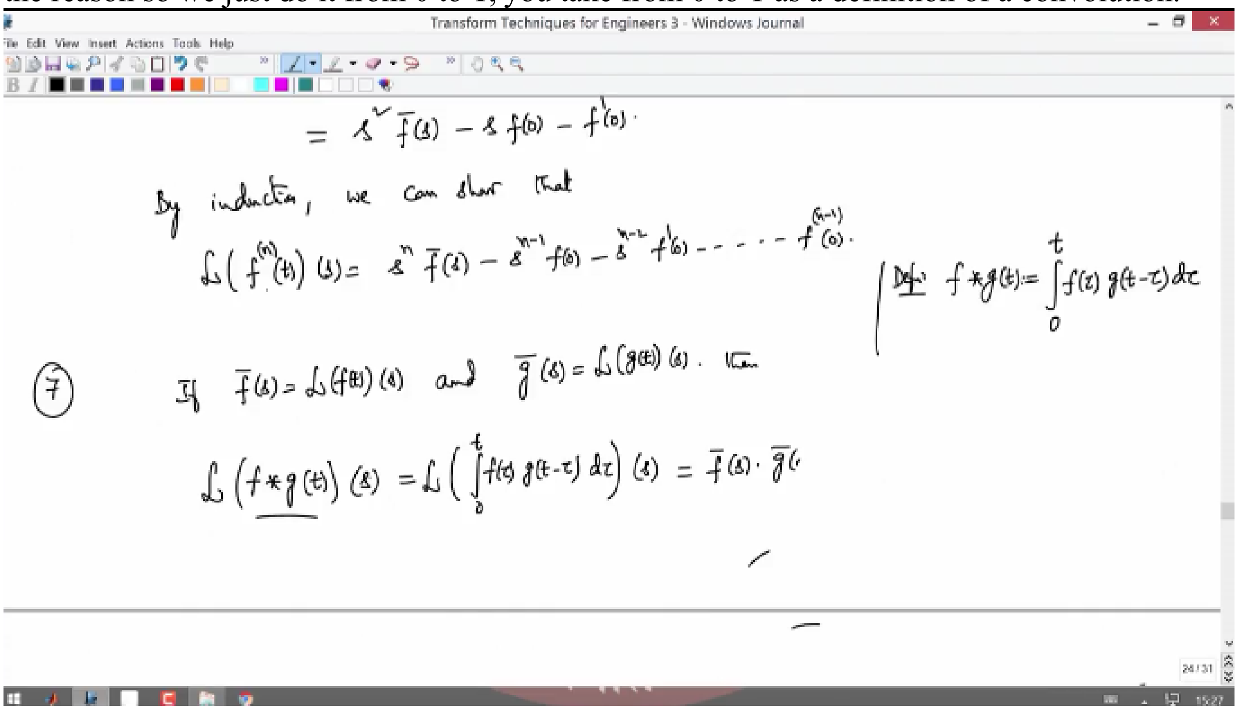


So this is second one, I repeat N times you can easily see this, so how to show this one, what you have to show assume that this is true for N = K, you do one more integration so you can bind, one integration by parts will give you at K+1 this is true, so by induction one can show the other result, okay, by induction we can show the generalized, you can show that Laplace transform of N, N derivatives of functions of T, as a function of S will be S power N times Laplace transform of F –F(0) S power N-1 times F(0) -S power N-2 times F dash(0) and so on



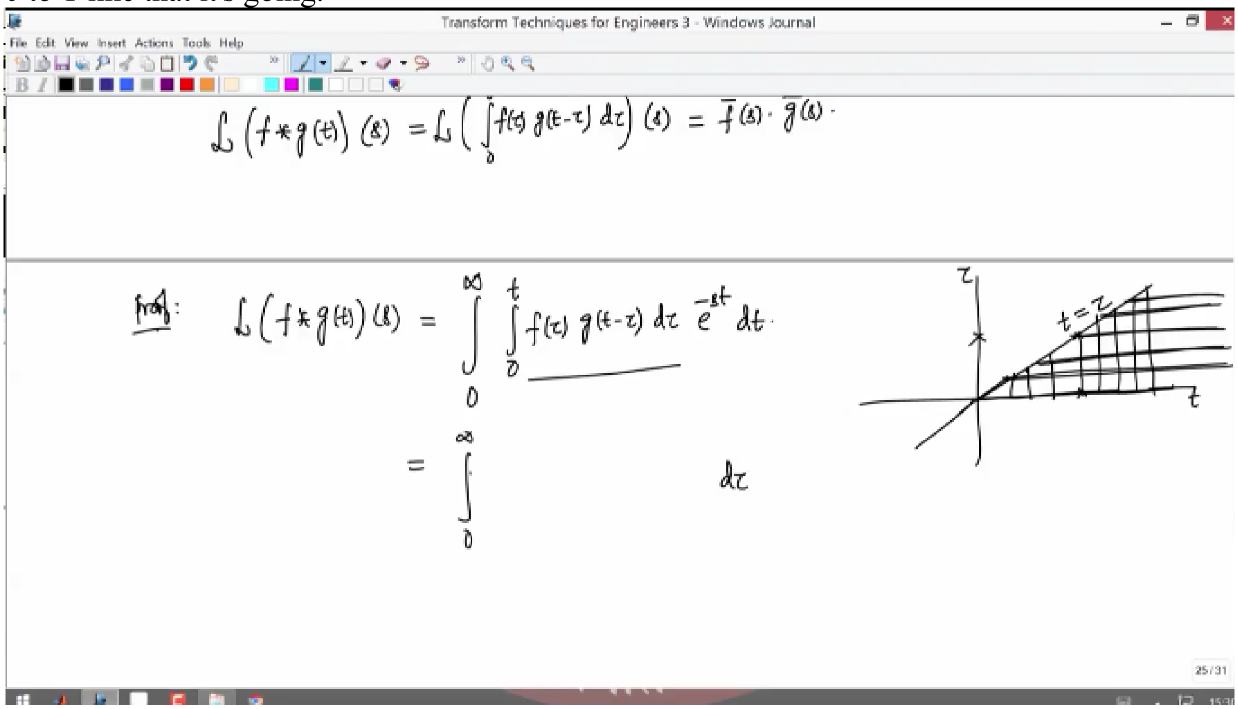
finally S power 0 into F(n-1) derivative (0), so this is what you can easily see, okay, so this is one property, using this one can show, we'll look at the other properties, for example convolution of 2 functions, how do you find, what is its Laplace transform? So this is property

number 7, that is convolution, convolution property if F bar(s) is the Laplace transform of F(t) with a function of S, and suppose G bar(s) is the Laplace transform of some function G(t) as a function of S, then what we have is a Laplace transform of F convolution with G as a function of T, then once you take the Laplace transform this will be function of S.
What is the meaning of this? Let me write what is the definition of convolution, F convolving with G as a function of T, you have two functions what we do is we integrate from 0 to T, F(tau) G(t-tau) D tau, so if you take like this this is the definition of convolution, so we don't do it from 0 to infinity because you have to end up so you always, if you do like this T-tau, T is fixed, T is bigger, T is always, but Tau is always less than T so you have T-tau is always positive, and tau is anyway positive so that sense because of this, this integral makes sense, if you do it from 0 to infinity if T-tau you cannot guarantee that is always positive, okay, so that is the reason so we just do it from 0 to T, you take from 0 to T as a definition of a convolution.

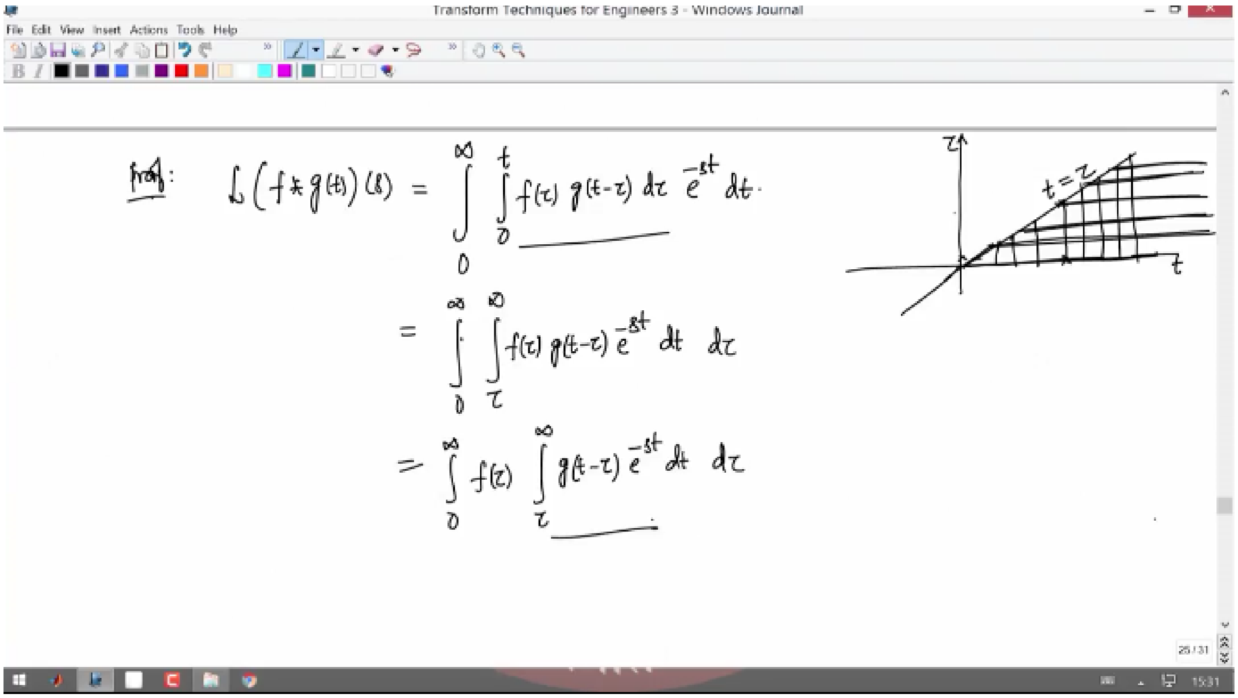


So if you do that what is the convolution, so Laplace transform of that convolution 0 to T, F(tau) G(t-tau) D tau this as a function of S will be simply product of a Laplace transforms that is F bar(s) times G bar(s), this you can easily show so what is the proof of this, proof is not difficult, so you just start with you take this inverse transform, so if you take the Laplace inversion both sides assume that F and G are such class of functions, class of continuous functions for which you have, which are exponential order then you have a Laplace transform that exists and it is unique, so you have inversion, so it's inversion exists so such class of function you take the inverse, if you write take the inverse both sides how do I do this integral, so let me do a proof is let me take this Laplace on left-hand side which is a Laplace transform of convolution of this function of S, so if I do this integral 0 to infinity F of, in the place of F(t) I've to write this function that is 0 to T, F(tau) G(t-tau) D tau times E power -ST DT, so this is the definition of a Laplace transform.
So what we do is, if we have to see how we integrate this, what is this integration, double integration, if you view in a plane if this is your T axis and this is your tau axis let us say, so if this is the line that, that is T = tau, what happens? T is actually 0 to infinity so you take 0 to

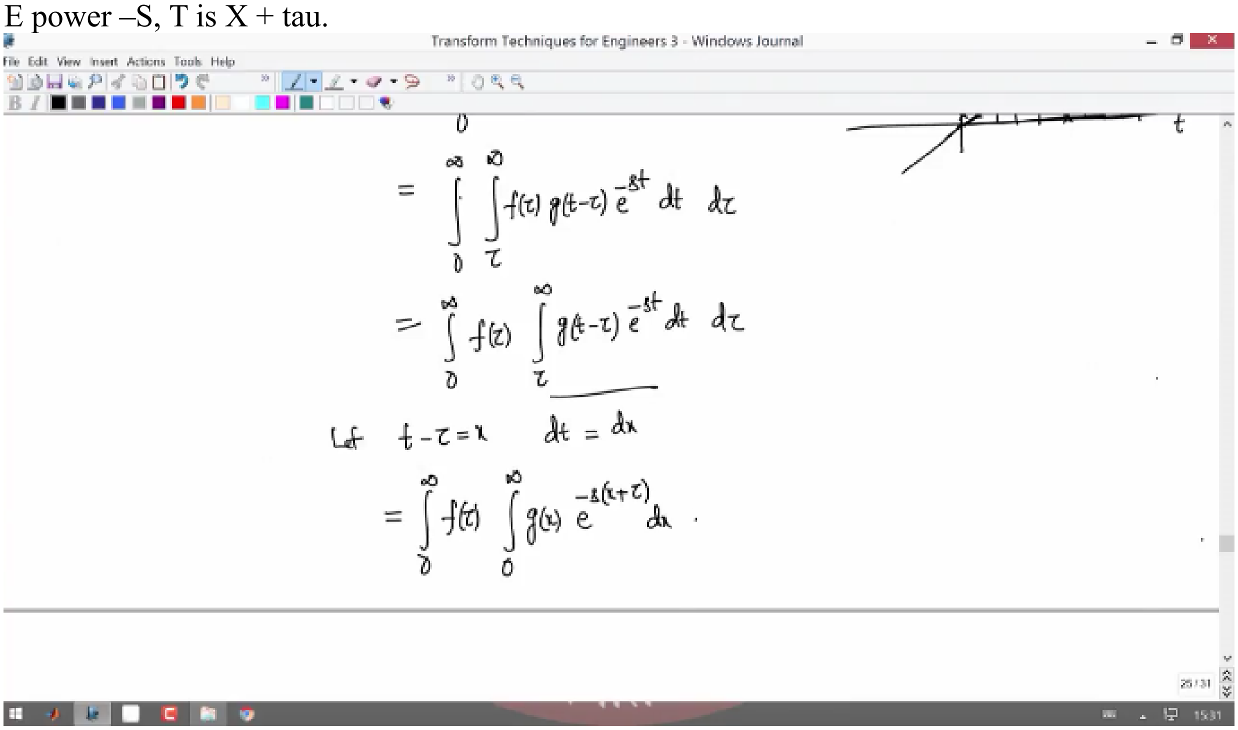
infinity here T, and when you fix T let us say fix T and what is tau? Tau is going from 0 to T, you are going like this, so you are integrating like this, you fix your T you're doing like this, you fix your T you are trying to integrate like this, so fix T first you integrate tau variable from 0 to T like that it's going.



So instead of doing this what I do is I try to fix my, what is T? T is tau this is how you look at this way, if you look at this way that means tau is from 0 to infinity this is same, right, so this is D tau, so if you view, if you integrate like this each piece like this and finally you are adding up, then what you are integrating is, what you are integrating is basically D tau that is, tau is from, okay, you're fixing tau, you are fixing tau, that is from 0 to tau, 0 to infinity, where you're fixing? Any one of these places, every place tau is fixed, so here I've fixed the tau, if you are doing here if you start with here I fix the tau here, so it's anyway in any case tau is 0 to infinity, so 0 to infinity T tau and if I do this if once I fix this, which is between 0 to infinity what I am doing is T is from tau to infinity, so tau to infinity and inside the integrand as it is, G(t-tau) E power -ST DT, okay, so this is nothing to do with tau you can write outside this integral 0 to infinity F(tau) you can put it outside, and because this is tau to infinity G(t-tau) E power –ST

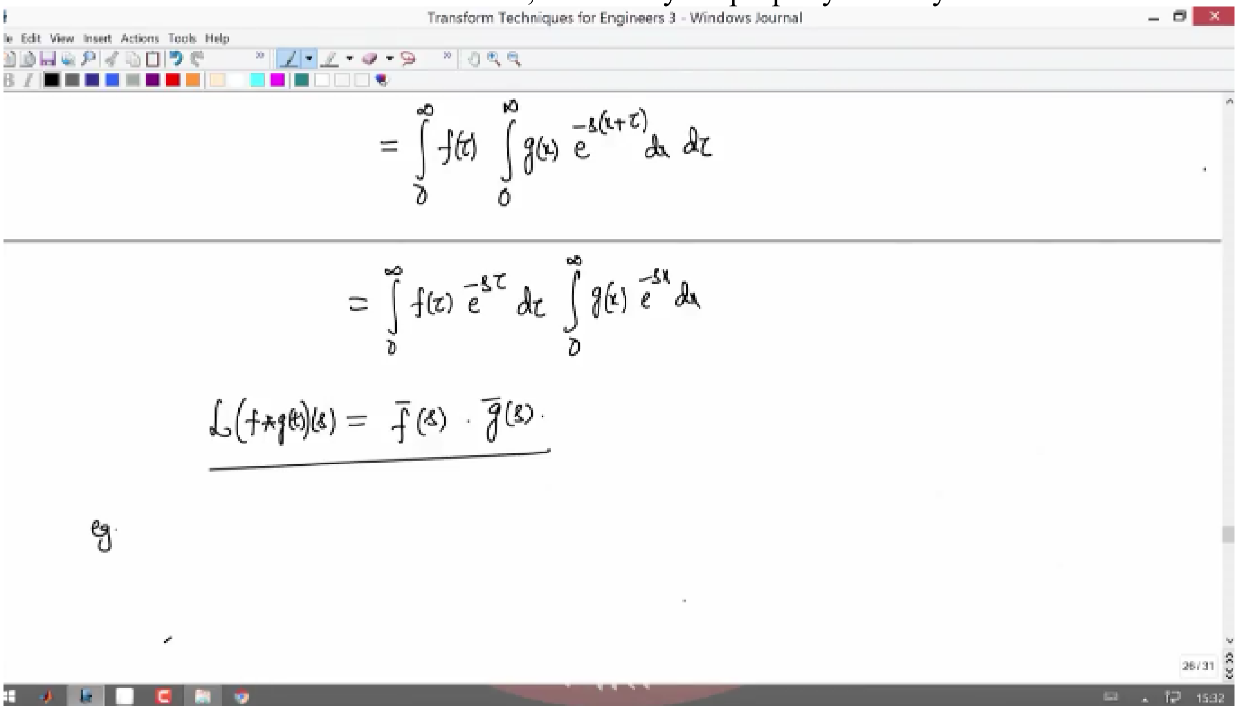


DT into D tau, so this integral you can evaluate by putting T-tau has some X variable, new variable so that you have DT which is equal to DX because tau is fixed so tau is constant, so you have 0 to infinity F(tau) so this integral, inner integral if you can write like when you put T = tau, T = tau, X is 0, when you put T = infinity, X = infinity for fix tau, so you have G(x) times E power –S, T is X + tau.

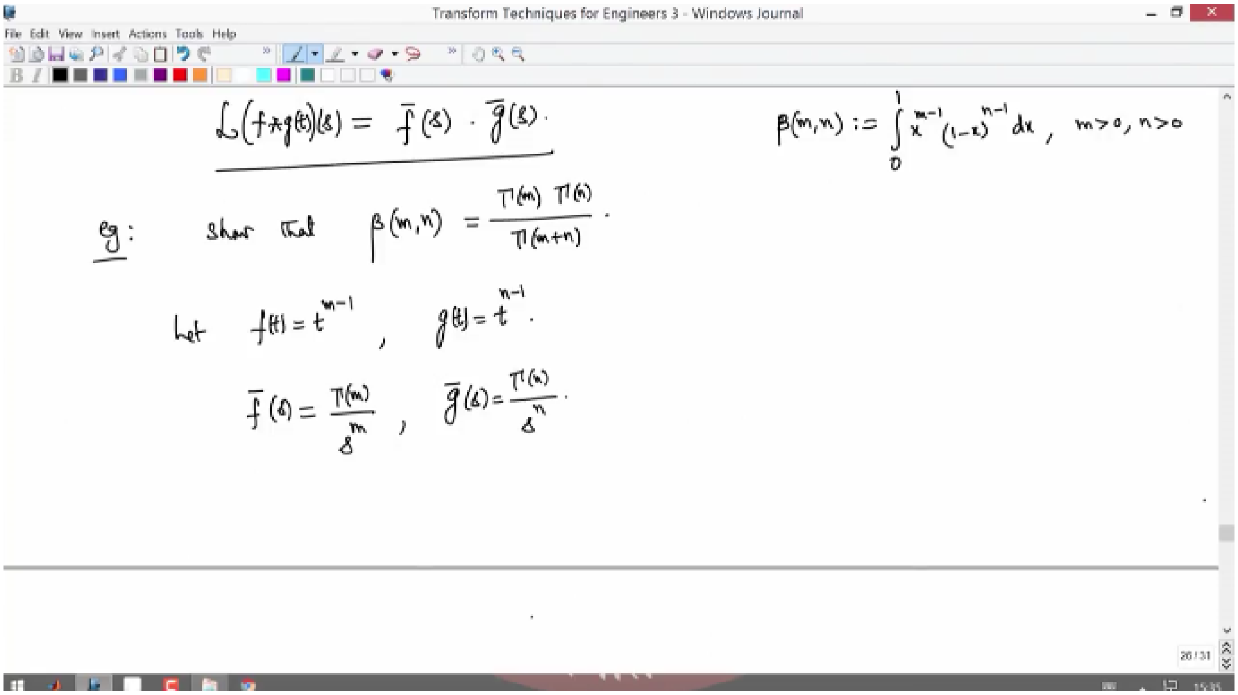


And then you have DT is DX, we have d tau, so this is nothing but 0 to infinity, F(tau) E power -S tau D tau they are separated now, so integrand is a function of tau and function of X, so you can write them, or you can separate them outside so you have G(x) E power -SX DX, X is the dummy variable you can put it as a T, X as T so what you see is they are nothing but the

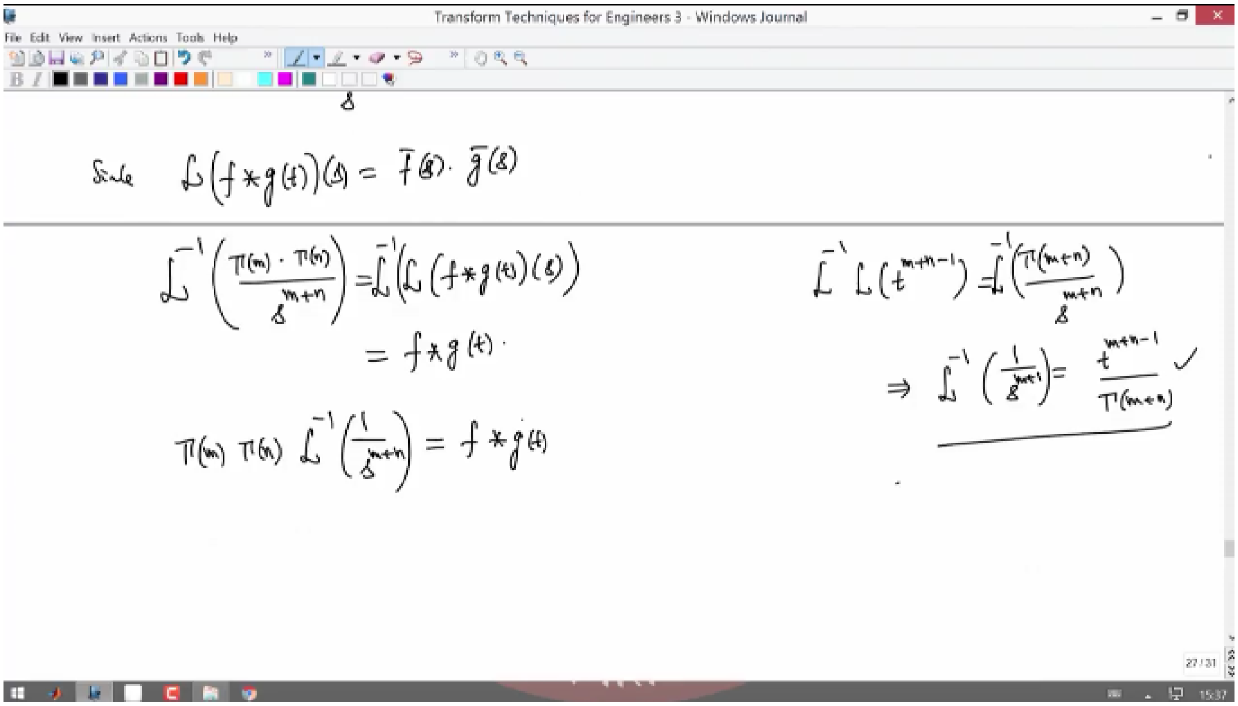
Laplace transform of F times, this integral is Laplace transform of G, so what we have is Laplace transform of it's a convolution this is nothing but F convolve with G was a function of T as a final form is a function of S is this, so this is your property which you can use later on.



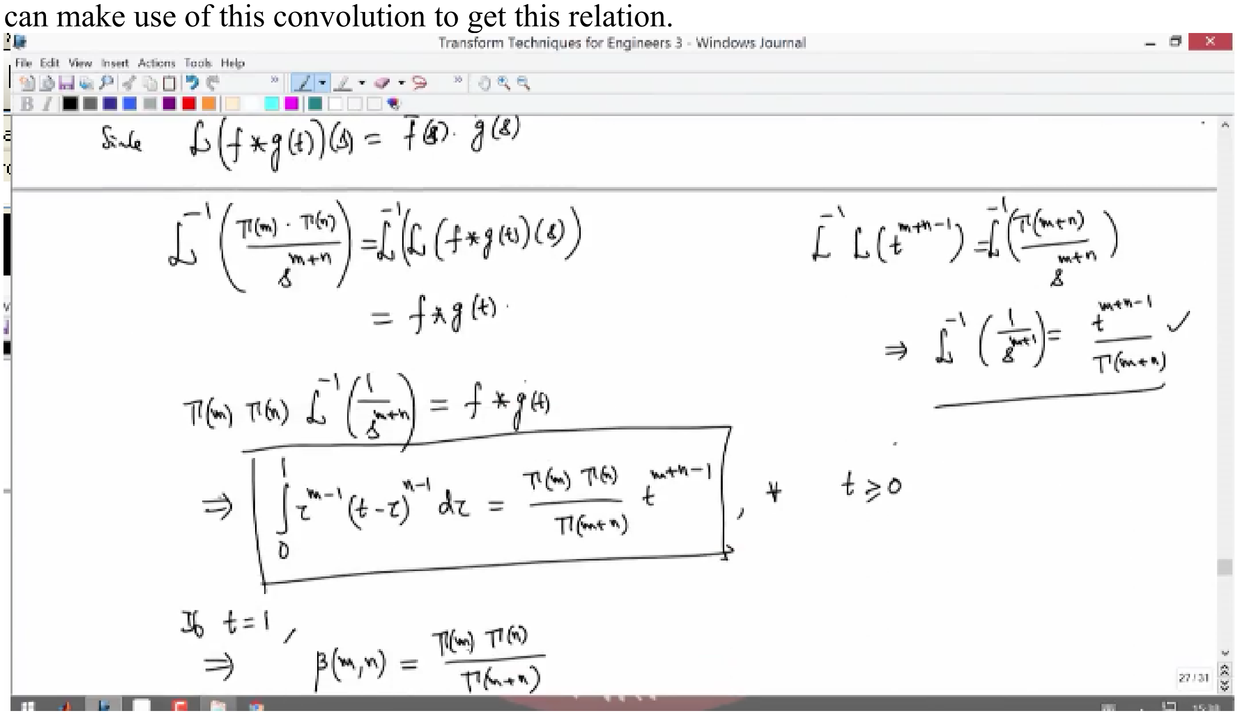
So using this property we can do some example, we can evaluate some nice relation between beta function and gamma function, we can use that, so use the convolution to, we can do many things, many convolutions we can do, we can make use of this convolution thing to do any problem, so before, so first one example that I want to do here is a relation between beta function and gamma function, so you have seen what is gamma function, let me define what is a beta function, beta function if I write beta(m,n) let us say, by definition this is an integral like this, this is a special function which is an integral between 0 to 1, X power M-1 times 1-X power N-1 DX such a definition with M positive, its well-defined function, so if this is your beta function you have a relation between to show that, what we show is of this beta function you can write in terms of, if M, N are integers or positive numbers, positive real, so you have beta(m,n)is actually nothing but gamma M, gamma N divided by gamma(m+n), so this is what we can show just by using this convolution, so what I do is I consider, I use convolution to show this one, so let me use F(t) as X power M, M-1 let us use, and G(t) is X power, G(t) is T power N-1, this is T power N-1, so if I use this what is the F bar(s)? F bar(s) is we have already seen if it is T power alpha is going to be gamma alpha + 1 divided by S power alpha + 1, so this is alpha, in the place of alpha you have M-1, so you have gamma M divided by S power M.



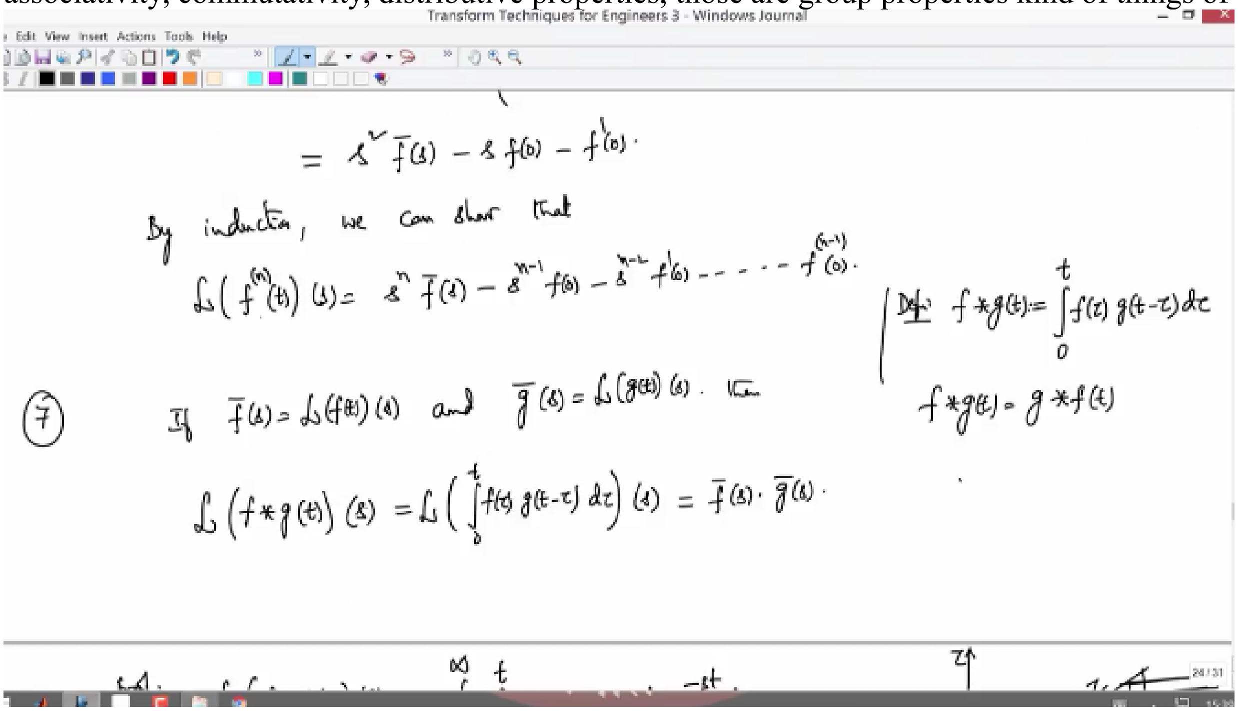
Similarly G bar(s) is gamma N divided by S power N, so we can use the convolution that is F convolving with G for which you take the Laplace transform, this is F bar(s) times G bar(s), since this is the case what we have is right hand side is gamma M, gamma N/S power M + N is a Laplace transform of F convolving with G(t) which is function of S, so you take the inversion both sides, inverse transform of this so that you have inverse transform of this that makes it right hand side F convolving with G as a function of T, so we need to know what is this one, this is a gamma M, gamma N which is a constant, so you can take it out, gamma M, gamma N, so Laplace inversion of 1/S power M+1 which is equal to F convolving with G(t), so we are using the same result that Laplace transform(t) power, so this is M+N, T power M+N -1 = gamma(M+N) divided by S power M+N, so this implies gamma Laplace inversion of 1/S power N+1 is T power M+N-1 divided by gamma M+N, so just take the inversion both sides, so this gets cancelled, so you have this one and this is just a constant so that comes out and you can divide it, so this is what is the result.



So if you use it here, so what you get is left hand side integral 0 to 1 T power M-1 that is your F(t) and G of, this is tau, okay, so we are using convolution and G(tau) so this is T-tau times N-1 D tau that is the right hand side which is equal to gamma M, gamma N, this is gamma M+N times T power M+N-1, so this is the relation that works for every T, where is this T validity? Validity of T is, T is between 0 to infinity, so you can take it as 0 to infinity, so what we require is if I put T = 1, if T = 1 what we get is a left hand side is the beta function, beta (m, n), on the right hand side simply gamma M, gamma N divided by gamma M+N, this is what is true, so we can make use of this convolution to get this relation.

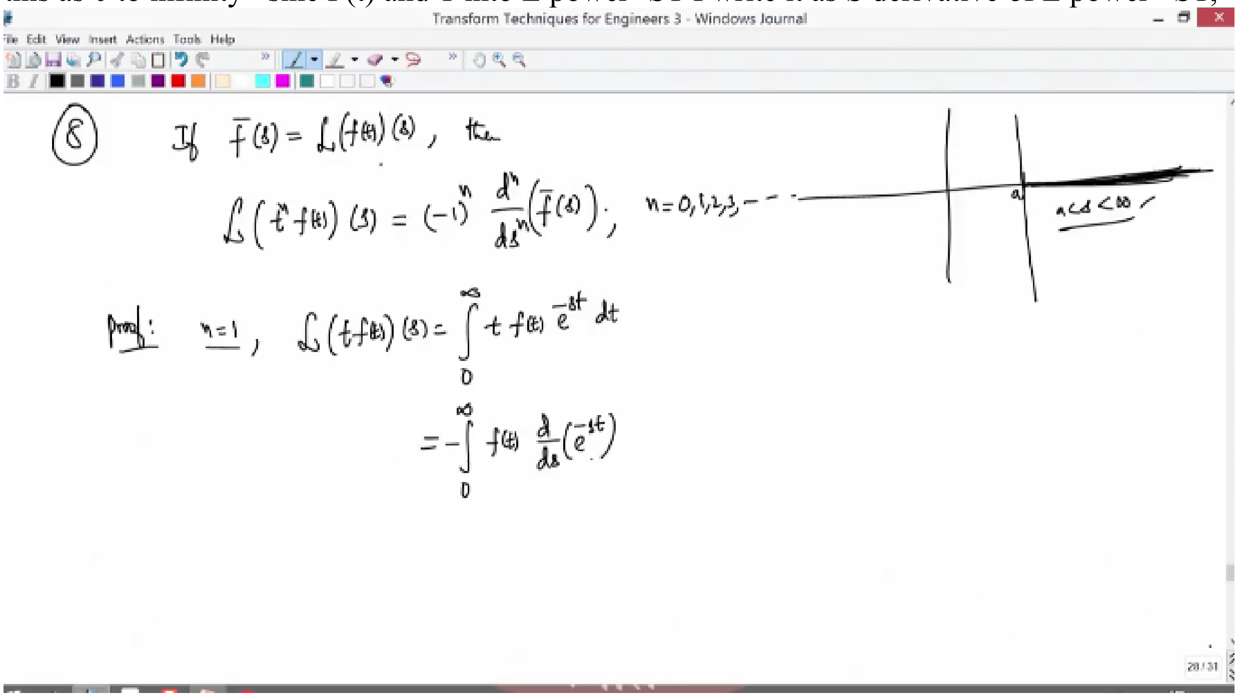


What we do is now, and we can also see many of the properties of this convolution, convolution we have defined what is convolution, many of this properties that F convolving with G(t) is same as G convolving F(t), and many of these trivial properties we can show, it satisfies certain associativity, commutativity, distributive properties, those are group properties kind of things of



that algebraic properties that we need not worry, so what is important maybe this one at least, you can write this or this both are same that you can easily see, okay, just by changing the variables you can easily see that this is true, okay, so you have this property and then we move on to get some more properties of this Laplace transform that is this property number 8, we'll see property number 8 that is we have seen earlier Laplace transform of the derivatives, now we can see that derivative of the Laplace transform, if F bar(s) is a Laplace transform of F, F(t) as a function of S then what is the Laplace transform of T power N times F(t) as a function of S this is actually -1 power N, nth derivative of S, S derivative, S derivative Nth derivative of with respect to S if you differentiate the Laplace transform.
I think of Laplace transform exists for S greater than A, real part of S greater than A, so you have a domain S is here, so S is bigger than some A, S is bigger than some A, let us say A is here so everywhere you have a domain here so you think of, on this real line AS is between A to infinity, you have a real values, in this domain you can differentiate as a real variable, for differentiate this is what you will get, so this is the Laplace transform, what you gain is this is the Laplace transform of some T power N times F(t), so this is the important property we use to solve differential equations, certain type of differential equations with variable coefficients, special variable coefficients like you have N derivatives, YN derivatives of X and multiplied with X power N, and such integrals so nth derivative, so for example if I write DN Y/DX power N this whole thing you can take the Laplace transform you can write Nth derivatives of Laplace transform of Y which of course some negative sign depending on what is your N, even or odd. So this is the property we use to convert the ordinary differential equation into algebraic equations when you try to solve variable coefficient equations we use this property, so let me show this one, what is the proof of this? This is also straightforward you can easily prove this, and this is valid from 0, 1, 2, 3, onwards anything you take this is true, so if N is 0 this is

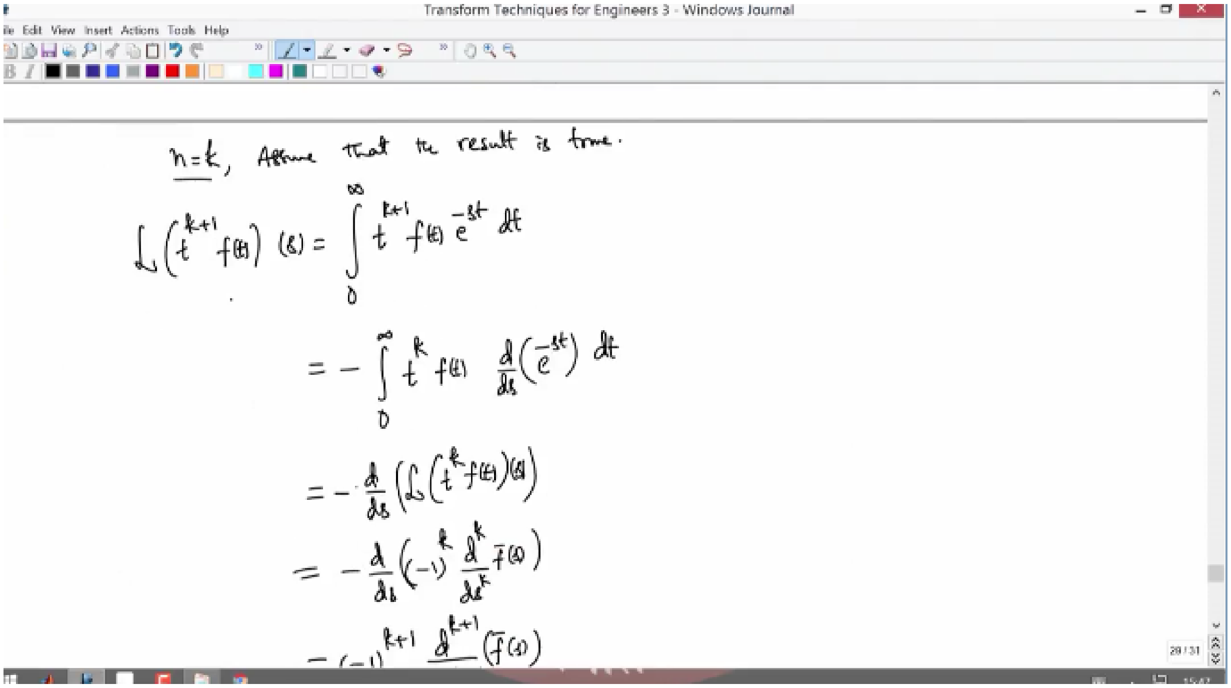
anyway both sides are same that is actually this, if N = 1 you can do this, do it for N = 1, N = 1 we show F(t) times, F(t) is a function of S by definition 0 to infinity TF(t) E power -ST DT, okay, what is this one? So this is if you differentiate so what is this one, so this is I can write this as 0 to infinity –sine F(t) and T into E power -ST I write it as S derivative of E power –ST,



what is wrote is again, -T E power -ST so that minus minus plus so this is exactly same as this DT, so because this is integral, this integral F(t) this, F(t) into E power -ST is uniformly convergent integral, this integral this derivative with respect to S, I can take it inside, so outside in inside both are same so I can write D/DS of F(t) times E power -ST DT, so this is nothing but F - derivative of D/DS of F bar(s) that's your Laplace transform of TF(t) as a function of S, so this is exactly your first one.

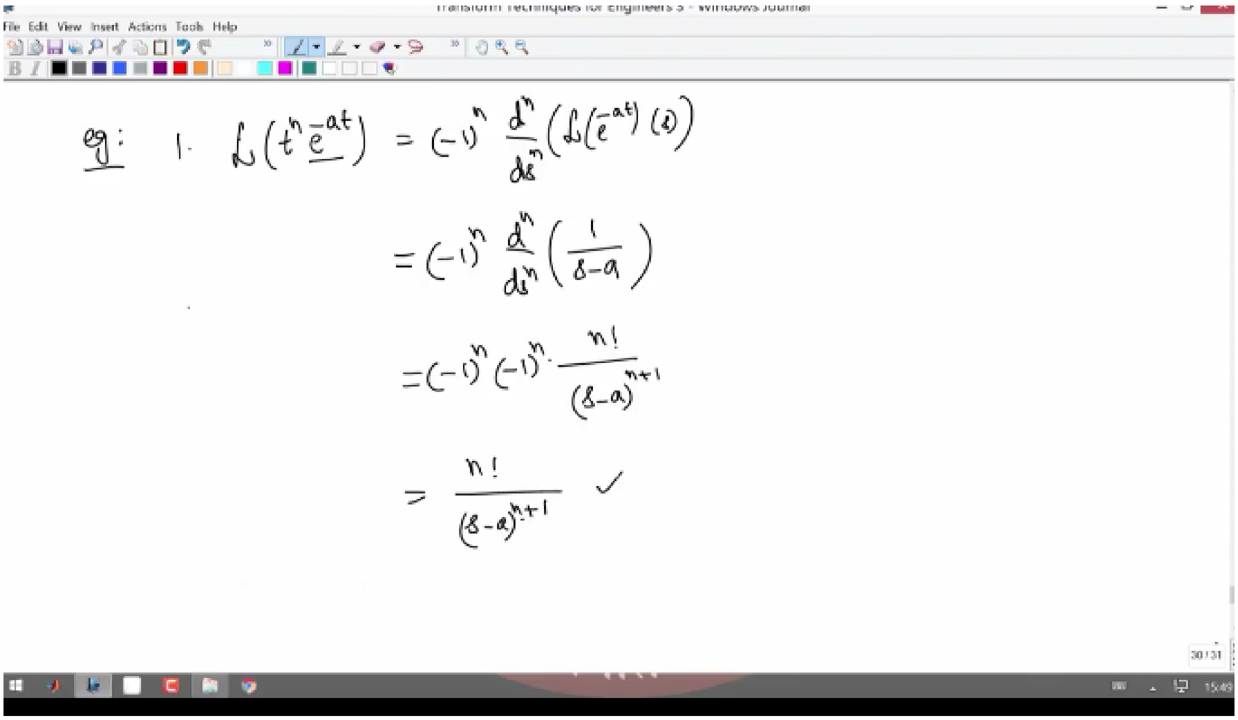
You repeat the same thing so N = K if it is true, suppose N = K assume that it is true this result is true then assume that the result is true, then we can show that N = K, K + 1 this result will be true, so how do I do this? So Laplace transform, so N = K if I take T power K + 1 times F(t) as a function of S will be -1 power K+1 rather, let me use by definition 0 to infinity T power K+1 F(t) E power – ST DT this is the definition of the Laplace transform of this, and this you can write like again you do the same technique that 0 to infinity T power K times -T I am writing as a derivative of D/DS of E power - ST DT, this is same as this, so you can take this derivative outside that is – D/DS of, this is Laplace transform of T power K F(t), what is left is there.

So you assume that this result is true, so –D/DS of Laplace transform of this, right, so in the place of this you can put it -1 power K times D power K/DS power K of F bar(s), of course this is a function of S that's why you are able to do this, okay, this is equal to so -1 power K+1

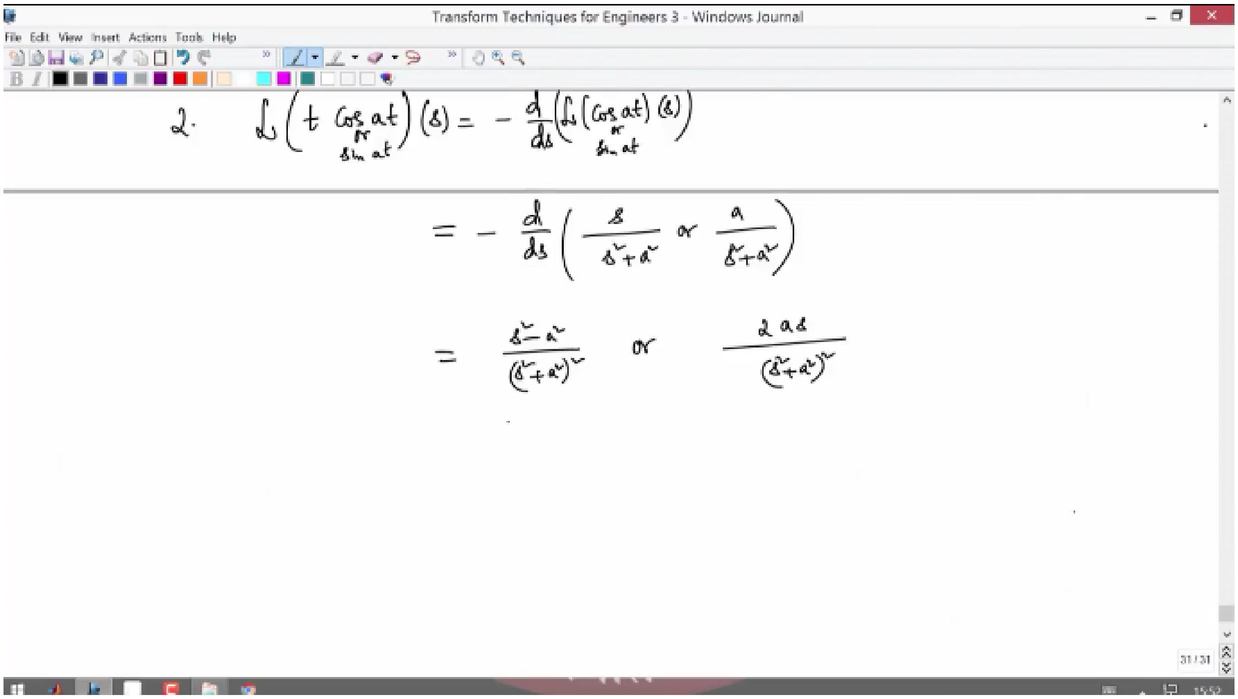


derivatives you have now K +1 derivatives of F bar(s), so this is exactly what is true, so if N is true, N = K it is true, N = K+1 the result is true, hence by induction that result is true, okay, so this is another property we use later on when we solve the differential equations.

Property 9 and I can do some examples I will do some examples just to give you a feeling, so an example is a Laplace transform of T power N times E power –AT, what is this one? This is equal to, so you want to find T power N when you multiply what happens, T power N E power AT, so this is if you use that formula this is going to be -1 power N, N derivatives of S which is all functions are Laplace transform of E power –AT, which is function of S, so this is -1 power N, N derivatives of with respect to S, what is this Laplace transform of this is 1/S-A, so you differentiate this you end up getting, you N times differentiate every time you get a negative sign so you have -1 power N times 1/S-A power N + 1 and you have N factorial comes out, comes up in the numerator, so this is going to be 1 so you have N factorial divided by S –A power N + 1 so this is one example where you can use.

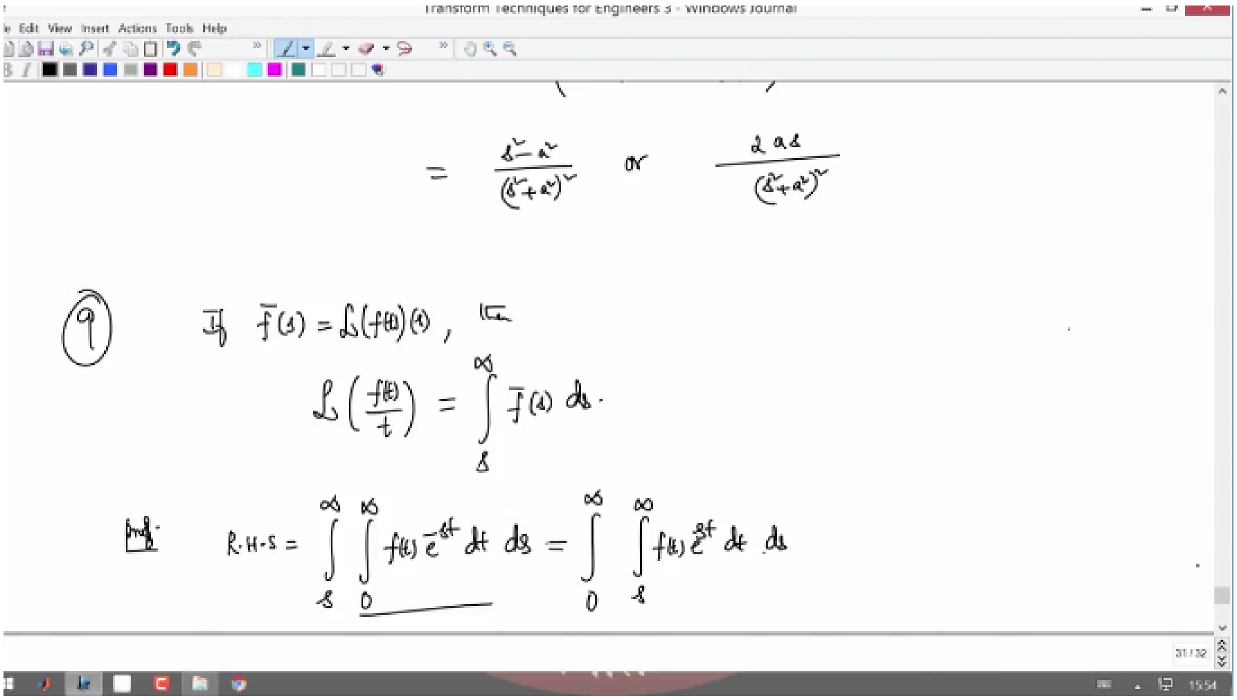


So if you have a T power N multiplication with known function whose transform if you know you can easily work out a product of these functions, so let me do some more examples one more example you can think of a Laplace transform(t) times cos AT so if you have a function like this as a function of S this is minus because this is T multiplication with T so you have – times 1 derivative of F bar, so Laplace transform of cos of AT which is function of S, so this is equal to –D/DS of, what is the Laplace transform of cos AT that we know that it is actually S divided by S square + A square, if it is a sine AT, if you have a sine AT if instead of cos you have a sign AT what we have is we will write here this or cos AT or sine AT let us say, okay, so sine AT or cos AT, cos AT or sine AT both the things you can look at it, so either this or A divided by S square + Asquare so this is what you have to do, either this or this depending on cos AT or sine AT, so what you end up is a -S square + A square whole square, so you have S square + A square - S times 2S, 2S square so you have a -S square comes out, so A square - S square, so with minus sign you have S square –A square divided by S square + A square whole square, whether this or and you have a minus derivative is S square + A square whole square, S square + A square times A that is 0, so you have minus minus plus and you have A times 2S, so 2AS, 2AS divided by S square + A square whole square, so either this or this depending on your Laplace transform of T times, cosine(AT) or sine(AT), so you will see this result 2.

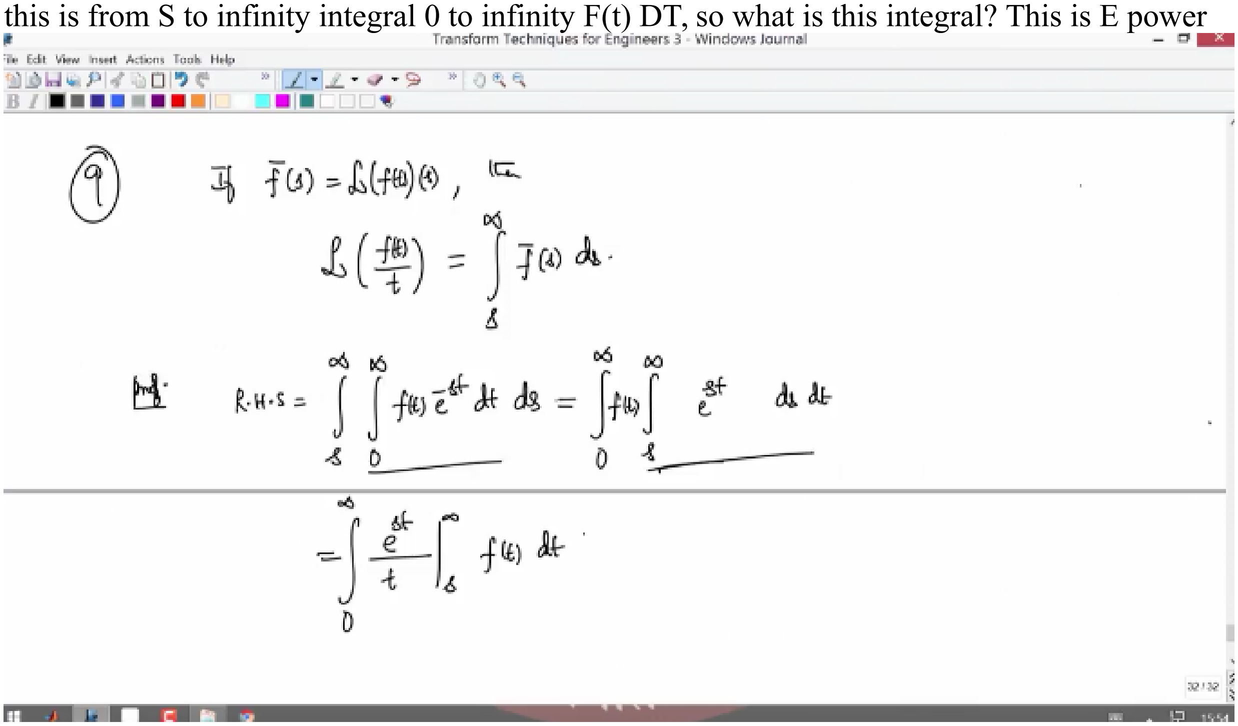


So we have used derivative so far, we can also get some more property of, one more property of Laplace transform that is in terms of the integral, integral of the Laplace transforms, now we can work out, so let me take this as a property 9, so property 9 is so we'll write the properties, if I have a Laplace transform which is a function of Laplace transform of T, F bar(s) is Laplace transform of F(t) then Laplace transform of F(t)/t, now we have this one if you have a division with T instead of multiplying what you get is actually integration from S to infinity F bar(S)/DS as a real variable, S is a real variable which you are integrating from S to infinity.

So this is where we use a property of uniform convergence of the Laplace transform integral, okay, so this proof is also simple so you start with, what you want to have let us start with the right hand side, so if you start with the right hand side that is from S to infinity, F bar(s) you have 0 to infinity, F(t) E power -ST DT this is DS, so you see this integral is uniformly convergent integral that F bar(s), so you are differentiating, you are integrating with respect to S from S to infinity, so this integral I can take it inside because of uniform convergence, so 0 to infinity I can do from S to infinity F(t) E power -ST DT DS, so this S integral I can take it

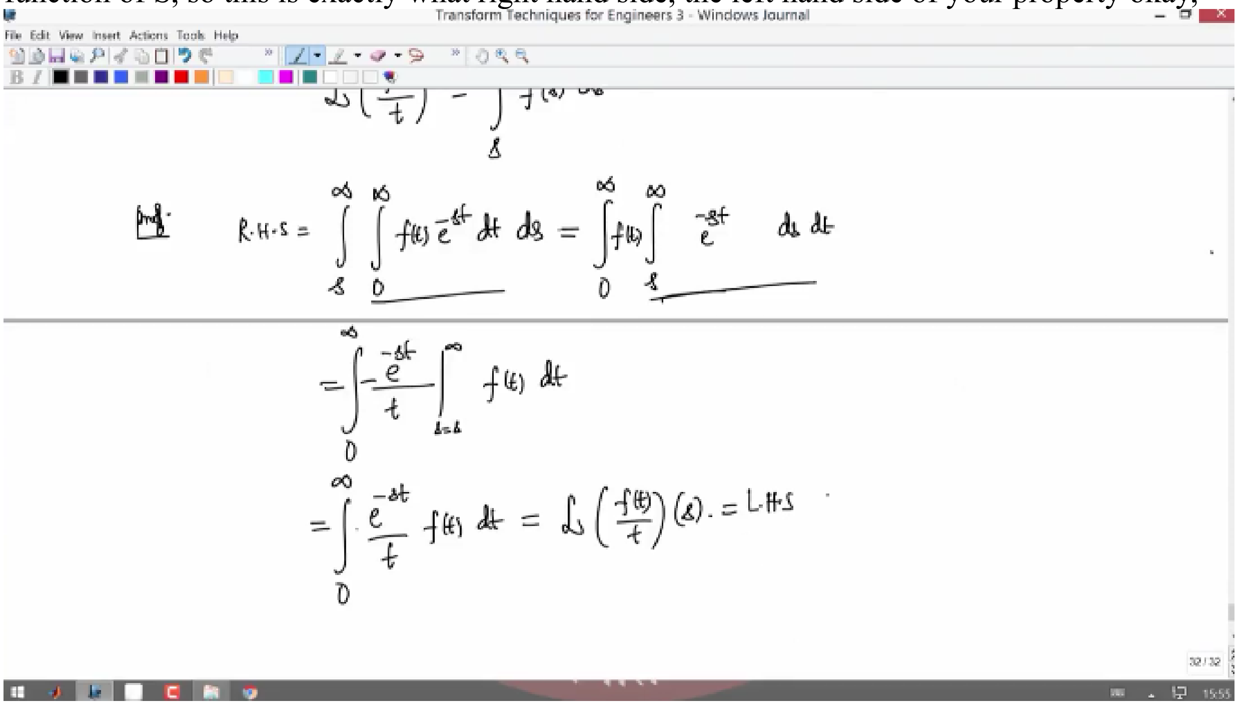


inside, so that is what, this integration we are doing inside so that you can write this here, and E power ST so F(t) you can put it here, so this integral you can evaluate if you do that evolution so what you get is, if you evaluate that you see that 1/S E power 1/T, so rather E power ST(t) this is from S to infinity integral 0 to infinity F(t) DT, so what is this integral? This is E power

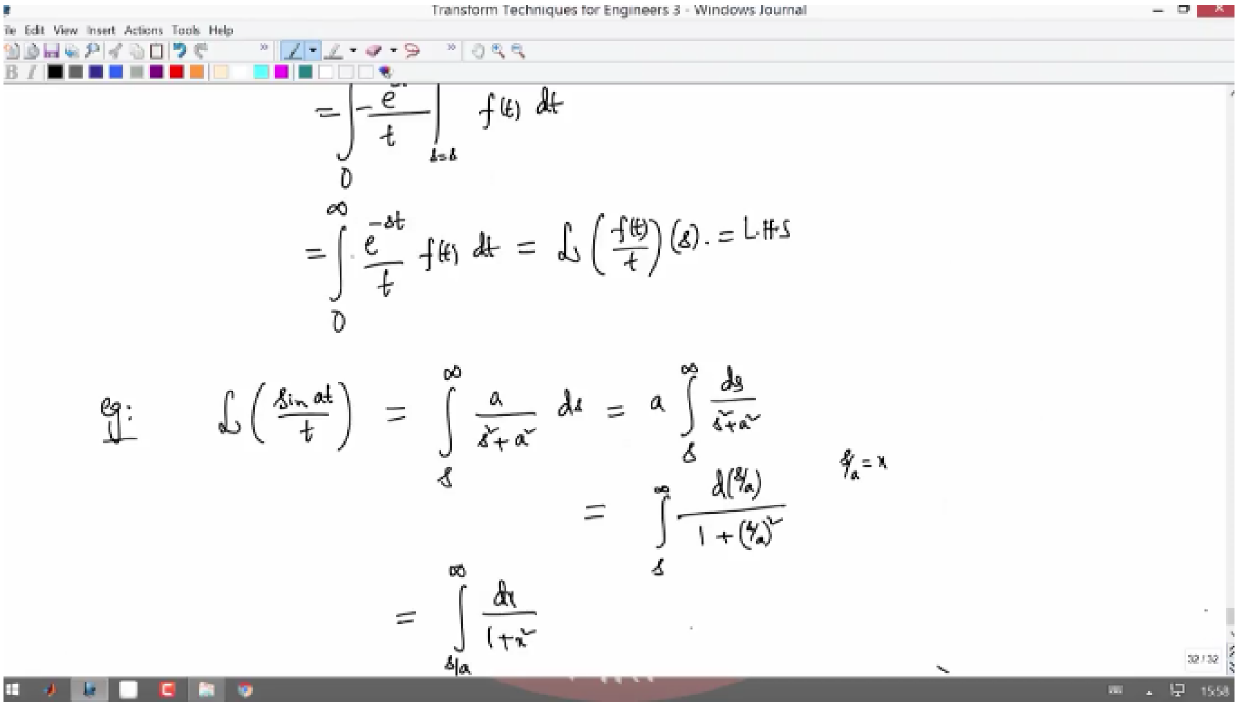


-ST right, so that’s -ST so that's missing so you have -ST you know minus, so you have minus here.

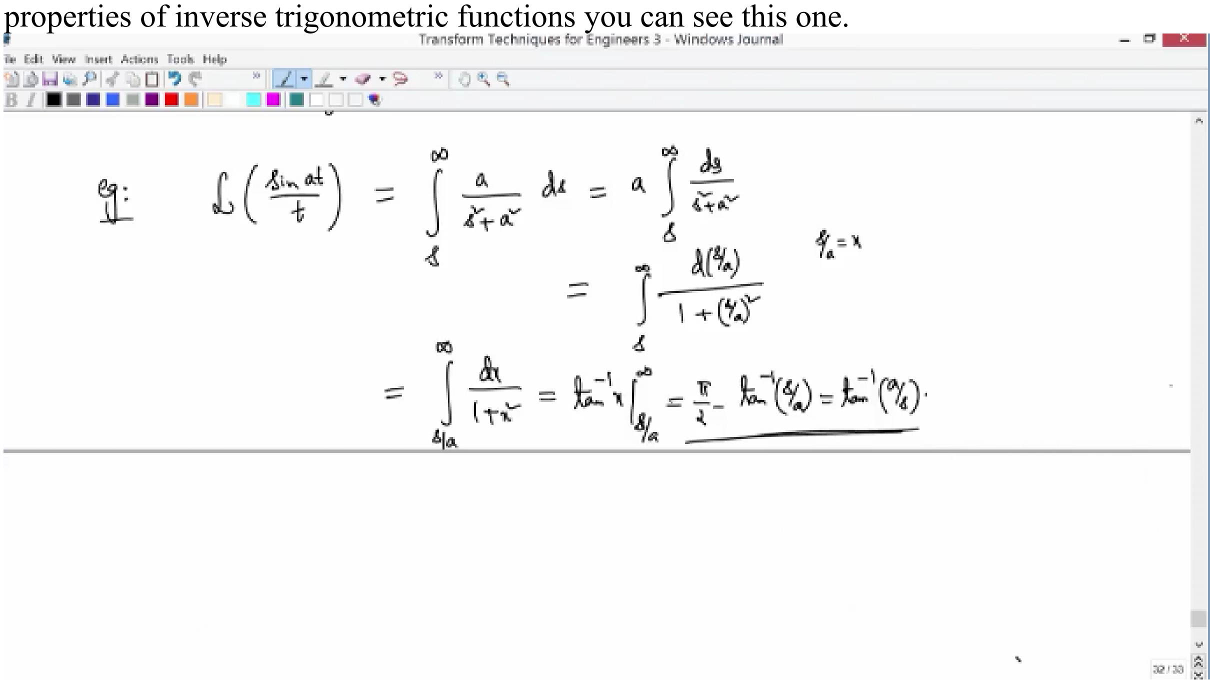
So at infinity again of course we are doing S integration so S = S to infinity, so when you put S = infinity this E power -ST because T is always positive, so T is positive so that means this at infinity is going to be 0 and minus minus plus you have E power minus S = S only, ST/T this is from integral 0 to infinity, and F(t) DT so this is nothing but Laplace transform of F(t)/T as a function of S, so this is exactly what right hand side, the left hand side of your property okay,



so this is how we show that if you have integration or if you have 1/T times of function which you know already what is the Laplace transform of F, then you simply integrate that to get the Laplace transform of 1/T times F(t).
So we can do, similarly we can do some examples example such as what is the Laplace transform of sine AT/T, this is again so by directly if you use S to infinity Laplace transform of S sine of, a sine AT that is A divided by S square + A square you have DS, so A comes out, S to infinity DS/S square + A square so this is equal to, so A square you take it out so you have 1/A comes out, S to infinity DS divided by 1 + S/A whole square, okay, so this is nothing but so you can also write take this and write D(s/a), so this is nothing but tan inverse S/A right, so this is equal to, you write S/A equal to some X, so you have DX/1+X square integral S/X is that so X is when you put X = S so you have S/A, okay.



So this is if you actually properly do it and put S/A = X, so you have DX/1+X square, X is now when you put S = S/A, S = S, so you have S/A to infinity and you put S = infinity that is actually infinity, so this is nothing but a tan inverse X from S/A to infinity, so what is this one? Tan inverse infinity that is pi/2 - tan inverse S/A, so this is also same as tan inverse A/S by the properties of inverse trigonometric functions you can see this one.



So like this you can use this property if you know the Fourier transform Laplace transform of some function if you want to find the Laplace transform of such function whose multiplied with 1/T you can use this property to evaluate the Laplace transform, so I'll stop here and we will see some more properties and how to find what are the ways to find the inverse transform, so far we are dealing with how to find the Laplace transforms, and we will see the properties how to find the inverse Laplace transforms. And finally before we move on to apply this Laplace transform technique to solve in the applications of solving differential equations another areas okay. So we'll see in the next video. Thank you very much

Online Editing and Post Production

Karthik
Ravichandran
Mohanarangan
Sribalaji
Komathi
Vignesh
Mahesh Kumar

Web-Studio Team

Anitha
Bharathi
Catherine
Clifford
Deepthi
Dhivya
Divya
Gayathri
Gokulsekar
Halid
Hemavathy
Jagadeeshwaran
Jayanthi
Kamala
Lakshmipriya
Libin
Madhu
Maria Neeta
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Muralikrishnan
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Parkavi
Poornika
Premkumar
Ragavi
Renuka
Saravanan
Sathya
Shirley
Sorna
Subhash
Suriyaprakash
Vinothini

Executive Producer
Kannan Krishnamurty
NPTEL Coordinator
Prof. Andrew Thangaraj
Prof. Prathap Haridoss
IIT Madras Production
Funded by
Department of Higher Education
Ministry of Human Resource Development
Government of India
www.nptel.ac.in