

NPTEL  
NPTEL ONLINE COURSE  
Transform Techniques for Engineers  
Laplace Transform of Elementary  
Functions  
With  
Dr. Srinivasa Rao Manam  
Department of Mathematics  
IIT Madras

# Transform Techniques for Engineers

## *Properties of Laplace Transforms*

Dr. Srinivasa Rao Manam  
Department of Mathematics  
IIT Madras



We have seen Laplace transform of elementary functions in the last video, in this video we will see how to, what are the properties of this Laplace transforms, some very few of its properties will derive, as in the case we have done for Fourier transform.

And then before we move on, before we apply these techniques, these properties and Laplace transform to applications, okay, before we apply to differential equations and solve them we'll just derive some properties of this Laplace transform in this video.

We can also do some other, some more complicated functions we can calculate Laplace transform, but let's refrain from doing that because it's as and when it is required we can do, we can find the Laplace transform of certain functions, other functions, other than these elementary functions what we have done so far.

So let me start with the properties, first properties is properties of Laplace transform, so to start with this kind of shifting property, if  $F(s)$  is the Laplace transform which is a Laplace transform of  $F(t)$  which is function of  $S$ , then Laplace transform of  $E^{-At}$  times  $F(t)$  is actually equal to Fourier Laplace transform of at  $S+A$  where  $A$  is a real, or  $A$  is a real number, so you can easily see this one if  $F(s)$  is actually representing the Fourier Laplace transform of  $F$  which is assumed to be exponential function of some order, then we will see what is its product with the exponential function  $E^{-At}$  for which if you take the Laplace transform, and then it's a function of  $S$ , so by definition we can write  $0$  to infinity  $E^{-At}$

$F(t) e^{-st}$  power  $-ST$  DT, so you can combine these two to write  $0$  to infinity  $e^{-st}$  times  $F(t)$  DT, so this is clearly the Laplace transform of  $F(s+A)$  so as simple as this, if  $A$  is not real complex then you may have to worry, so even otherwise there should not be a problem because  $e^{-st}$ , so if  $A$  is a real constant then this is the case, if  $A$  is a complex number for example if it is like this,  $A$  is a complex number and then  $S$  we know that  $S$  is greater than  $A$ ,  $S$  is greater some, suppose if you look at this function  $e^{-st}$  times  $F(t)$  which is exponential of order, some order, so  $S$  is greater than beyond that number, so let us say  $A$  is exponential, is exponential of order and let us say this is  $-A$  so you have, if you call this  $C$ ,  $C-A$  let us say, if this is order  $S$  is greater than  $C-A$ , so this is the number, so  $S$  is greater than this, so real part of  $S$  has to be, so  $S$  is complex numbers such that real part of  $S$  is greater than  $C-A$ ,  $C-A$  if it's a complex number, so if  $A$  is also complex number so you can see that real part of  $C-A$  you can put it this way.

Properties of Laplace transforms:

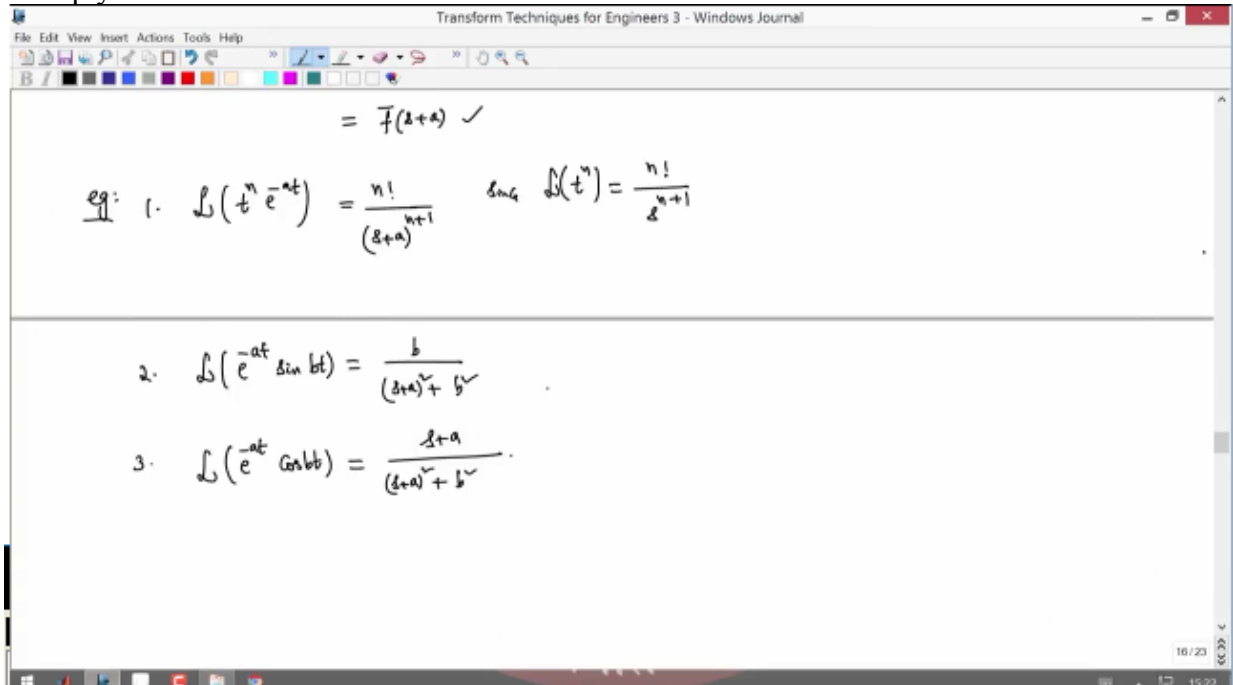
1. If  $F(s) = \mathcal{L}\{f(t)\}$  then  $\mathcal{L}\{e^{-at} f(t)\} = F(s+a)$ ,  $a \in \mathbb{R}$ .

$$\begin{aligned}
 \mathcal{L}\{e^{-at} f(t)\}(s) &= \int_0^{\infty} e^{-at} f(t) e^{-st} dt \\
 &= \int_0^{\infty} e^{-t(s+a)} f(t) dt \\
 &= F(s+a) \checkmark
 \end{aligned}$$

$e^{-at} f(t)$  is exponential of order  $C-a$ .

And then so  $S$  is, so you can think of  $S$  is greater than  $C-A$ , so that only real part of  $S$ , so that real part of  $S$  is bigger than that, okay, so it's still possible if it is a complex, but so anyway so if it is a real this is  $a$ , you can easily see, so if  $A$  is real this is simply a  $C-A$  so real part of  $S$  so that is greater than  $C-A$ , so this is clear as a property one so, what it says is if you multiply your function, some function with the exponential function with  $-AT$ , so if it Laplace transform of it, if you know the Laplace transform of  $F$  that is product is simply Laplace transform with  $S$  is translated to  $S+A$ , so this is one property through which you can give some example, so for example what is the Laplace transform of, you can represent Laplace transform like this or bar so you have  $T$  power  $N$   $e^{-AT}$ , so  $T$  power  $N$  you know what is its Laplace transform, Laplace transform of  $T$  power  $N$  is, since Laplace transform of  $T$  power  $N$  is  $N$  factorial divided by  $S$  power  $N+1$ , so because of this the Laplace transform this by this property you have  $N$  factorial divided by  $S+A$  power  $N+1$ , okay, so this is one example where you can use. And the second example so we have already seen many elementary examples for which you have Laplace transforms, so you can write for example  $e^{-AT}$  sine or cos  $BT$  for example in this case, for sine because we know that is  $B$  divided by  $S^2 + B^2$  because of this exponential multiplication we have to replace  $S$  by  $S+A$ , so I'll write  $S+A$  whole

square, similarly Laplace transform of  $E \cos BT$ ,  $\cos BT$  if you calculate that is  $S$  divided by  $S^2 + B^2$  that is with cosine, without this exponential this is the Laplace transform, so if I multiply this with  $E^{-AT}$  so you have to replace  $S$  by  $S+A$ , so you have  $S$  by  $S+A$  whole square by the property, so this is how you can, elementary functions if you multiply with the exponential functions you can easily without calculating its interval so you can directly write replacing  $S$  by  $S+$  some constant, whatever exponential order which you multiply.



Second property that we use another shifting property here, if Laplace transform of  $F(t)$  which is function of  $S$  is  $\bar{F}(s)$  then Laplace transform of  $F(t-a)$  times Heaviside function with  $T-A$ , this is actually equal to  $E^{-AS}$  Laplace transform of  $F$ , so this is what you have, so this way that also you can easily prove, you start with the left hand side a Laplace transform of  $F(t-a)$  times Heaviside function of  $T-A$  as a function of  $S$  this should be, by definition  $\int_0^{\infty} F(t-a) H(t-a) e^{-st} dt$  this is the definition, so because  $H$  is 1 if  $T$  is greater than  $T-A$  is positive so that is  $T$  is  $A$  to infinity, this is 1, so you have  $F(t-a)$  times  $E^{-st} dt$ , so put this  $T-A$  as a new variable,  $S$  we call this  $X$  so you have  $DT = DX$  so that your, when you  $T = A$ ,  $X$  is 0, when you put  $T = \infty$ ,  $X$  is infinity,  $F(x) E^{-ST}$  is  $A+X$ , so  $A+X$  so this  $SA$  comes out and you have  $DT$  is  $DX$ , so this is nothing but  $E^{-SA}$  times  $\bar{F}(s)$ ,  $X$  is the simply the number, this is exactly what we need.

2. If  $\mathcal{L}\{f(t)\}(s) = \bar{f}(s)$ , then  $\mathcal{L}\{f(t-a)H(t-a)\}(s) = e^{-as}\bar{f}(s)$ .

$$\begin{aligned} \mathcal{L}\{f(t-a)H(t-a)\}(s) &= \int_0^{\infty} f(t-a)H(t-a)e^{-st} dt \\ &= \int_a^{\infty} f(t-a)e^{-st} dt \end{aligned}$$

$t-a = \tau$   
 $dt = d\tau$

---


$$\begin{aligned} &= \int_0^{\infty} f(\tau)e^{-s(\tau+a)} d\tau \\ &= e^{-sa}\bar{f}(s). \end{aligned}$$

If you're using this you can easily see for example if  $F$  of simple example you can take this, if it's a constant function what is this one, so you can see that Laplace transform of Heaviside function simply, if  $F(t)$  is 1,  $\bar{F}(s)$  is  $1/s$  we know, and then  $E$  power  $-sa$ ,  $E$  power  $-sa$  because we know that  $F(a)$  Laplace transform is  $1/s$ , so this is what you can see Laplace transform or Heaviside function is this one, okay, if  $A$  is 0 and you have this  $1/s$  is the Laplace transform of Heaviside function which is anyway 1 okay, Heaviside function doesn't make any sense if you take from 0 to infinity it's 1, and if it is 0 it's 1 here because it's a constant function we're only looking at positive side if you are applying Laplace transform, the positive side is just a constant function you don't say anything so Laplace transform, translated version of Laplace transform that is 0 here it's power 1 here and 0 here so a little bit this positive between 0 to some  $A$ , if it's 0 and then if it is 1 here that make sense, so the translated version of this Laplace Heaviside function, Laplace transform is multiplied with the exponential function  $E$  power  $-sa$  okay.

Other example which you can do is if  $F(t)$  is this, let's say 1 if between  $T$  less than 1, -1 if it's  $T$  is less than 2, 1 between 1 and 2, 0 if  $T$  is greater than 2 for example, so what you have is the function  $F$  which is 0 to 1, it's 1, and it is -1 and again so everywhere else it is 0, so you have this, this is what is you have between 1, 2 this is -1, between 0 to 1 it is 1, okay, so this is what you have, so if this is the case what is the Laplace transform of  $F(t)$  which is a function of  $S$ , I can write this function as an application of this property we can write this function  $F$  is as Heaviside function, so this is for example 1, you can write how we can represent this  $F(t)$  as  $1 - 2$  times  $H(t-1)$  and then you make it  $+ 2$  times  $H(t-2)$ , so if you have a wave function repeated wave function for example like this 0 to 1, and this again if it repeats like this you go on adding, adding and subtracting, for example if I have this and this I may have to add  $-2$  times  $H(t-3) + 2$  times  $H(t-4) + 0$  and so on, if you have as many as you want as a periodic function if you have we can write, represent like this because we have only these two so you can easily verify this one.

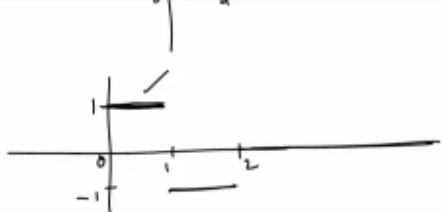
Transform Techniques for Engineers 3 - Windows Journal

$= e^{-sa} f(s)$

eg: 1.  $f(t) = 1$ ,  $\mathcal{L}\{H(t-a)\} = e^{-sa} \frac{1}{s}$  ✓

2.  $f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 0, & t > 2 \end{cases}$  ✓

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\left(\right)$



$f(t) = 1 - 2H(t-1) + 2H(t-2)$

$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$

17/23

If T is between 0 to 1 what happens? T is between 0 to 1, if T-1 is negative that is 0 this is, this will be 0 and this will be between 0 to 1, and this is going to be between, T is between 0 to 1 this is also negative, this is 0 so it is 1, so both are 0 here so it is 1, so this is clear between, T is between so we can easily see F(t) is which is 1 if T is between 0 to 1, if T is between 1 to 2 what happens here, so this is 1-2 times H(t-1) when you have T is between 0 to 2 that is between 0 to 1 that is 1, so -2 and this will be 0 because 0 to -1 so this is 0 so you have -1 here and between greater than 2, 1-2 between 2 greater than 1 so this is always 1-2 and then this plus this is also 2, so if it's greater than 1 if it's T is greater than 2 what happens, this is 1 1-2 times, T is greater than 1 2-1 so which is always 1 will be 1 so that is -2 + here 0 to positive side this is also 2, so this is also 1, so you have canceled this is 1, so this is actually 1, if T greater than 2 this is 1 this kind of representation is actually 1 here, okay if such a function if you choose what is the Laplace transform of this is actually you can represent like this that function what you have is both are same these two are the same so you can write  $1 - 2H(t-1) + 2H(t-2)$  as a function of S, so because it's a linear operator this integral so you can split this into, as this is your function 3 terms so you can write it as 3 integrals, each integral will represent the Laplace transform of 1 that is, this is the Laplace transform of 1 as a function of S- 2 times the Laplace transform of H(t-1) this is function of S +2 times H Laplace transform of H(t-2) as a function of S, so you can write  $1/S$  if it's a constant function, and -2 times H(t-1) now we can use this property, E power -SA so we have E power -SA is 1, A is 1 so you have by  $S + 2$  times here E power -2S divided by S, so this is exactly what we have as  $F(s)$  if F is given like this, okay.

Transform Techniques for Engineers 3 - Windows Journal

File Edit View Insert Actions Tools Help

B / [color palette]

Ex: 1.  $f(t) = 1$ ,  $\mathcal{L}\{H(t-a)\} = e^{-sa} \frac{1}{s}$  ✓

2.  $\mathcal{F}\{f(t)\} = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$  ✓

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{1 - 2H(t-1) + 2H(t-2)\}(s)$   
 $= \mathcal{L}\{1\}(s) - 2\mathcal{L}\{H(t-1)\}(s) + 2\mathcal{L}\{H(t-2)\}(s)$

$= \frac{1}{s} - 2 \frac{e^{-s}}{s} + 2 \frac{e^{-2s}}{s}$

$f(t) = 1 - 2H(t-1) + 2H(t-2)$  ✓

$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

18/23

So what you have is it's not 0 here, what you have is it's completely 1 here, such a thing okay that is your  $F(t)$ , but instead of this if you have again if it repeat it's like a wave you may have to keep on adding it, you may have to add and so on like this alternatively, okay, so minus, first minus and then plus and so on we'll get, so you keep on adding with terms  $T-3$ ,  $T-4$  and so on,  $H(t-3)$   $H(t-4)$ , the multiplication factor to which you add subtract, add subtract, as an infinite series that represents this wave function, periodic wave function that you can see.

Transform Techniques for Engineers 3 - Windows Journal

File Edit View Insert Actions Tools Help

B / [color palette]

Ex: 1.  $f(t) = 1$ ,  $\mathcal{L}\{H(t-a)\} = e^{-sa} \frac{1}{s}$  ✓

2.  $\mathcal{F}\{f(t)\} = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$  ✓

$\mathcal{L}\{f(t)\}(s) = \mathcal{L}\{1 - 2H(t-1) + 2H(t-2)\}(s)$   
 $= \mathcal{L}\{1\}(s) - 2\mathcal{L}\{H(t-1)\}(s) + 2\mathcal{L}\{H(t-2)\}(s)$

$\mathcal{F}(s) = \frac{1}{s} - 2 \frac{e^{-s}}{s} + 2 \frac{e^{-2s}}{s}$

$f(t) = 1 - 2H(t-1) + 2H(t-2)$

$f(t) = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \\ 1, & t > 2 \end{cases}$

17/23

So we'll move on to another property, these are the examples for this property that what you have seen, this shift property so next we'll look at the other property of this Laplace transform that is a scaling property this is simple so if a Laplace transform of  $F(t)$  is  $\bar{F}(s)$  then Laplace



power  $-S/A$ , if  $A$  is negative minus here, so look onto this special case later on, I'm not getting exactly the same form, so we'll see if  $A$  is positive this is the case, we will stop here, so if  $A$  is negative case we will see later, so using this so let's choose this as  $A$  positive, okay, so if  $A$  positive,  $A$  positive, so let me choose this way, okay, so this is what is clear.

Transform Techniques for Engineers 3 - Windows Journal

3. If  $\mathcal{L}\{f(t)\} = \bar{f}(s)$ , then  $\mathcal{L}\{f(at)\} = \frac{1}{|a|} \bar{f}\left(\frac{s}{a}\right)$ ,  $a \neq 0$

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(at) e^{-st} dt = \int_0^{\infty} f(x) e^{-\frac{s}{a}x} dx \quad \begin{matrix} at = x \\ a dt = dx \text{ if } a > 0 \end{matrix}$$

$$= \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$$

If  $a < 0$ ,

$$= \frac{1}{a} \int_0^{-\infty} f(x) e^{-\frac{s}{a}x} dx \quad \begin{matrix} at = x \\ a dt = dx \text{ if } a < 0 \\ -x = t \\ -dx = dt \end{matrix}$$

$$= -\frac{1}{a} \int_0^{\infty} f(-t) e^{\frac{st}{a}} dt \quad \checkmark$$


---


$$= \frac{1}{a} \int_0^{\infty} f(t) e^{-\frac{s}{a}t} dt \quad \checkmark$$

19/23

So using this so we can do some examples such as if you take like earlier you have this wave function 0 to infinity, you have square wave function repeated like this and so on, okay, if you have like that so your  $F(t)$  is that  $H(t) - 2 \text{ times } H(t - A) + 2 \text{ times } H(t - 2A) - 2 \text{ times } H(t - 3A) + 2 \text{ times } H(t - 4A) - \dots$ , so suppose this is with  $A$ ,  $A$  is wave period so every time this is  $A$ , every time this is  $A$ , so  $A + 2A, 3A$  and so on,  $4A$ , so these are the values it was, every time you are adding so let me write it as  $T - A + 2 \text{ times Heaviside side function as } T - 2A$ , again now repeats with this you can easily see this represents this one and this one, so if you add two more it'll represent this one, so we will add two more so for example  $H(t - 3a) + 2 \text{ times } H(t - 4a)$  and so on, you go on get it, so you'll go on getting this as a series, series represents this wave function, periodic wave function from  $T$  positive side, so what happens to this Fourier transform of Laplace transform of this, is actually you can see, if you actually see this one  $F(t)$  and you have this series here, some series  $F(t)$  into  $E$  power  $-ST$  DT because of this  $E$  power  $X -ST$  which is exponential function, exponentially decaying function, this series if you put it, when you put together this function you can put it into the series  $F(t)$  and that series is uniformly convergent so you can take this integral inside and that sense you can do term-by-term integration, so if you do that you can easily see that the Laplace transform of  $H(t)$  that is  $1/S$  that is the first one.

Next is 2 times  $E$  power  $H$  Laplace transform of  $T - A$  that is  $E$  power  $-AS$  divided by  $S$ , again 2 times  $E$  power minus of, this is  $2AS$ ,  $2AS$  divided by  $S$ , and next one is  $-2 \text{ times } E$  power  $-3AS$  divided by  $S$  and so on, so what this represent is?  $1/S$  comes out you have  $1 - 2 E$  power  $-AS$  if you take common so we end up getting one here minus this one  $E$  power  $-AS + E$  power  $-2AS$  and so on, you will get, you will be getting this series inside, so we get  $1/S$  times  $1 - 2 \text{ times } E$  power  $-AS$  this series is this is less than 1,  $E$  power  $-AS$  is always, so if you write like this  $E$  power  $-2AS$   $1$  over  $1 + E$  power  $-AS$ ,  $1$  over  $1 + E$  power  $-AS$  is this series okay, because modulus of  $E$  power  $-AS$  you can assume that this is less than 1 as a geometric progress series,



for sufficiently big S you can always choose, S you have in your control and your given function F, S you can choose in such a way that S is bigger than this function exponential order and S is you can always choose somewhere big so that this quality is less than 1, so you have the series is actually geometric progress series, so that is this.

The image shows a screenshot of a software application window with a toolbar at the top. The main area contains handwritten mathematical work. It starts with an example function  $f(t)$  defined as a series of rectangular pulses:  $f(t) = H(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + 2H(t-4a) - \dots$ . Below this, the Laplace transform  $\bar{f}(s)$  is calculated as an integral from 0 to infinity of  $f(t)e^{-st}$  dt. This leads to a series of terms:  $\frac{1}{s} - 2\frac{e^{-as}}{s} + 2\frac{e^{-2as}}{s} - 2\frac{e^{-3as}}{s} + \dots$ . The series is then factored into  $\frac{1}{s} (1 - 2e^{-as} (1 - e^{-as} + e^{-2as} - \dots))$ . A note  $|e^{-as}| < 1$  is written to the right. The final simplified result is  $\frac{1}{s} (1 - 2e^{-as} \frac{1}{1+e^{-as}})$ .

So if you actually simplify this will become  $1/S$  times  $1 + E$  power  $-AS$   $1 + E$  power  $-2AS$  that is going to be  $1 - E$  power  $-AS$ , so this is nothing but  $1/S$  times you can represent this as  $E$  power  $-AS/2 + AS/2$  or rather  $+ A/2$  minus minus, and so that here also you can write  $AS/2 + S/2$  so this together is, this is same as the earlier one, so let me write separately this is one way, this is what you have earlier so what you do is  $1/S$  times you take  $E$  power  $-AS/2$  out here so you get  $E$  power plus here, to get 1 you have to add  $AS/2$  and here because you have taken this out another half that remains  $AS/2$  divided by again here also you do the same technique  $AS/2$  if you take so you have  $AS/2$  here, plus here you have  $-AS/2$ , so this gets cancelled what you have is  $1/S$  times,  $1/S$  times this is nothing but  $\tan$  hyperbolic  $SA/2$ , so this is exactly your Laplace transform of this function, such a function this periodic wave function which repeats, so that is what is the Laplace transform we can calculate easily.

Transform Techniques for Engineers 3 - Windows Journal

$$\begin{aligned} \bar{f}(s) &= \int_0^{\infty} f(t) e^{-st} dt = \frac{1}{s} - 2 \frac{e^{-as}}{s} + 2 \frac{e^{-2as}}{s} - 2 \frac{e^{-3as}}{s} + \dots \\ &= \frac{1}{s} \left( 1 - 2e^{-as} (1 - e^{-as} + e^{-2as} - \dots) \right) \quad |e^{-as}| < 1 \\ &= \frac{1}{s} \left( 1 - 2e^{-as} \frac{1}{1+e^{-as}} \right) \\ &= \frac{1}{s} \frac{1 - e^{-as}}{1 + e^{-as}} = \frac{1}{s} \frac{e^{\frac{as}{2}} (e^{\frac{as}{2}} - e^{-\frac{as}{2}})}{e^{\frac{as}{2}} (e^{\frac{as}{2}} + e^{-\frac{as}{2}})} \\ \bar{f}(s) &= \frac{1}{s} \tanh\left(\frac{sa}{2}\right) \end{aligned}$$

So you can easily see that this example what we used this scaling, actually we have not used the scaling here, right, we have not used this is one scaling property, scaling property we have not

Transform Techniques for Engineers 3 - Windows Journal

3. If  $\mathcal{L}\{f(t)\} = \bar{f}(s)$ , then  $\mathcal{L}\{f(at)\} = \frac{1}{a} \bar{f}\left(\frac{s}{a}\right)$ ,  $a > 0$ .

$$\begin{aligned} \mathcal{L}\{f(at)\} &= \int_0^{\infty} f(at) e^{-st} dt = \frac{1}{a} \int_0^{\infty} f(x) e^{-\frac{s}{a}x} dx \quad \begin{matrix} at=x \\ a dt=dx \end{matrix} \quad \text{if } a > 0 \\ &= \frac{1}{a} \bar{f}\left(\frac{s}{a}\right) \end{aligned}$$

eg:  $f(t) = \underline{f(t)} - 2 \underline{H(t-a)} + 2 \underline{H(t-2a)} - 2 \underline{H(t-3a)} + 2 \underline{H(t-4a)} - \dots$

$$\begin{aligned} \bar{f}(s) &= \int_0^{\infty} f(t) e^{-st} dt = \frac{1}{s} - 2 \frac{e^{-as}}{s} + 2 \frac{e^{-2as}}{s} - 2 \frac{e^{-3as}}{s} + \dots \\ &= \frac{1}{s} \left( 1 - 2e^{-as} (1 - e^{-as} + e^{-2as} - \dots) \right) \quad |e^{-as}| < 1 \end{aligned}$$

used, so this is a scaling property this is not an example for scaling property, so this is the general example, so example of periodic function, okay, Fourier transform, example of periodic function, this is not actually application of this property. So anyway, so you have these are the functions we don't need to scale here so directly if you calculate so you end up getting this one, so in general if F is a periodic function another example is general periodic function if you take if you know that F(t) is a periodic function, there is a periodic function with period let us say A then such is this one, okay, so this is a periodic function with period A it repeats right, it's

actually this is actually repeating, this is a periodic function of period  $2A$  here, okay, and that is not  $A$ , so here we choose this periodic function with period  $A$  and if Laplace transform of  $F$  exists, then what is the Laplace transform of  $F(t)$  when exists, when it exists is Laplace transform of  $F(t)$  which is equal to  $1 - e^{-AS}$  for which you take the inverse, so  $1$  over this,  $1$  over  $1 - e^{-AS}$  times  $F$  so you have this integral  $0$  to  $A$  with your period, and  $e^{-sT} F(t) dt$ , so this you can easily see directly by calculating what is the Laplace transform, so what is given solution for this is if  $F(t)$  is a function with period  $A$ , so this is  $F(t+a)$  for every  $T$  positive this is the case, and assume that  $A$  is positive so  $A$  is also positive period  $A$ .

So what happens if you calculate the Laplace transform of  $F(t)$  as a function of  $S$  which is  $\bar{F}(s)$  as  $0$  to infinity  $F(t) e^{-sT} dt$  by definition, if it exist as an integral what you do is you write this as between  $0$  to  $A$ ,  $F(t) e^{-sT} dt + A$  to infinity I split this into two integrals, so what you do here is I try to put this as  $T+A$  as  $X$ , okay, or  $T-A = X$ , if I use this change of variable in this integral, okay, so what you get is  $dt$  is  $dx$  so you have this as it is,  $F(t) e^{-sT} dt +$  this integral  $F$  of,  $T$  is  $A+X$ ,  $X+A$ ,  $F(x+a)$  is nothing but  $F(x)$  because

Transform Techniques for Engineers 3 - Windows Journal

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-AS}} \int_0^A f(t) e^{-st} dt$$

Assume: If  $f(t) = f(t+a)$ , for  $t \geq 0$ ,  $a > 0$ .

$$\mathcal{L}\{f(t)\} = \bar{F}(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$= \int_0^A f(t) e^{-st} dt + \int_A^{\infty} f(t) e^{-st} dt$$

$t - a = x, dt = dx$

$$= \int_0^A f(t) e^{-st} dt + \int_0^A f(x+a) e^{-s(x+a)} dx$$

of periodicity, so this times  $e^{-sT}$  is  $X+A$  times  $dt$  is  $dx$  when you put  $T = A$ ,  $X = 0$ , and  $T$  equal to infinity this is infinity, so you end up getting  $0$  to  $A$ ,  $F(t) e^{-sT}$  times  $dt +$  here  $e^{-sA}$  comes out this integral is  $F(x+a)$  is  $F(x) e^{-sX}$   $dx$  this is nothing but Fourier Laplace transform of  $F$ , so this is  $\bar{F}(s)$  this is also  $\bar{F}(s)$ , so if you combine this is the result,  $\bar{F}(s)$  is  $1$  over  $1 - e^{-AS}$  times this integral  $F(t)$  times  $e^{-sT}$

Transform Techniques for Engineers 3 - Windows Journal

$$= \int_0^a f(t) e^{-st} dt + \int_a^\infty f(t) e^{-st} dt$$

$t - a = x, \quad dt = dx$

$$= \int_0^a f(t) e^{-st} dt + \int_0^\infty f(x+a) e^{-s(x+a)} dx$$

$$\bar{f}(s) = \int_0^a f(t) e^{-st} dt + e^{-sa} \bar{f}(s)$$

$$\Rightarrow \bar{f}(s) = \frac{1}{1 - e^{-sa}} \int_0^a f(t) e^{-st} dt.$$

21 / 23

DT, so if you know that is Apriori it's a periodic function you can simply if you evaluate this integral that is enough to get your Laplace transform, okay, that's what, this technique you can apply here rather here also you can use in this example where you choose this, this series infinite thing you have, this series, this is a periodic function with period  $2A$  so if you use this periodic function  $2A$  so you end up getting this one, you can directly calculate from this integral from this formula also you can get the same result, okay.

Transform Techniques for Engineers 3 - Windows Journal

$a$

④. . . If  $f(t)$  is a periodic function with period ' $a$ '.

Then  $\mathcal{L}\{f(t)\}$ , when it exists, is

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t) dt.$$

th: If  $f(t) = f(t+a)$ , for  $t \geq 0$   $a > 0$ .

$$\mathcal{L}\{f(t)\} = \bar{f}(s) = \int_0^\infty f(t) e^{-st} dt.$$

19 / 23

So this property for a periodic function what is its Laplace transform, that we can call this as property number 4, okay, so 4 is the this property if you have a periodic function the Laplace transform of it is given like this, and for which you have seen this example and property 3 is

Transform Techniques for Engineers 3 - Windows Journal

File Edit View Insert Actions Tools Help

3. If  $\mathcal{L}\{f(t)\} = \bar{F}(s)$ , then  $\mathcal{L}\{f(at)\} = \frac{1}{a} \bar{F}\left(\frac{s}{a}\right)$ ,  $a > 0$ .

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(at) e^{-st} dt = \int_0^{\infty} f(x) e^{-\frac{s}{a}x} dx \quad \begin{matrix} at = x \\ a dt = dx \end{matrix} \quad \text{if } a > 0$$

$$= \frac{1}{a} \bar{F}\left(\frac{s}{a}\right) \quad \checkmark$$

Example of periodic function:

$$f(t) = f(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + 2H(t-4a) - \dots \quad \checkmark$$

$$\bar{F}(s) = \int_0^{\infty} f(t) e^{-st} dt = \frac{1}{s} - 2 \frac{e^{-as}}{s} + 2 \frac{e^{-2as}}{s} - 2 \frac{e^{-3as}}{s} + \dots \quad \checkmark$$

18 / 23

this I'm rounding up whichever is the property, examples without rounding up, so second is this property, and the first one is, these are the examples and this is our first property, so we'll see one by one and if you see look at the property 3 that is if  $\bar{F}(s)$  is the Laplace transform of  $F(t)$  and this is what we have seen if  $A$  is positive, if  $A$  is negative you can work out similarly and see that, you will see the same thing so this is actually valid for any  $A$  positive, so for any  $A$  real number.

Transform Techniques for Engineers 3 - Windows Journal

File Edit View Insert Actions Tools Help

$$f(s) = \frac{1}{s} - 2 \frac{e^{-as}}{s} + 2 \frac{e^{-2as}}{s} - \dots$$

3. If  $\mathcal{L}\{f(t)\} = \bar{F}(s)$ , then  $\mathcal{L}\{f(at)\} = \frac{1}{a} \bar{F}\left(\frac{s}{a}\right)$ ,  $a \in \mathbb{R}$

$$\mathcal{L}\{f(at)\} = \int_0^{\infty} f(at) e^{-st} dt = \int_0^{\infty} f(x) e^{-\frac{s}{a}x} dx \quad \begin{matrix} at = x \\ a dt = dx \end{matrix}$$

$$= \frac{1}{a} \bar{F}\left(\frac{s}{a}\right) \quad \checkmark$$

Example of periodic function:

$$f(t) = f(t) - 2H(t-a) + 2H(t-2a) - 2H(t-3a) + 2H(t-4a) - \dots \quad \checkmark$$

$$\bar{F}(s) = \int_0^{\infty} f(t) e^{-st} dt = \frac{1}{s} - 2 \frac{e^{-as}}{s} + 2 \frac{e^{-2as}}{s} - 2 \frac{e^{-3as}}{s} + \dots \quad \checkmark$$

18 / 23

So let me say this whatever may be  $A$ , so it just doesn't matter so this is actually true, so we will see that  $1/A$  so this property is true for every  $A$  belongs to real numbers, so that's property 3, and you have seen a periodic function if you have, what is its Laplace transform and that is by

this formula as this, so if you work out if you apply that formula to this function with the period 2A so you can come to the same conclusion that is this, so you can verify that, that we can give is an exercise, so directly you calculate it as a series and you use the property of linear property of this Laplace transform, that is obvious property that we used and the series, a converging uniformly so you have this integral you can do term by term integration, so I have not shown why this is uniformly convergent this integral, Laplace transform and it exists it's uniformly

convergent for some, from S greater than some number, some real number, real part of S is bigger than, let us say some A, where A is order of exponentiality of the function F(t) if that is the case you can do, once this is uniformly convergent if this series is also kind of such thing if you have a series like this and you can do the term by term, so this is the kind of series you have, if the series is you can see that is uniformly convergent, you can see that this is actually true because it's a periodic function with period 2A, so it's between 0 to 2A is actually finite thing, this is valid and this is a continuous function on a finite this is on an interval, this is a bounded so that is actually this uniformly convergent that's what you can easily see, so because of that is uniformly convergent series so if you integrate so you can take this integral inside that's what, that is the property we have used and see that this is your transformation, so this is a Laplace transform of such series, such uniformly convergent series.

Instead if you see that function, given function which is a series which is a periodic function of period 2A and you apply this formula, so you end up getting you calculate only this integral you will see that you get the same result okay, so and we can also see some property that is a property of the Laplace transform, that is Laplace transform is uniformly convergent, so let me do that, so we use this in the derivatives differentiation we use this property that is let me call this some, this is the property number 4, so let me see call it 5, this property 5 is, if F bar(s) is a, is the Laplace transform of F(t) as a function of S then rather let me put it, so if F(t) is exponential function of order E power AT, as T goes to infinity then the Laplace integral that is a Laplace transform then this integral E power -ST or F(t) times E power -ST is uniformly convergent, uniformly convergent with respect to S, for S is greater than or equal to, or rather S is greater than A, okay, so this you can easily, this is a kind of form so you have, you might

have seen a series  $N$  is from 0 to infinity some  $FN(x)$  this is a series of functions, if this is converging uniformly means as a function of this is, you can see that this is after integration this is a function of  $S$  so you can see that this is a sum, this is also kind of sum, and you see that this is uniformly convergent means is a function of  $X$ , so this can be made less than, this converges to or rather if you put some  $K$  this converges to the series,  $N$  is from 0 to infinity,  $FN(x)$  as  $K$  goes to infinity, okay, so what does it mean so this means this minus this can be made less than epsilon whenever  $N$  is bigger than some big number, okay, so that's what it means, rather  $K$  is, this is  $N$  is this, so  $K$  is bigger than some big number  $N$  let's call this, okay, this is actually  $K$ ,  $N$  is actually depending on both epsilon and this also function value  $X$ , this value set  $X$ . So if I can find such a  $N$ 's for every  $X$  value that is valid wherever  $X$  belongs to let us say some domain, some domain  $D$ , okay, so for every  $X$  if I can find such a big maximum sum  $N$  epsilon for each  $X$  if you have a maximum value of  $N$  epsilon of, this  $N$  existing big number, epsilon of  $X$ ,  $X$  belongs to  $D$ , if this is your big number, okay, then this  $N$  if such a maximum exists are the supremum exist, supremum says something which need not be exact, if it's a finite you call this, let's call this Maxima if you don't follow, so or something bigger than all these numbers, okay, if it exists such a number that works for every  $X$  in  $D$ , so that is the meaning of uniform convergence, then you say that this series is converging as  $K$  goes to infinity converges uniformly.

5) If  $f(t) = O(e^{at})$  as  $t \rightarrow \infty$ , then

$\int_0^{\infty} f(t) e^{-st} dt$  is uniformly convergent w.r.t  $s$  for  $s > a$ .

$x \in D$

$$\left| \sum_{n=0}^k f_n(x) - \sum_{n=0}^{\infty} f_n(x) \right| < \epsilon$$

$k > N, \forall x \in D$

$$N = \max_{x \in D} N(\epsilon, x)$$

So the same way you can see this one also, so the same argument holds good, once this is uniformly convergent this series you can do term by term differentiation and term by term integration, so we see the same thing as a function of  $S$  if it is bounded with some other integral which is independent of this function, independent of this variable  $S$  then you say that is uniformly convergent, so this is simple to prove, proof let me give you since  $F$  is of this order and you see that  $F(t)$  times  $E$  power  $-ST$  this one, this if you estimate this will be less than because  $F(t)$  is  $K$  times some constant times  $E$  power  $AT$ , as  $T$  goes to infinity then you have  $E$  power  $-ST$  this is the modulus, so this is less than or equal to  $K$  times  $E$  power  $-T$  times, this is going to be  $S-A$ , these all positive.

Now if  $S$  is bigger than  $A$ , so you choose some  $A_1$ , let  $A_1$  is bigger than  $A$ , then mod of  $F(t)$  times the above this is less than or equal to  $K$  times  $E$  power  $-T$  times  $A_1 - A$ , okay, so this is true, where is this valid?  $S$  is bigger than or equal to  $A_1$ , for every  $S$  if you take  $S$  is, for every  $S$  bigger than or equal to  $A_1$  this is always, this quantity is always bigger than that, so this is this, this is always bigger than or equal to  $K$  times  $E$  power  $-T$  times  $A_1 - A$ , if  $A_1$  is less than  $S$  or equal to  $S$ , okay, because of equality, so that is what I have chosen, so if I choose  $A_1$  bigger

Transform Techniques for Engineers 3 - Windows Journal

$\int_0^{\infty} f(t) e^{-st} dt$  is uniformly convergent w.r.to  $s$  for  $s > a$ .

Proof:  $|f(t) e^{-st}| \leq |k e^{at} e^{-st}| \leq k e^{-t(s-a)} \leq k e^{-t(a_1-a)}$ , if  $a_1 \leq s$  ✓

Let  $a_1 > a$ . Then

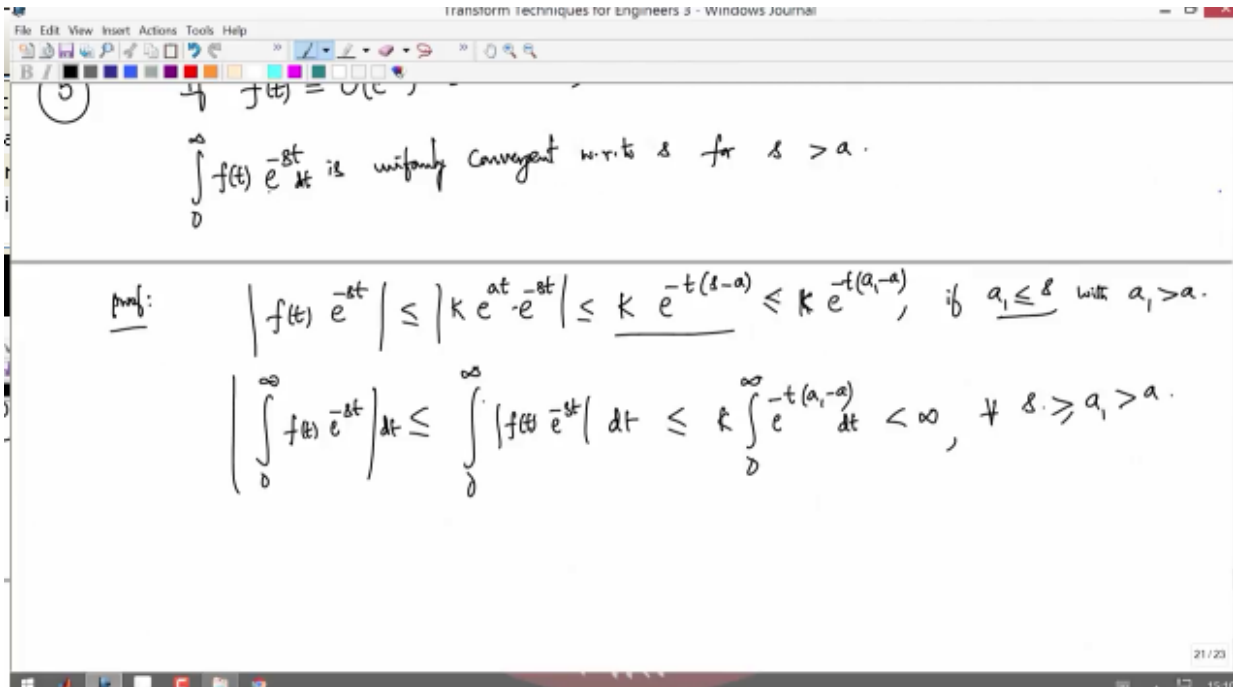
$|f(t) e^{-st}| \leq k e^{-t(a_1-a)}$ ,  $\forall s \geq a_1$

22 / 23  
15:09

than  $A$  then you can write this, this quantity is, because  $S$  is bigger than  $A$ , and you can write it like this, so instead of writing this you can easily, you can just use this one, if  $S$  is bigger than or equal to  $A_1$ , okay, so where  $A_1$  with  $A_1$  is bigger than  $A$  let us say, so such a thing this is always true, so that means left-hand side is independent of, depending on  $S$ , this is independent of  $S$ , so you can take the integration, integration of  $F(t) E$  power  $-ST$  0 to infinity, this is less than or equal to, so this is uniformly convergent, so this exists, okay.

This is because this integration will be the less than or equal to 0 to infinity, modulus of  $F(t)$  times  $E$  power  $-ST$   $DT$  this is less than or equal to  $K$  times integral 0 to infinity  $E$  power  $-T$  times  $A_1 - A$   $DT$  this is actually finite, this is true because this is true for every  $S$  bigger than or equal to  $A_1$  which is bigger than  $A$ , that is the meaning, okay, so this is the meaning of





uniformly convergent function of  $S$ , independent of  $S$  beyond some  $S$  bigger than  $A$  you have uniform convergence, so because of this I can take  $F(s)$  which is  $\int_0^{\infty} f(t) e^{-st} dt$  this is uniformly convergent integral, so I can differentiate with respect to  $S$ , with respect to  $S$  so that I can take this derivative, inside this integral just like how I take, if it is uniformly convergent series is a function of  $X$ ,  $N$  is from 1 to infinity let us say, 0 to infinity I can differentiate this whole thing as I can take this derivative inside this is same as  $N$  is from 0 to infinity,  $F(s)$ , so same thing, so this is same as  $\int_0^{\infty} f(t) e^{-st} dt$  okay, this is true only if this is true, so this is uniformly convergent, because this is uniformly convergent we can prove that this is true and you can also do the integration of  $F(s) ds$ .

Now let us say some, wherever  $S$  is the domain, if  $S$  is bigger than  $A$  so you can think of  $S_2$ , wherever  $S$  is  $S$  to infinity,  $S$  to infinity you can integrate as this function, and this is integration of  $S$  to infinity,  $\int_0^{\infty} f(t) e^{-st} dt$ , so this is again I can take this integral inside this integral, okay, so that is same as  $\int_0^{\infty} \int_{S_2}^{\infty} f(t) e^{-st} ds dt$ , so these are same, both are same only when it's a because that integral

Transform Techniques for Engineers 3 - Windows Journal

$$\left| \int_0^{\infty} f(t) e^{-st} dt \right| \leq \int_0^{\infty} |f(t) e^{-st}| dt \leq k \int_0^{\infty} e^{-t(a_1-s)} dt < \infty, \quad \forall s \geq a_1 > a.$$

$$\frac{d}{ds} \sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} f_n'(s)$$

(i)  $\frac{d}{ds} \bar{f}(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} \frac{d}{ds} (e^{-st} f(t)) dt. \quad \checkmark$

(ii)  $\int_s^{\infty} \bar{f}(s) ds = \int_s^{\infty} \int_0^{\infty} f(t) e^{-st} dt ds = \int_0^{\infty} \int_s^{\infty} e^{-st} ds f(t) dt$

is uniformly convergent, so this is the property important property which is analogous to what we had in series functional series, function series that is where if you talk of this notion called uniformly convergent, convergence because of this you can do term by term differentiation, term by term integration in the series, the same thing is you can do for if it is the integral, so summation is also, the integration is after all is the summation limit of a summation that's what, that's the reason we can do these two results based on that, so you can one can show that, okay.

Transform Techniques for Engineers 3 - Windows Journal

(i)  $\Rightarrow f(t) = O(t^c)$

$\int_0^{\infty} f(t) e^{-st} dt$  is uniformly convergent w.r.t  $s$  for  $s > a$ .

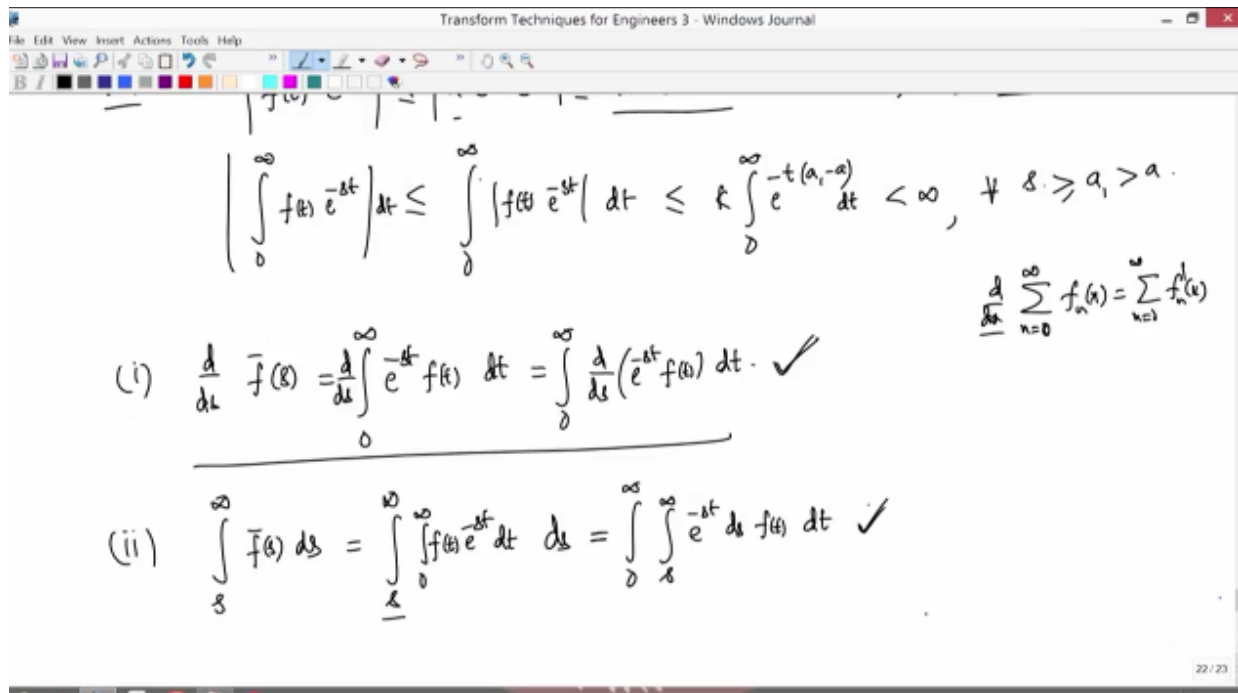
Proof:  $|f(t) e^{-st}| \leq |k e^{at} \cdot e^{-st}| \leq k e^{-t(s-a)} \leq k e^{-t(a_1-s)}, \text{ if } a_1 \leq s \text{ with } a_1 > a.$

$$\left| \int_0^{\infty} f(t) e^{-st} dt \right| \leq \int_0^{\infty} |f(t) e^{-st}| dt \leq k \int_0^{\infty} e^{-t(a_1-s)} dt < \infty, \quad \forall s \geq a_1 > a.$$

$$\frac{d}{ds} \sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} f_n'(s)$$

(i)  $\frac{d}{ds} \bar{f}(s) = \frac{d}{ds} \int_0^{\infty} e^{-st} f(t) dt = \int_0^{\infty} \frac{d}{ds} (e^{-st} f(t)) dt. \quad \checkmark$

So you take this as an property and then we can do differentiation of Laplace transforms, so look at the Laplace transform of its derivative, so if I have so far we have seen certain



properties of Laplace transform that is to start with what is the scaling property, shifting properties, I start with the shifting properties and then 1 or 2 or the first two shifting properties and then we had a scaling property, and then what is the Laplace transform of the periodic function, how do you get the, what is the formula for that, there is another property and also we have seen this Laplace transform is, if this Laplace transform is uniformly convergent integral which is independent, that means is a finite for all, so it is bounded with some number, some integral that is independent of  $S$  so that is kind of uniform convergence so that you can do term by term differentiation, term by term integration of this integral, so that means so you can differ, derivative you can take it inside this integral or integration when you do the integration of this Laplace transform you can take that integral inside the other integral, okay, inside the integral of the Laplace transform.

So these are the properties we had so far, so we'll look at derivatives and other properties of this Laplace transform in the next video. Thank you very much.

### Online Editing and Post Production

Karthik

Ravichandran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh Kumar

### Web-Studio Team

Anitha

Bharathi

Catherine

Clifford

Deepthi

Dhivya  
Divya  
Gayathri  
Gokulsekhar  
Halid  
Hemavathy  
Jagadeeshwaran  
Jayanthi  
Kamala  
Lakshmipriya  
Libin  
Madhu  
Maria Neeta  
Mohana  
Mohana Sundari  
Muralikrishnan  
Nivetha  
Parkavi  
Poornika  
Premkumar  
Ragavi  
Renuka  
Saravanan  
Sathya  
Shirley  
Sorna  
Subhash  
Suriyaprakash  
Vinothini

**Executive Producer**

Kannan Krishnamurthy

**NPTEL Coordinator**

Prof. Andrew Thangaraj

Prof. Prathap Haridoss

**IIT Madras Production**

Funded by

Department of Higher Education  
Ministry of Human Resource Development

Government of India

[www.nptel.ac.in](http://www.nptel.ac.in)

Copyright Reserved