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Transform Techniques for Engineers

Laplace Transform of Elementary Functions

With
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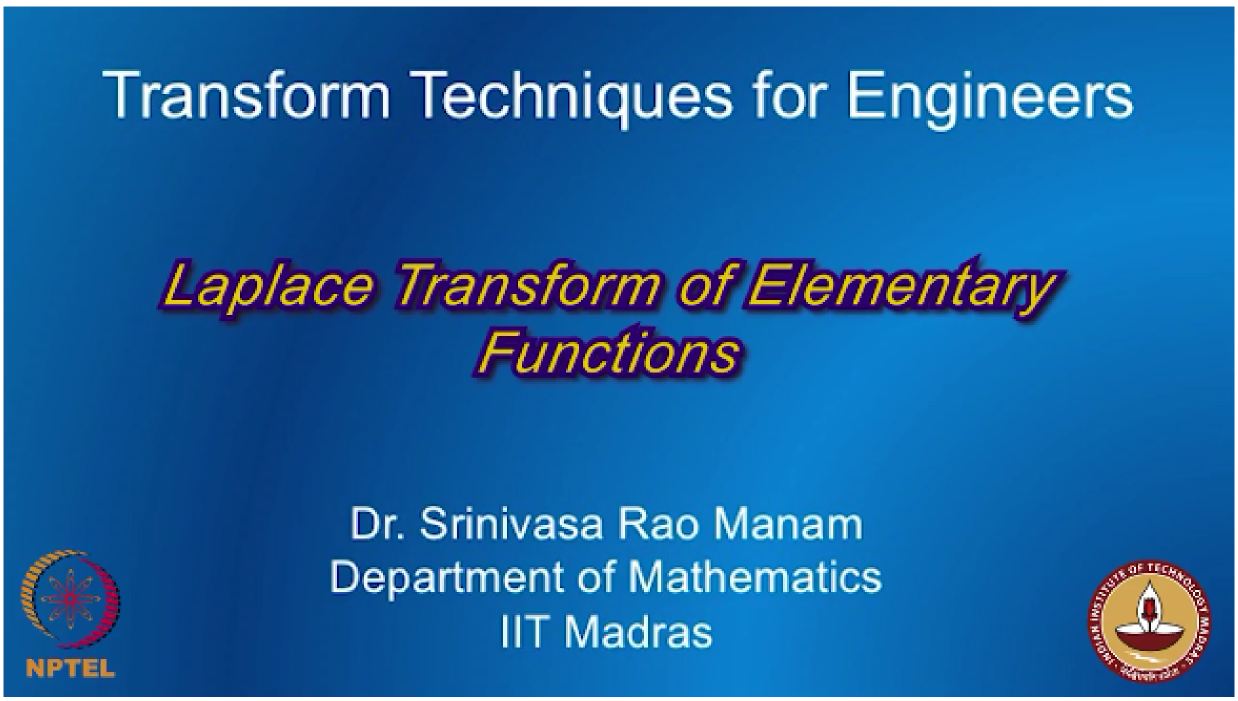
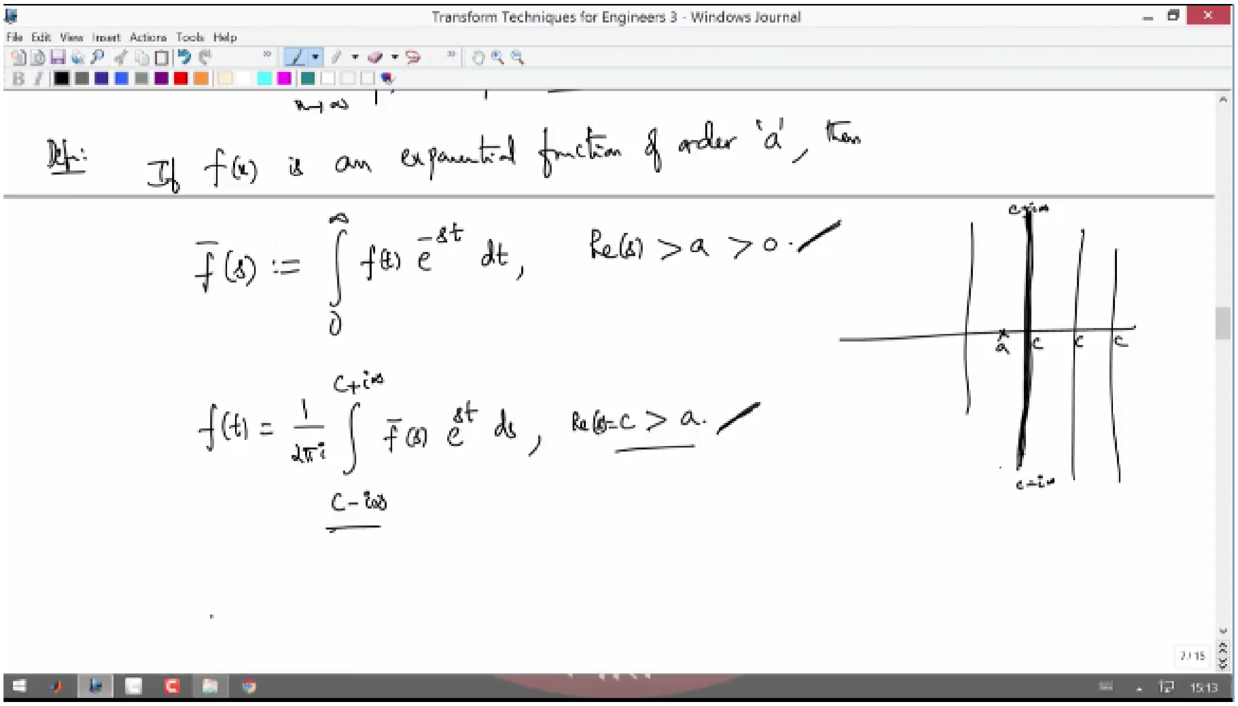


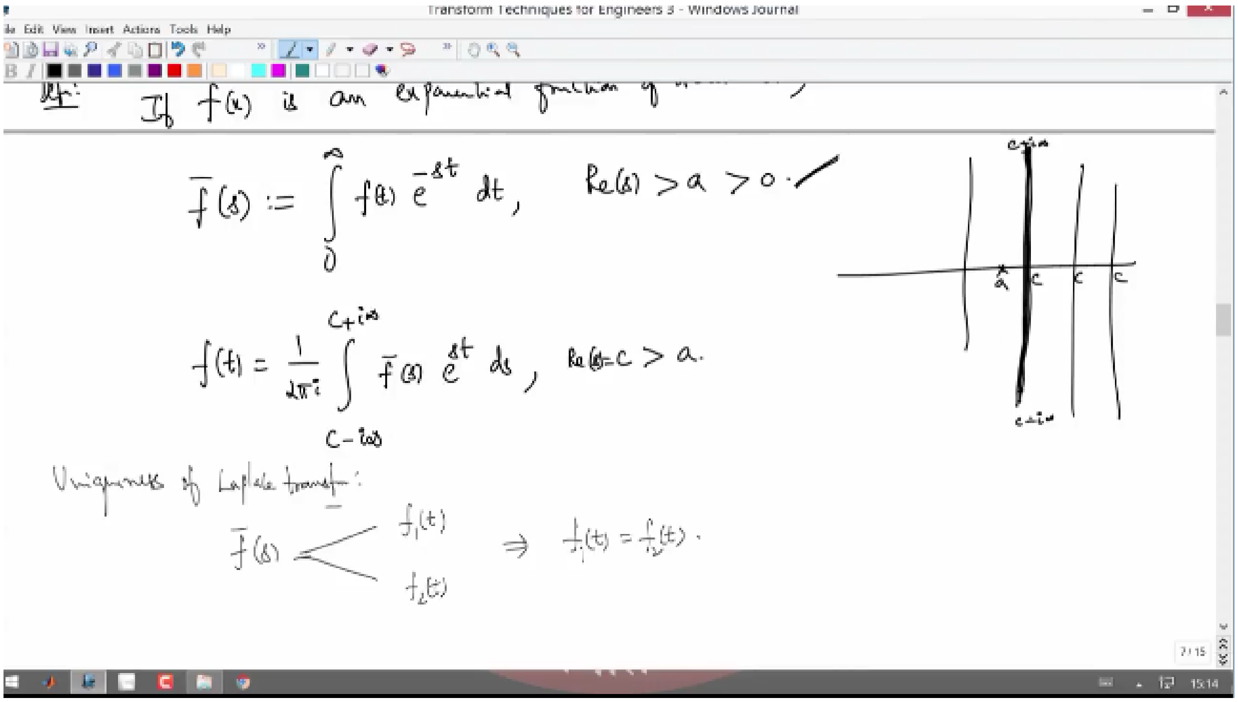
Welcome back, in the last video we have seen how to define Laplace transform for an absolutely integrable function, which is a smooth function rather because we have used Fourier integral theorem to define this Laplace transform, so the function F, the function F has to be a piecewise smooth R because we used only F(x) on the left hand side and the Fourier integral theorem, so what you expect is a smooth function that means it's a continuously differentiable function or at least differentiable function, differentiable function with only, it's a differentiable function and that is absolutely integrable function for which you have this Fourier integral theorem is valid, so you have defined Laplace transform for such a function F.

So you have, so the way you modified this integral theorem what you end up is either the function for which you have defined the Laplace transform is not just absolutely integrable function and a differentiable function or smooth function it is rather, it's much bigger class such as including those functions plus any function of exponential order. And also we have seen if such a exponential function, for example here if it is a function of order A the inverse



transform, what you have is inverse transform, so in the inverse transform you have integral, integral over contour C-I infinity to C+I infinity here like here, so that is what is C? C is actually a real part of S, this variable S that's what you have seen yesterday, so that C has to be bigger than A, because your function is absolutely integrable function and because the way you have seen F(x) into E power -AX that has to be, for that is going to 0, okay, so E power -CX into F(x) that goes to 0 implies F(x) has to be a function of, exponential function of order A, A is less than C, so in that sense C has to be greater than A.

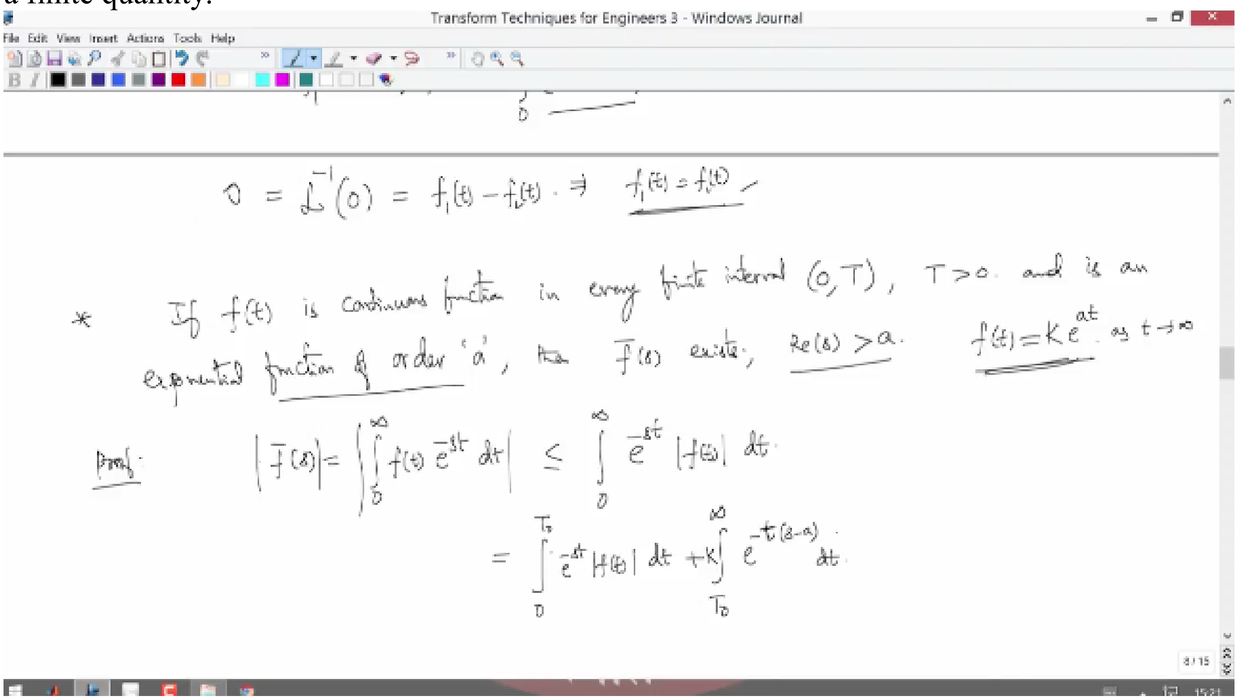
So this is how you define this Fourier Laplace transform and it's inverse transform, and clearly because if it is a smooth function that is what we considered, so you have this F(s) is actually unique function, so this Fourier transform suppose you have F bar(s) that is there for two different functions the F1(t) and F2(t) then this once you have this Fourier transform, same Fourier transform for each of these functions implies F1(t) has to be equal to F2(t) so this is what uniqueness, uniqueness of Laplace transform.



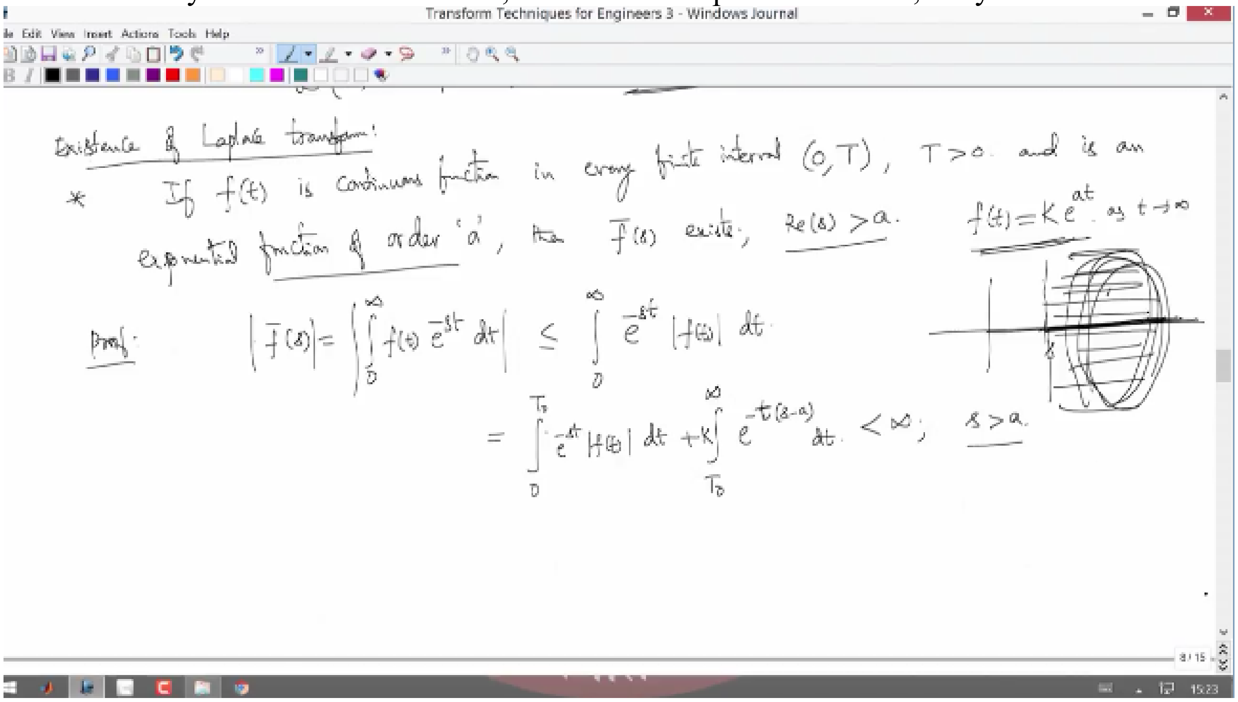
So if you take 2 different functions, if they are smooth continuous function so you can easily see that this is a, that is you can easily see because if it is the same and you have F1 bar(s) is same as F2 bar(s) that implies integral 0 to T, F1(t) - F2(t) times E power -ST DT = 0, rather this is going to be infinity, okay and then that implies the inverse Laplace transform of 0, the right hand side is 0 = F1(t) – F2(t), but we know that inverse Laplace transform of 0, 0 if you take a Laplace transform of 0 is 0 which is a function of S that is also 0, so this implies you can say that this is inverse transform 0 itself is 0, okay, if you are looking for only continuous or smooth functions which are differentiable functions is 0 so that makes it F1(t) = F2(t), so that way in that sense these two are unique, if you have for 2 different functions if your Laplace transform is unique and these functions are piecewise or continuous functions rather, at least continuous functions that is enough, then it has to be equal to this, okay, but the way we have defined they are actually with the way we've started if they are actually piecewise, they are actually smooth functions that is at least differentiable functions, once differentiable functions.

So you have a uniqueness of this Laplace transform okay, and its existence of the Laplace transform also you can talk rather you cannot say this is the uniqueness of inverse transform by basically inverse Laplace transform, okay, so we can also show that if F is a exponential function of order A then you have exponential of order A you have this Fourier transform is defined, and so we will show that it actually exists, so let me write it as a result, if so let me write it as a result, small result so you have if F(t) is continuous, continuous function in every finite interval every finite interval 0 to T okay, so that T is positive, and is an exponential function of order A let us say, then that is E power AT F(t) is E power AT, exponential of order A means say, okay, so let me not write this, so then Laplace transform F bar(s) exists, this is you can easily see provided a real part of S has to be bigger than A, so this is what we have seen already, and how do we prove this?

Let me do small calculation and so that you can easily see this, so if you do this F bar(s) if you define, by definition this is so the way we have chosen if it is, if actually F is a smooth function so that makes it is continuous, so that is a sufficient condition basically this is so F bar(s), what we have actually has, what we considered the function is that is smooth function, there's more than continuity, so such functions 0 to infinity, F(t) times E power -ST DT, so you want to just take the modulus of this, this will be less than or equal to 0 to infinity, E power -ST that is positive quantity, and then unless it's a time being assume that S is real, for real values we are only looking at the real values then this will be simply this with modulus this is this, if S is real into modulus of F(t) DT, this will be less than or equal to, what you have is F(t) at infinity 0 to, so at infinity this is like once this is a exponential function of order, and what is your F(t) is, basically some constant times E power AT that is how it looks, okay, at higher values, so actually this is true as T goes to infinity, okay, so this is actually F is equal to this, as T goes to infinity, so you want to be rigorous so what you do is 0 to some fixed quantity, let's say T naught into E power -ST mod F(t) DT + T naught to infinity E power -ST in this case for bigger values of T sufficiently, T naught bigger so you can write this as K times E power –AT, so you can write - S-A DT, and this is because F is a continuous function and this is well defined, this is finite quantity, and you have a fixed quantity K and this is also finite quantity, so together it's a finite quantity.

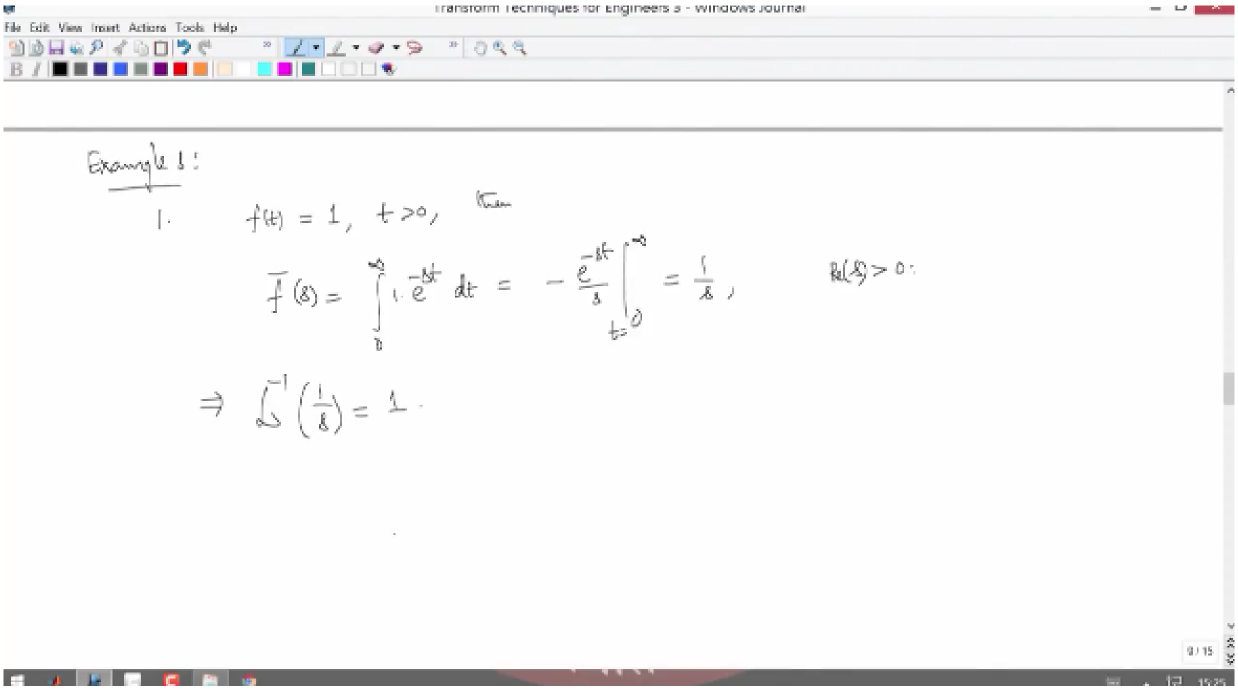


And where is it valid? If S is bigger than A that is true, that is valid, okay, so as of now because we have not taken the modulus for this when S is complex, so for real-valued this is true so once you have this real value F bar(s) S real value it exists Fourier transform exist S greater than A, for example S beyond this it exists, you can extend this as a analytical continuous way, so this function as such this one, one can show that we don't give the proof this Fourier in Laplace transform is actually analytic function in the as S, S greater than A in the, as S greater than A that means in this region this is analytic function, so that is a theorem that without proof you can assume, and because see this calculation you have done for S real, S greater than here, so because it is analytic function as such so we can see that, okay, that is an independent result that it is an analytic function, because it is analytic function once you know this exists here you can extend, analytical continuation you can have, you can have the analyticity everywhere so that this is the Fourier transform exists everywhere, S greater than A even if S is complex, okay, so this is how you show that this exists, existence of Laplace transform, okay.



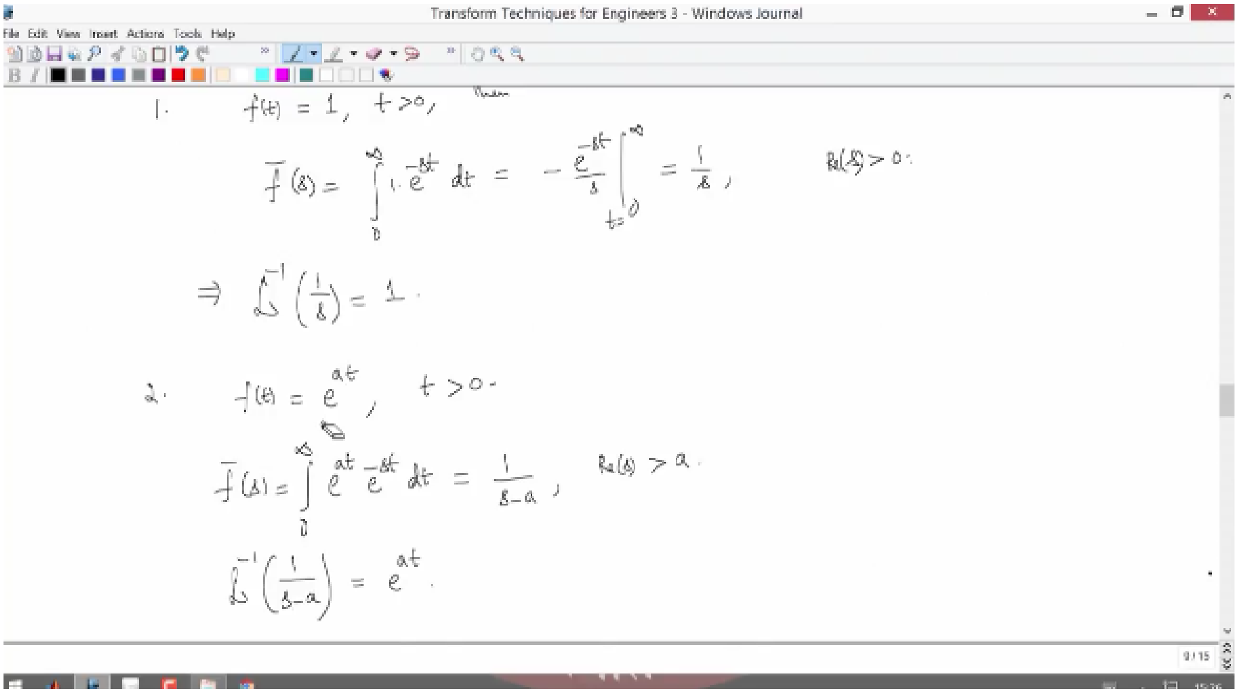
So now we have a, so for a bigger class of functions we have defined what is Laplace transform and it's inverse transform, these are unique because the moment you say this is your inverse the way we define, once we call this inverse that means it's mapping is unique, so you have a unique from here, from the domain to codomain, codomain to this, it’s unique, unique function both sides so that it's a mapping from space of functions is exponential, bigger class of functions to this analytic functions, F bar(s) site, okay.

So let's compute for elementary functions, what is your Laplace transform? So let's start with simple examples, examples for which we can calculate the Laplace transform, so start with letter 1, so if my F(t) = 1 for T positive then what is its Laplace transform, F bar(s) by definition 0 to infinity, F(t) is 1 times E power -ST DT this is function of exponential order 0, so you can expect this -E power –ST/S 0 to infinity if you do, you get this is 0, this infinity, infinity when you put because T is, T = 0, T = infinity, and only when S is positive, okay, because it's a function of exponential order of 0 so this has to be real part of S has to be 0, has to be positive that is why this quantity at infinity is 0, and minus minus plus then you put T = 0, this is what 1/S okay, so this is exactly what you have as your Laplace transform.



So immediately Laplace inversion of 1/S is actually 1, so this is what, so if you calculate for a certain function which you already appear you know, you calculate this Laplace transform and so what is the resulting? The resulting Laplace transform for which if you just write its inverse is actual function we started with, so that way inverse Laplace of 1/S is 1.

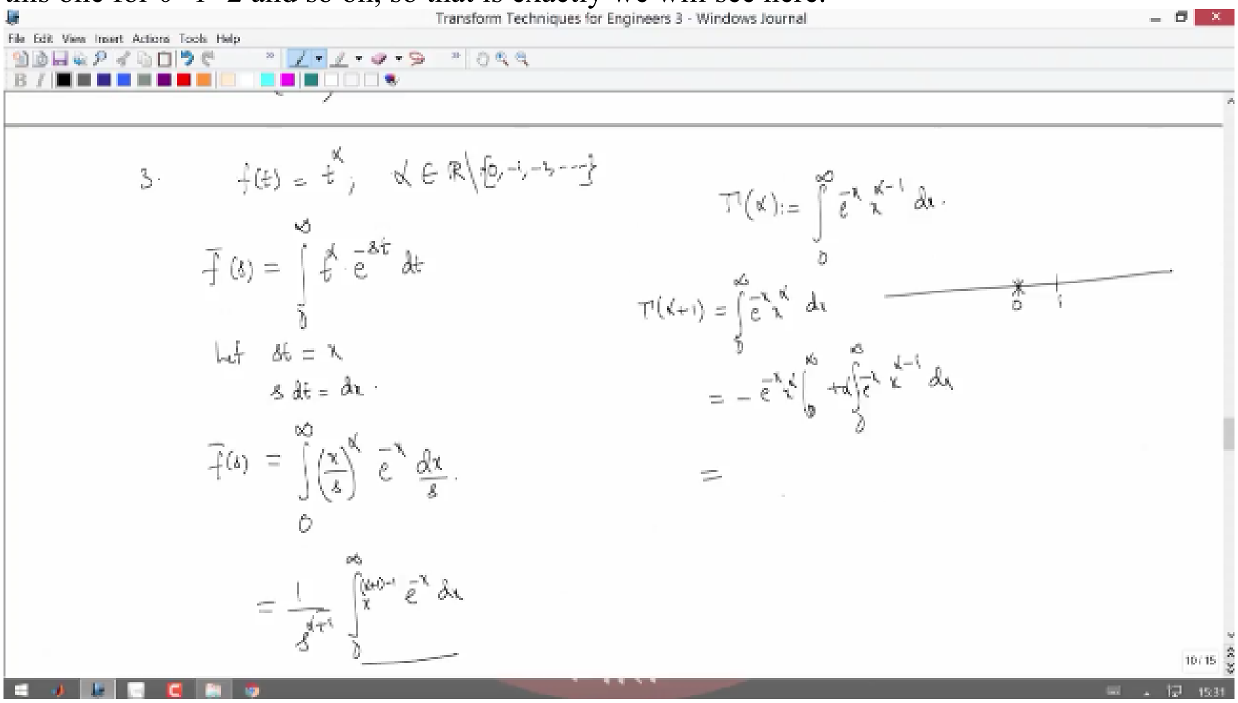
Second thing we can consider is let's, we have already seen what is F(t) is, this exponential function of order 0 if you consider exponential of order A for which Laplace transform if you calculate F bar(s) is 0 to infinity E power AT times E power –ST DT, the same way you can see that is going to be 1/S-A, real part of S has to be greater than A, so inverse Laplace transform of 1/S-A, so whatever S, 1/S-A you can immediately write this as E power AT okay, so this is how it is, okay.



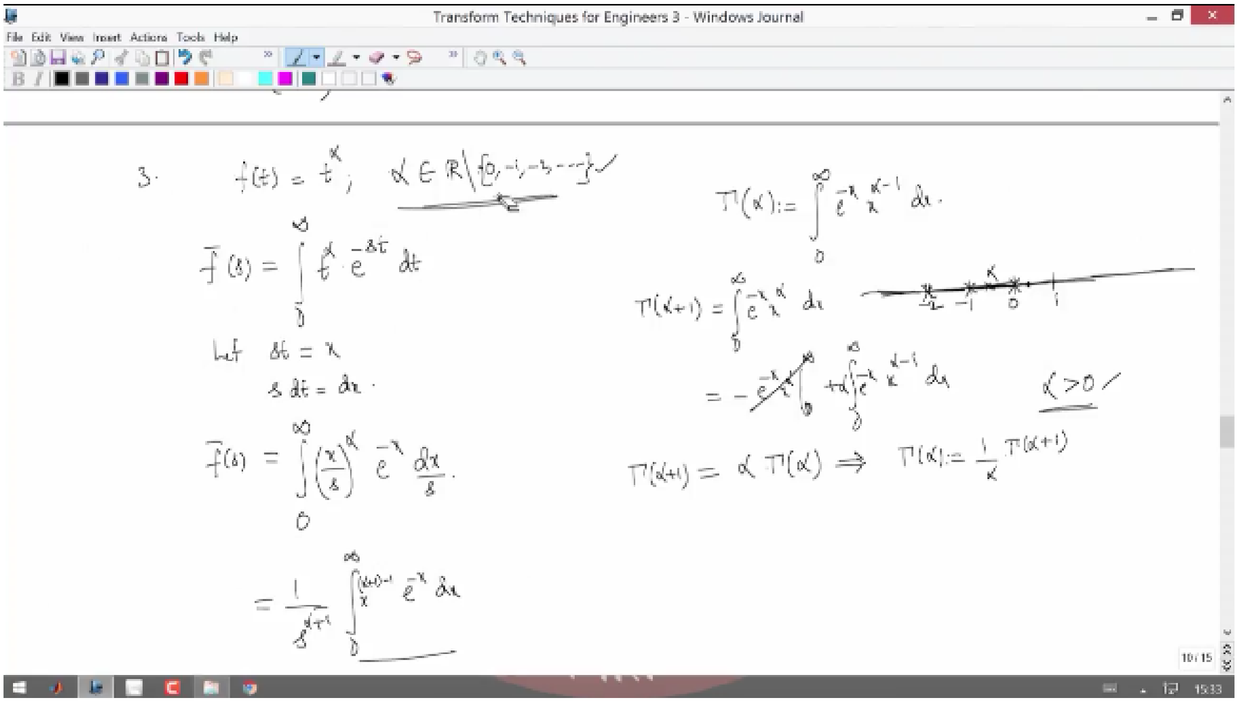
This can be even, A can be even and negative number let us see if it is a negative number what happens, this is going to be 1/S+A, real part of S is greater than –A you have to write in that ,case so it's basically you have, your analytic function will be the integral can be even you can think of just or positive okay, anything you can choose here or anywhere, the greater than –A, so if it is –A here anywhere you can choose as you are C, C-I infinity to C+I infinity, so that way so you can see that is actually valid for every A, A belongs to -infinity infinity okay, this is the case, so this is A.

So third example that you can do is, you can think of polynomials, exponential functions you have done, simple functions, now you can think of polynomials F(t) is let us say T power N, N is for that matter any alpha, alpha belongs to a real number, so if I take like this what is the Laplace transform? So this includes if alpha is integer, positive integer, a negative integer whatever, so if alpha is integer so you can have, if alpha = 0, one we have seen already, so you have 0 to infinity, T power alpha times E power -ST DT, so this will be, how do I evaluate this? This is equal to, if I choose let ST = let us say X, if let ST = X then SDT = DX, so this implies F bar(s), what you get is T is, this is going to be 0 to infinity, T is X/S power alpha E power -X DX, DT is DX/S, this is what you have, so this is equal to 1/S power alpha +1 and you have this integral X power alpha E power -X DX, so this X power alpha I can write like X power alpha +1 -1, so if you write like this, this is actually the definition of gamma function, so gamma of alpha = integral 0 to infinity, X power, E power -X into X power alpha -1, okay, DX so this is the definition of gamma function, if you define like this where is this valid? This is actually valid so if you look at only real line 1, 2, so 0 to 1 except 0 if you put alpha = 0, this is 1/X this integral will be infinite in that case, so at this point it is not defined.

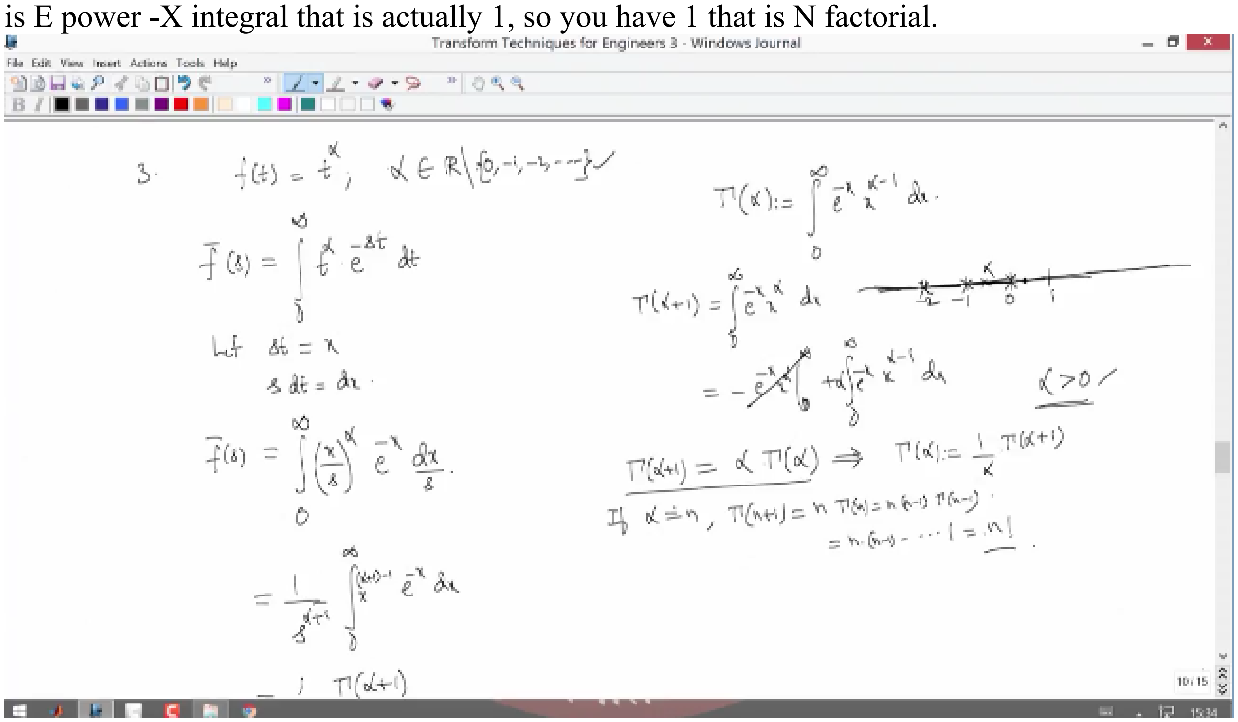
If it is alpha is between 0 to 1 it is well-defined, if alpha = 1 it is well known, and so how do I do this? So I try to, so you try to calculate gamma alpha + 1 you consider, so you have integral 0 to infinity E power –X, X power alpha -1 DX, X power alpha in this case do the integration by parts so that you have -E power -X into X power alpha, put the limits and this minus minus plus and you have alpha times X power alpha -1 E power -X DX, so what you have with this quantity? Because of E power -X this is 0, because of X power alpha, alpha is, yeah so here alpha has to be, let's use alpha is positive side, so it's not, alpha is positive, if alpha is negative you can see that this is not at 0, it's not 0, okay, so actually negative side you can go up to let us say -1 okay, so anyway we'll see, we will fix this alpha, alpha is R actually you cannot define this one for 0 -1 -2 and so on, so that is exactly we will see here.



So if alpha is, well let us say any number alpha is positive, if alpha is positive okay let's choose alpha is positive, if alpha is positive that means here, okay, so you have this is 0, this quantity is 0, so you have end up, this is the definition of gamma alpha, this is what is true for, so this way you can go on this side, so 0 to 1 it is well-defined, 1 to, this side positive side is well-defined, to get the negative side you use this relation to get gamma(alpha) = 1/alpha times gamma(alpha+1), so using this if I choose -1 which is not defined, if I choose any value here alpha if you take this as alpha and this is your definition of your day, you define this as using the values gamma(alpha + 1), this is alpha + 1 will be somewhere here, okay, if it is close to here alpha+1 will be here, so at this it is already defined we know, okay, so that is how you define all the values in between this, so that way you have all the negative side -1 -2 so once you have the values here defined in between you can, using them you can assign them for this here, like that you can go on getting it with 1/alpha of course, 1/alpha times the values here will give me the values here, like that you can go on defining except 0 -1 -2 and so on, so outside that you can use your, you have this gamma function is defined, so because I choose such a

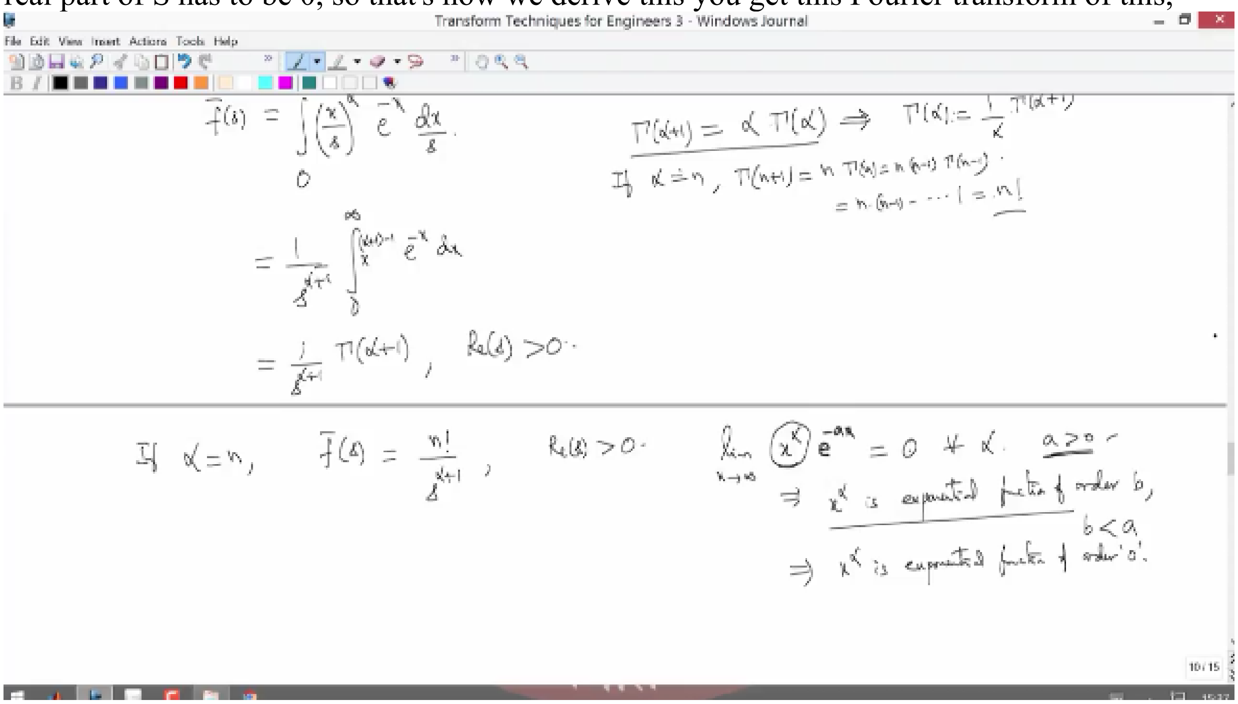


alpha, so now this integral is a gamma function so that is 1/S power alpha + 1 gamma(alpha+1) okay, so gamma(alpha+1) is actually, if alpha is integer one can easily see that you can go on using it if alpha is equal to N here, using this gamma(n+1) will be N times gamma(n), again N into N-1 into gamma N-1, again if I reuse keep on using it recursively what you end up finally is N into N-1 and so on, finally gamma(1), so gamma(1) is if is gamma(1) is alpha = 1, so that is E power -X integral that is actually 1, so you have 1 that is N factorial.

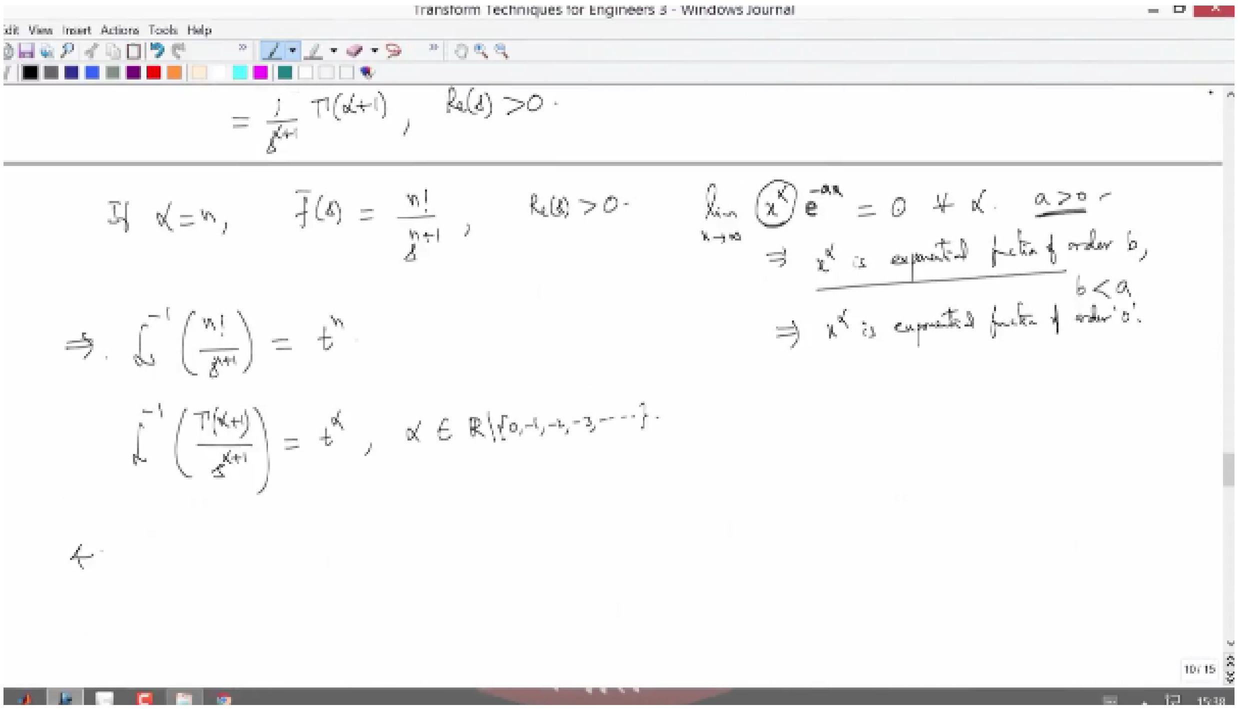


So this is the Laplace transform of any general alpha such that alpha cannot be non-negative, okay, it's not, it's all integers except all real numbers except this negative integer including 0, for which you have this is the Laplace transform, if alpha = N so Laplace transform of this equal to 1/S power alpha+1 times you have N factorial, okay, of course real part of S, real part of S is, what is this? Exponential function real part of S is, nothing is, real part of S, what is real part of S? T power alpha, what is T power alpha? If because, where is this valid real part of S. To see this you look at the X power alpha, alpha into is it exponential, is it exponential order, that means this should behave like this into E power some –AX, okay, times this limit, what is this limit? Limit X goes to infinity is always goes to 0 for every alpha, okay, including 0 so if alpha is 0, this is going to be 0 for A positive, okay, if A is positive exponential any A positive, okay, and you can see that if you choose this is your F(x) and you multiply E power –AX, I mean this limit is always going to 0 as X goes to infinity for every X, okay, for every alpha, alpha is, even if it's a positive bigger value so because of the polynomial, a kind of X power alpha divided by E power -A alpha will go on if you keep doing it you will end up with this, if you differentiate many times by using a Laplace it'll come down, finally that infinity is going to be 0, so that’s how we can show that this limit is 0.

So that implies this function X power alpha is exponential function of initial function of order certainly of something less than, harder less than A, of orders let us say B where B is less than A, A itself is true for every positive so that implies exponential of order 0, so that implies X power alpha is exponential function of order 0, okay, so that means this has to be greater than 0, real part of S has to be 0, so that's how we derive this you get this Fourier transform of this,

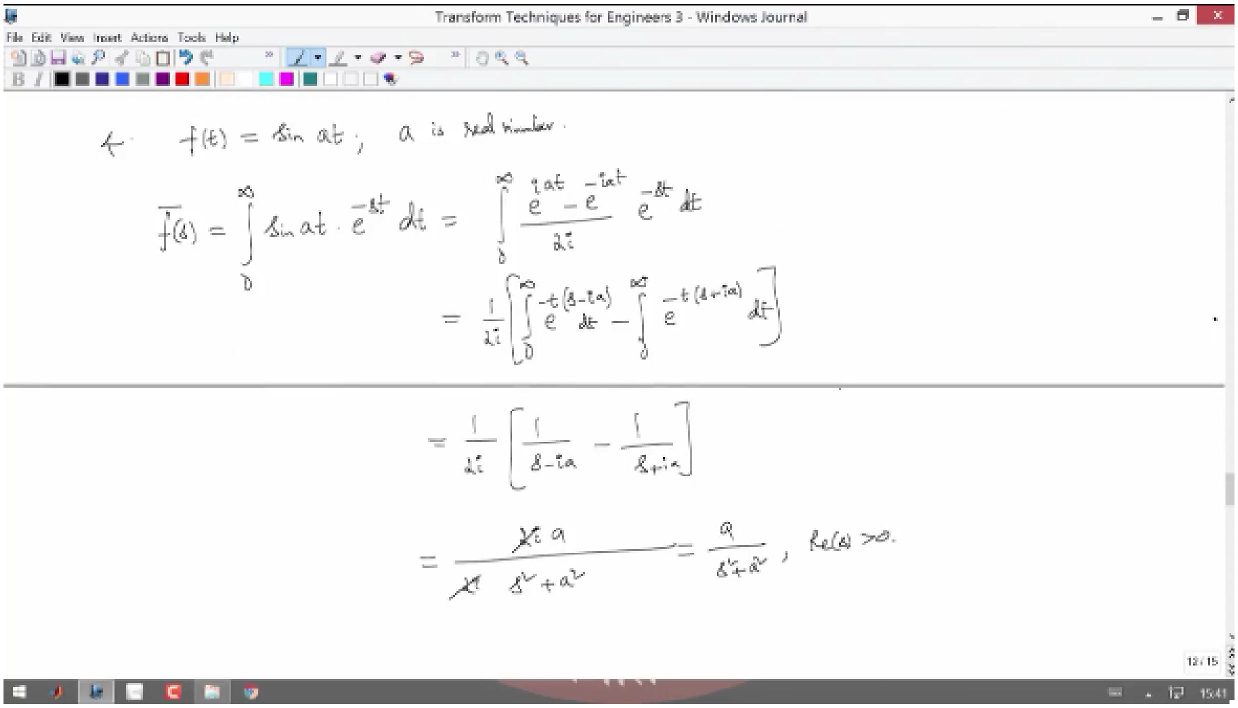


here also same way real part of S is positive, so immediately you can write the inverse Laplace transform of the N factorial/S power alpha and N, this is N, N+1 = X power N okay, or rather here T power N, or inverse Laplace transform of gamma and alpha+1/S power alpha+1 which is equal to T power alpha, did I use T power alpha? Yes, where alpha belongs to R -0 negative integers, including 0, that's it, so you have this result.

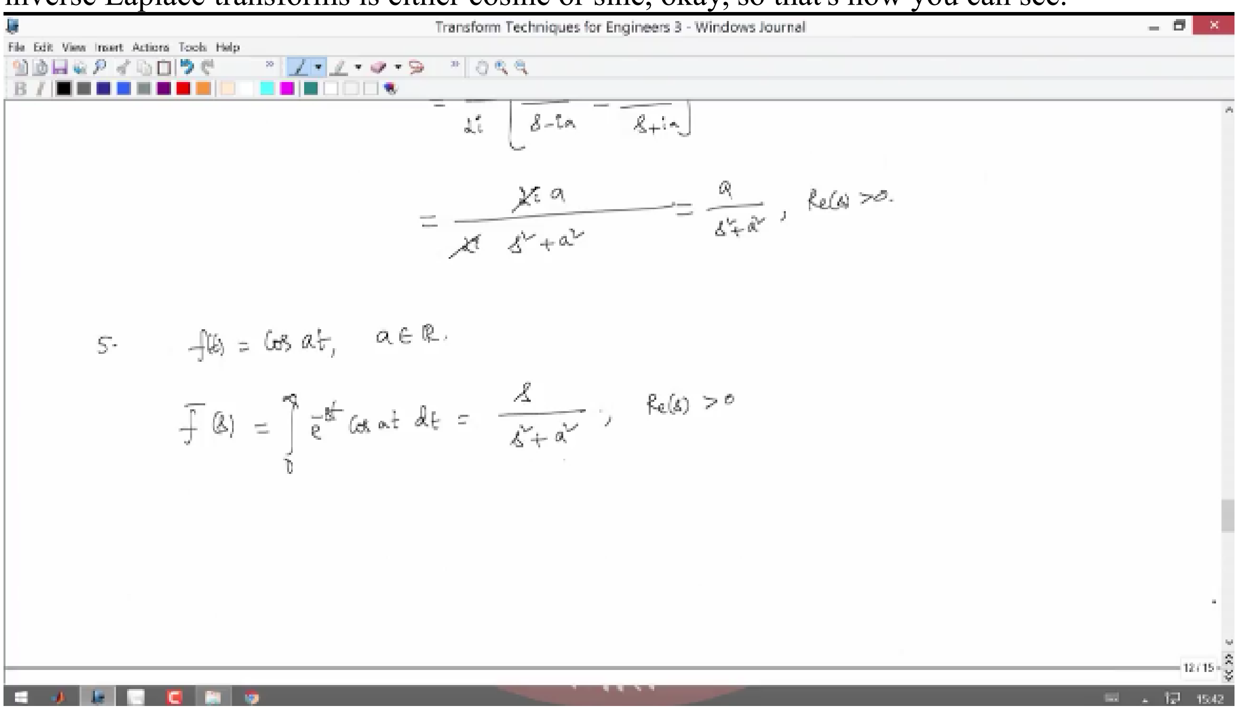


Then we can go on next example, we can see that it's going to be, let's use trigonometric functions, if F(t) is sine function, sine AT, if this is sine AT, A is real, A is a real constant, real number, if A is real number then what is it, this is also exponential of order 0, because if you multiply anything same way this E power -AX any A positive multiplied with the sine X that as X goes to infinity that is going to 0 so it is exponential of order 0, so real part of S has to be 0 when you define, once you get, once you calculate its Fourier transform.

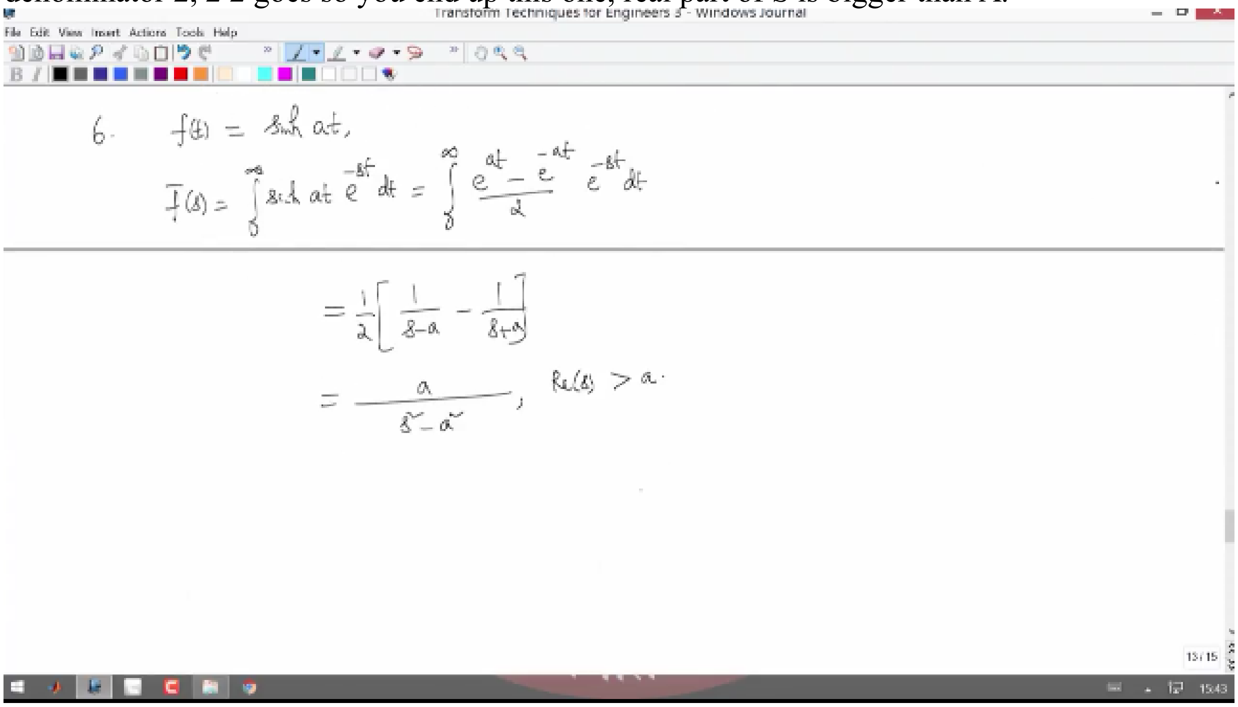
So Fourier Laplace transform of this is equal to 0 to infinity sine AT times E power –ST DT, so this one you can write this as an exponential function so you get 0 to infinity E power I AT -E power -I AT divided by 2I times E power –ST DT, so this is equal to 1/2I times integral 0 to infinity, first integral is E power T times -T times S-IA –integral, again 1/2I that you can put it as a bracket, so here you have integral 0 to infinity E power –T times S+IA DT, so this is equal to 1/2I what you get here is the value 1/S-IA, and here 1/S+I, so if you do this calculation so S square + A square in the denominator and here S+IA – S+IA so you have 2IA so that is 2IA and you have 2I, that’s gets cancels, so you end up getting A divided by S square + A square, so this is another way of evaluating this integral by, you can write sine as in terms of exponentials and use the integral directly, you see this, okay, so of course here real part of S is positive because it's bounded functions, all these bounded functions are exponential functions of order 0.



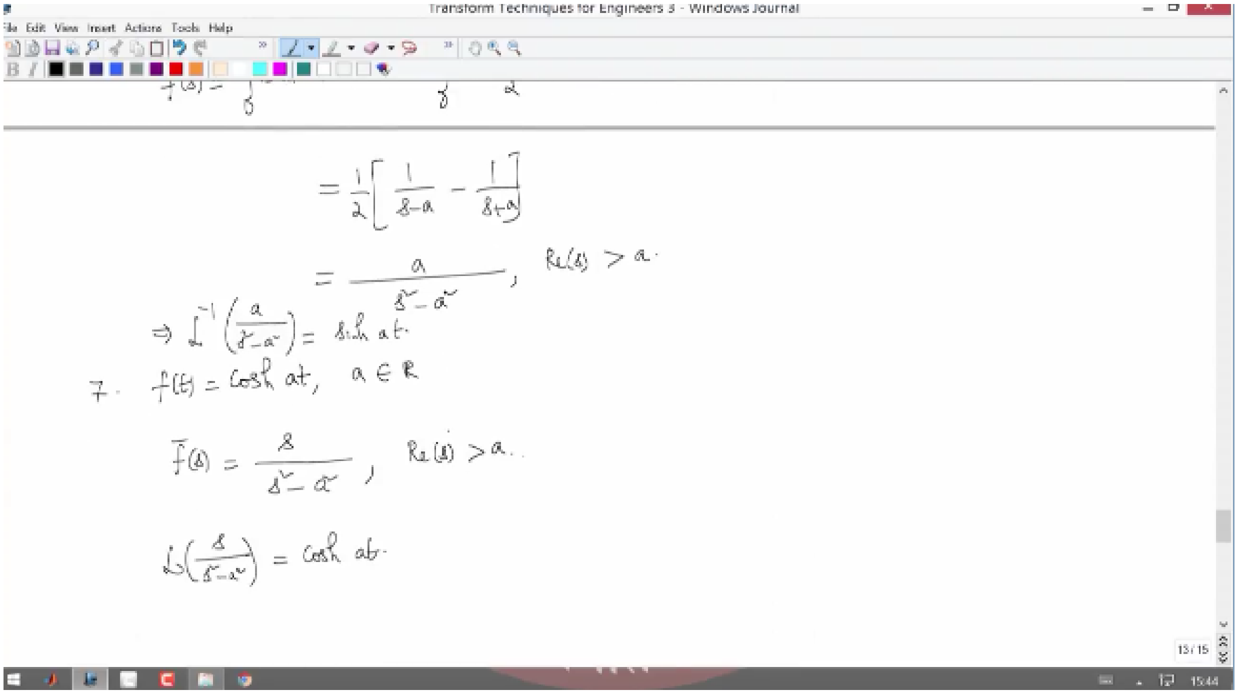
So same way you can get this cosine function also, so if you write this cosine function, so what is the Laplace transform of, if let's write this F(t) cos AT, A belongs to R, so for which if you write the same way F bar(s) this is 0 to infinity E power -ST cos AT DT this you can easily see that S divided by S square + A square, so again real part of S is positive so if you see this as your function of S anything like constant by S square + A square bar RS/S square + A square inverse Laplace transforms is either cosine or sine, okay, so that's how you can see.



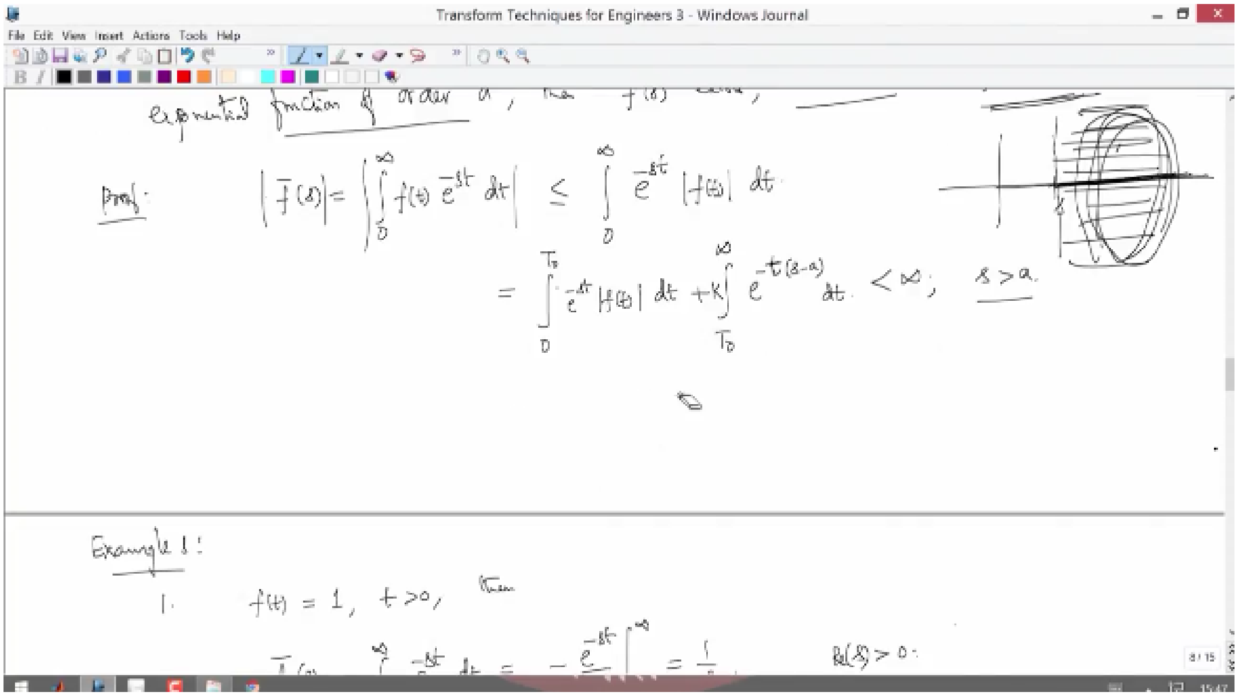
Similarly we can also work with hyperbolic functions, sine hyperbolic functions and cos hyperbolic, sine hyperbolic of AT which is a maximum, this is of exponential function of order A because it's E power AT -E power –AT/2 so this is a maximum of order A, so exponential of order A, so F bar(s) if you do the calculations routinely sine hyperbolic AT times E power –ST DT 0 to T, E power -AT + AT - E power –AT divided by 2 times E power –ST DT this is as equal to 0 to infinity E power, so I'll write directly so 1/2 times, 1/S-A and the other one is 1/S+A so this will give me S square - A square, S+A - S+A so you have 2A, so you have a denominator 2, 2 2 goes so you end up this one, real part of S is bigger than A.



Similarly you can also get F(t) is cos hyperbolic, cos hyperbolic AT, A is any real number so for this F bar(s) you do the similar calculation, so you can see that S divided by S square - A square, you observe that if it is plus it is cos or sine, if it is minus in the denominator S square +, S square – A square you have cos hyperbolic or sine hyperbolic, your real part of S is bigger than A, because cos hyperbolic AT is E power AT + E power –AT divided by 2, so this is exponential of order bigger on this one, okay order A, so that's how you can get these elementary functions its Laplace transform and it's inverse transform as well, once you know that transform, inverse transform is of this one S square -A square is equal to cos hyperbolic AT here, here inverse transform of this is inverse transform of this one A divided by S square -A square this is sine hyperbolic AT, like that we can get, so we can go on different some complicated functions we can also derive, we can also calculate Laplace transform of some complicated functions for example Bessel functions you can also construct, you can also calculate the Laplace transform of some, what is that error function, some special functions okay.

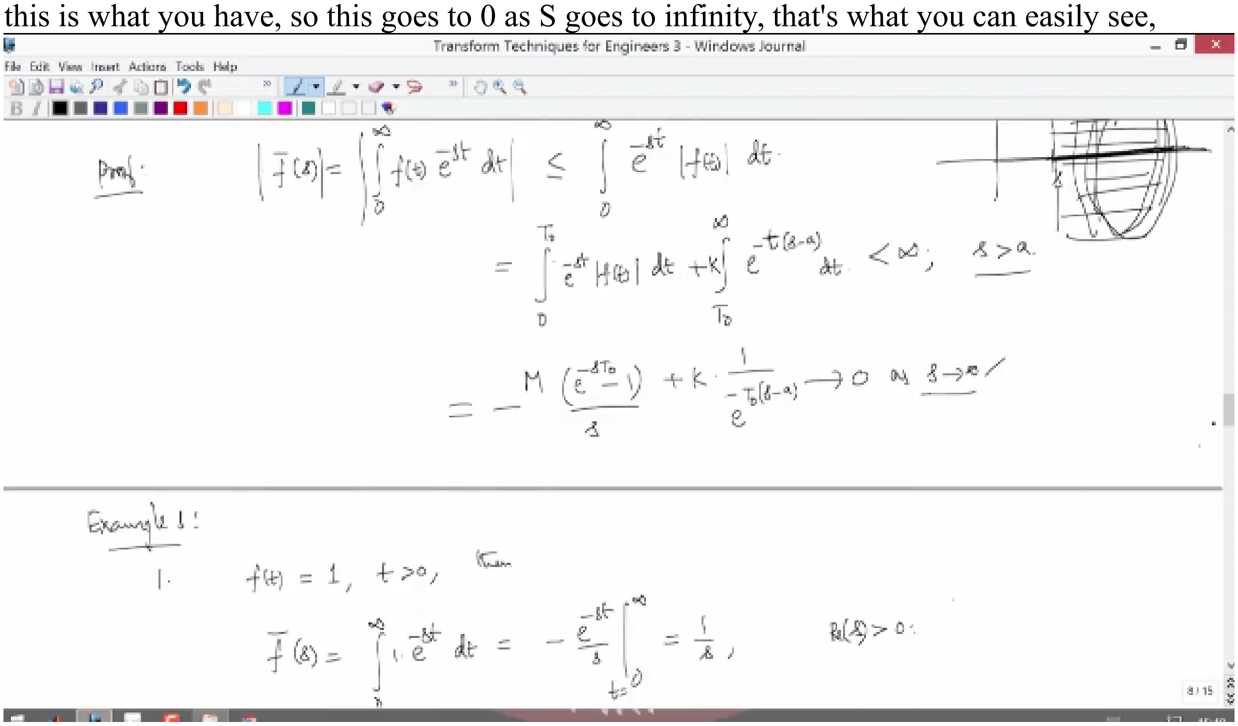


So before we do that we'll just, what we do is as and when we apply the Laplace transform, when we do the applications as and when it is required we will calculate the Laplace transform so that it's inversion, so that we calculate, so when you calculate the inverse if you see these functions then you can directly write as its inverse Laplace transform, otherwise there is a way to find the inverse Laplace transform that we will see in the next videos, and before I close today so what we do is you see that the way we have defined Laplace transform F bar(s) = integral 0 to infinity, F(t) into E power –ST DT okay, where F is exponential function of some order, okay, and what we have seen that this exists, if F is a continuous, smooth function with absolutely integrable or exponential function which is smooth for which this exists, that is what we have seen in the early starting of this video we have seen this exists, and also you can see that limit F bar(s) as S goes to infinity, what happens to this? If you consider this and you can easily see that the right hand side we have derived earlier that this is a, so from this one can show that, so with this one can show that from this you can easily see that this is going to be +K

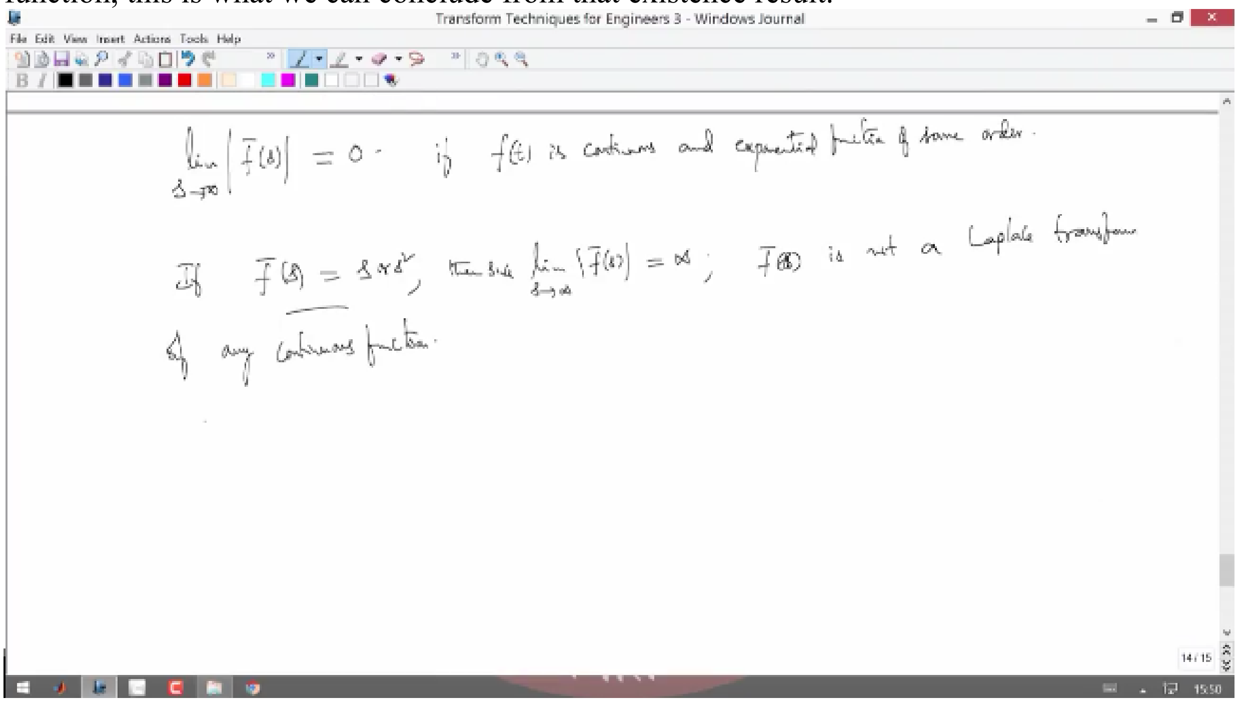


times 1 divided by S-A times E power T naught times -T naught times S-A, T naught is bigger T naught here, so this is what you have.

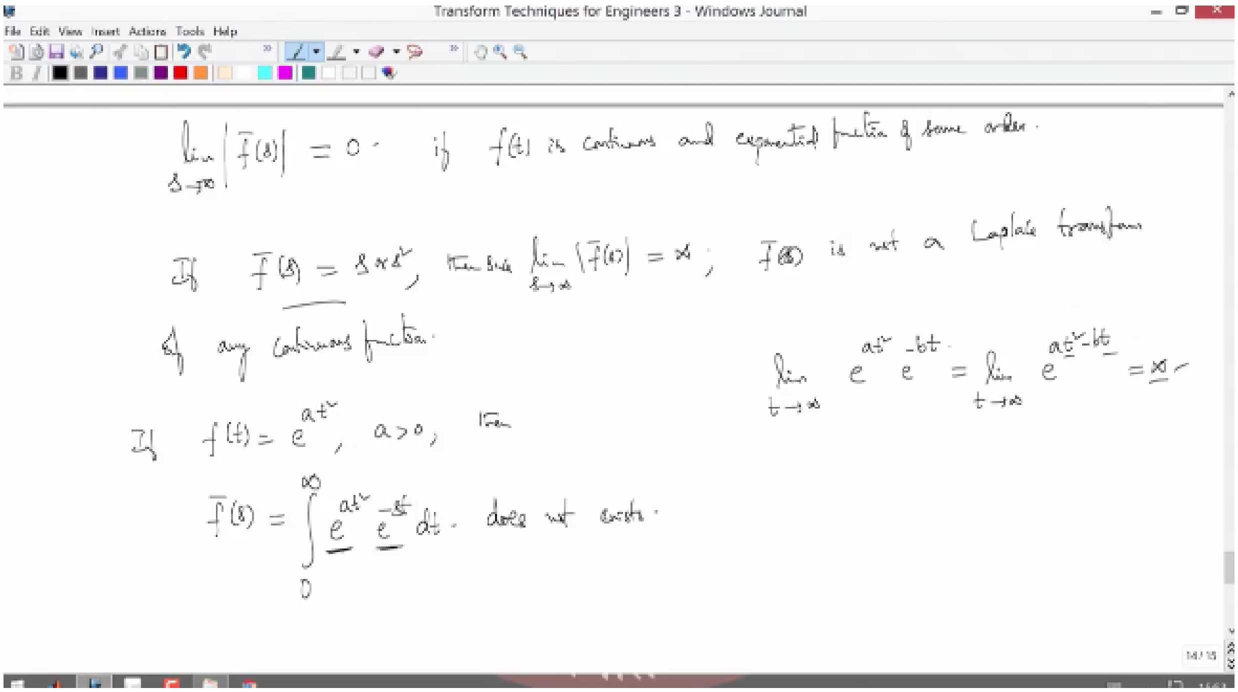
For this one is some constant but here also if you, once you get this so because if it is a continuous function, because F is continuous function this is some bounded, so E power -ST integral you will have E power –ST naught divided by -E power S, at 0 it is 1 divided by –S, so this is what you have, so this goes to 0 as S goes to infinity, that's what you can easily see,



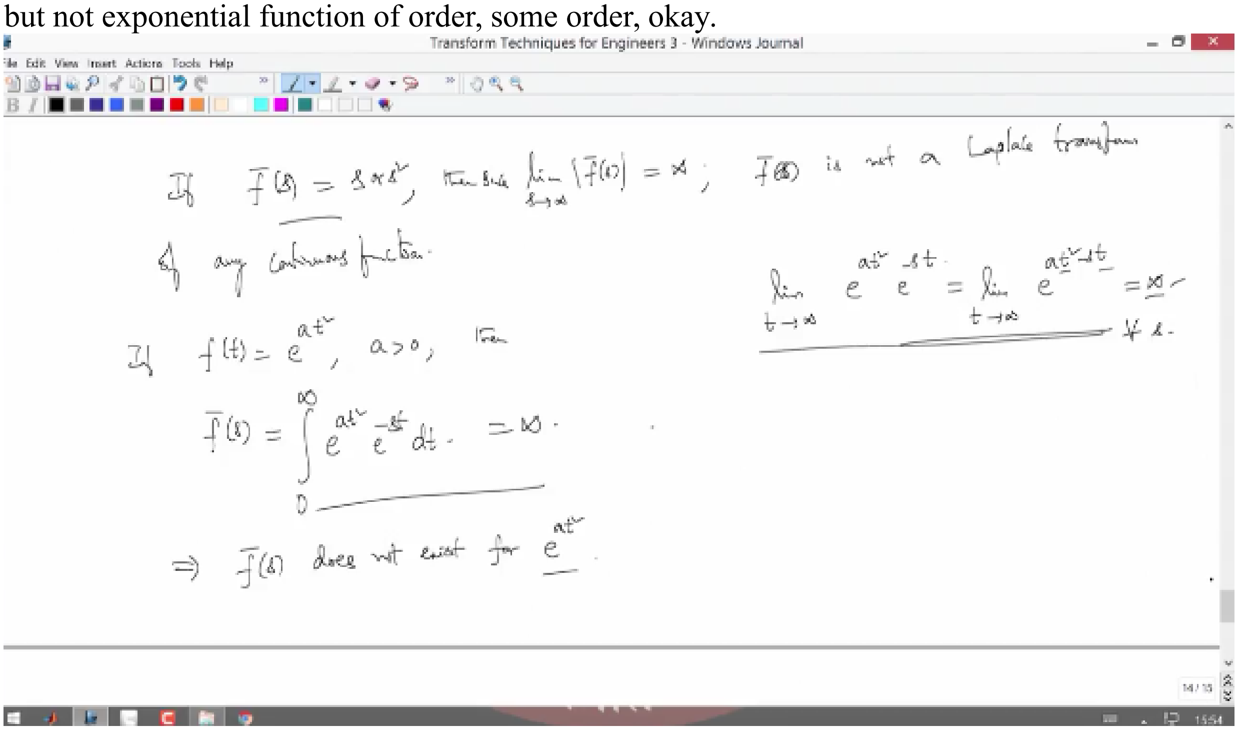
okay, so because of this we can remark that this is as S goes to infinity this is 0, so if F(t) is rather smooth or continuous function, continuous and exponential function of order okay, and exponential function of some order, this is what we have. If it is a function of exponential function of some order and discontinuous this has to be 0, from which if F bar(s) for example if this is simply so let us say S, clearly if this is the case and you have this result limit, okay, suppose if I have this one, limit since, limit S goes to infinity this F bar(s) this is infinity if I choose this, what we have is F bar(s) is not a Laplace transform of any continuous function, okay, so not just S or you can also write S square, any S square this is true, S or S square, S cube any polynomial in S, we can see that this is not a Laplace transform of any continuous function, this is what we can conclude from that existence result.



If F(t) is E power AT square with A positive then you can see that F bar(s) this is, if Laplace transform exists if it is, this is a continuous function and it is and what we have is it is not exponential function of some order, clearly it is not because if it is exponential of some order let us say B then some E power T square into this function into E power, if it is exponential function of order B then I have -BT this limit has to go to 0, is this really true? But this is not okay, this is actually limit T goes to infinity E power AT square –BT, so T square grows bigger than T, so as T goes to infinity this is going to be infinity which is not equal to 0, okay, if you have, if it is a function of, exponential function of some order we know that this is result is true, this limit has to go to 0, okay, and you see that this, because this is not exponential function of some order B, some order clearly because this is infinity we cannot say that this integral Laplace transform E power AT square, E power -ST DT, DT does not exist, because this is exponential order which is higher than this is so, its growth is bigger so that is why it is, for any S this is going to be infinity, at infinity as T goes to infinity which is infinity so this integral does not exist, okay, because of this result, because of this.



So E power –BT, instead of B you take some S, S can be anything you fix your S, any S this is true, for any S, for every S this is true, so in place this does not exist, this is actually infinity, so that implies F bar(s) does not exist for the function, for E power AT square which is continuous but not exponential function of order, some order, okay.



So in the next video we will see some of the properties of these Laplace transform, and we will state them, and we'll just prove as properties, and then we will try to prove some applications of this Laplace transforms in the next later videos, okay, thank you very much.

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