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NPTEL ONLINE COURSE

Transform Techniques for Engineers

Introduction to Laplace Transform

With
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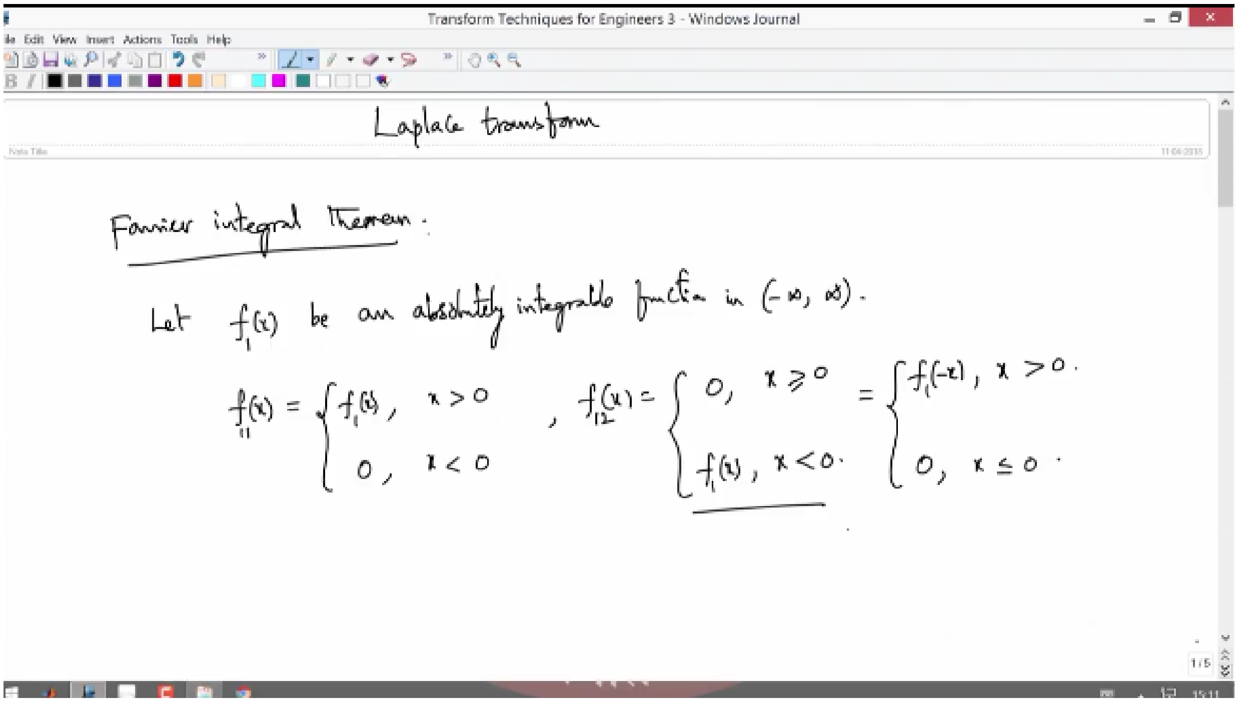
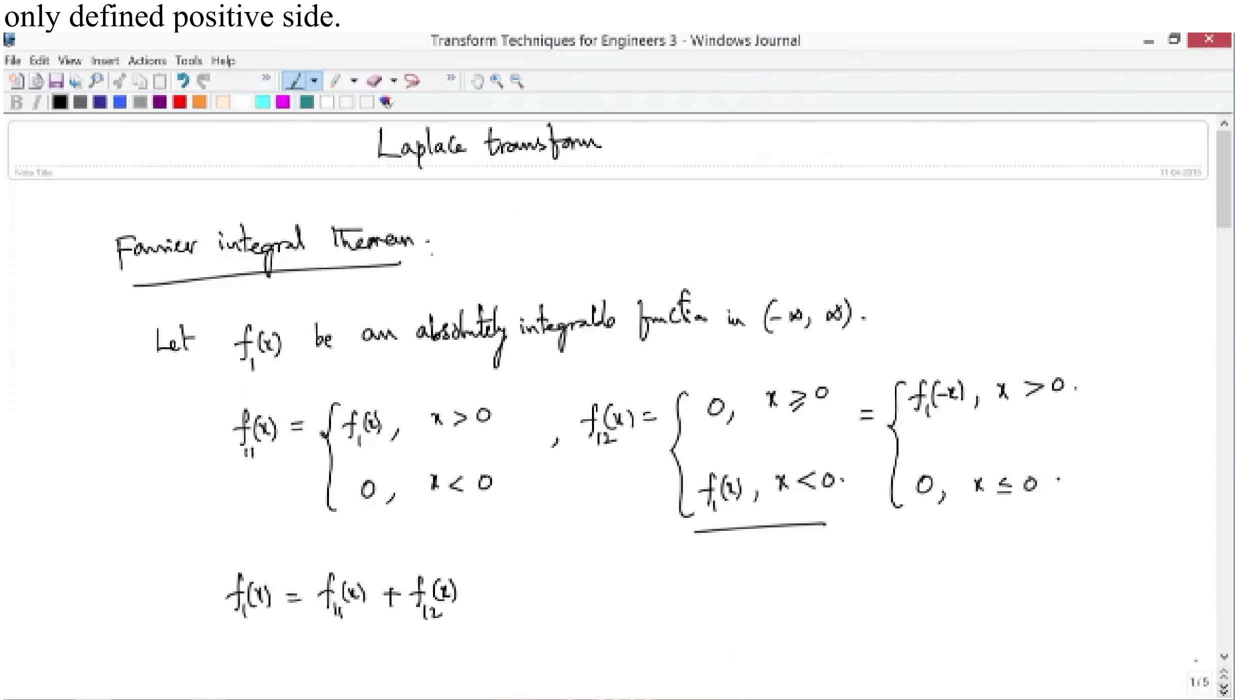
Welcome back, so far we have seen problems how to, we have seen only how to solve problems boundary value problems using Fourier transforms, are basically what we have seen is a Fourier transform and it's applications in the last component.

So we'll start, we will define what is called a Laplace transform in this video. So to start with we'll just consider Fourier integral theorem that we have proved earlier videos, so we just take this Fourier integral theorem start with that, Fourier integral theorem, so what is that you have? So what we have is if I have a function, let's call this F1(x) okay, Fourier integral theorem is basically you define a Fourier transform and it's inverse transform for an absolutely integrable function, so you consider F1 be such a function, be an absolutely integrable function, where it's defined as, its defined function in minus infinity infinity so full real line, okay, so if you are given a full real line function absolutely integrable function that is our full real line, what you have is, you can break this into this way, so F1(x) as a positive side, so you call this F1, F11 let us say okay, so F11(x) is F1(x) for X positive and then X this is 0 for X negative, okay.

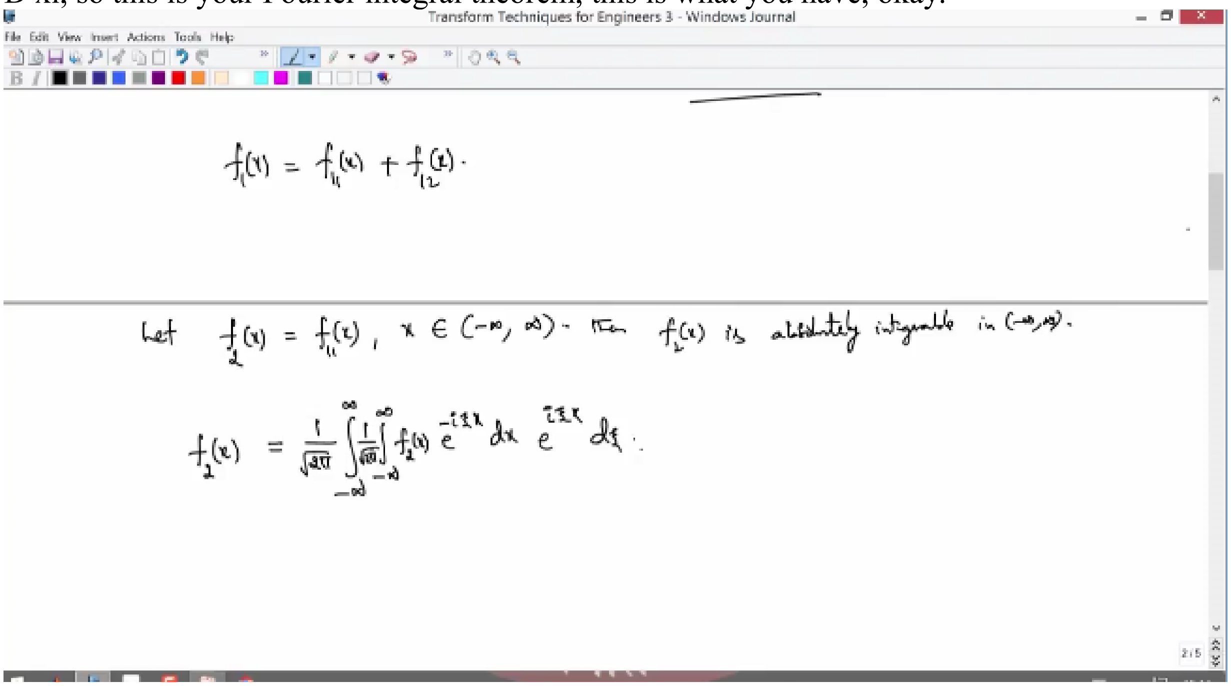
I break this F1(x) into 2 parts, that is F12(x) this is, this I define if it's 0 if is X positive and it's a negative side, negative side what you have is F1(x), when X is negative. So if you want this to be defined for positive side, so what you should have is so in an equivalent way you can write this as 0 if X is negative, okay, 0 if X is a negative, and this one here F1(-x) if X is positive, so this is same as saying this one F1 is, when X is positive F1(-x) is F1 of negative things, so that is exactly your function, so this is for X positive and this is for, negative side this is 0, so this is



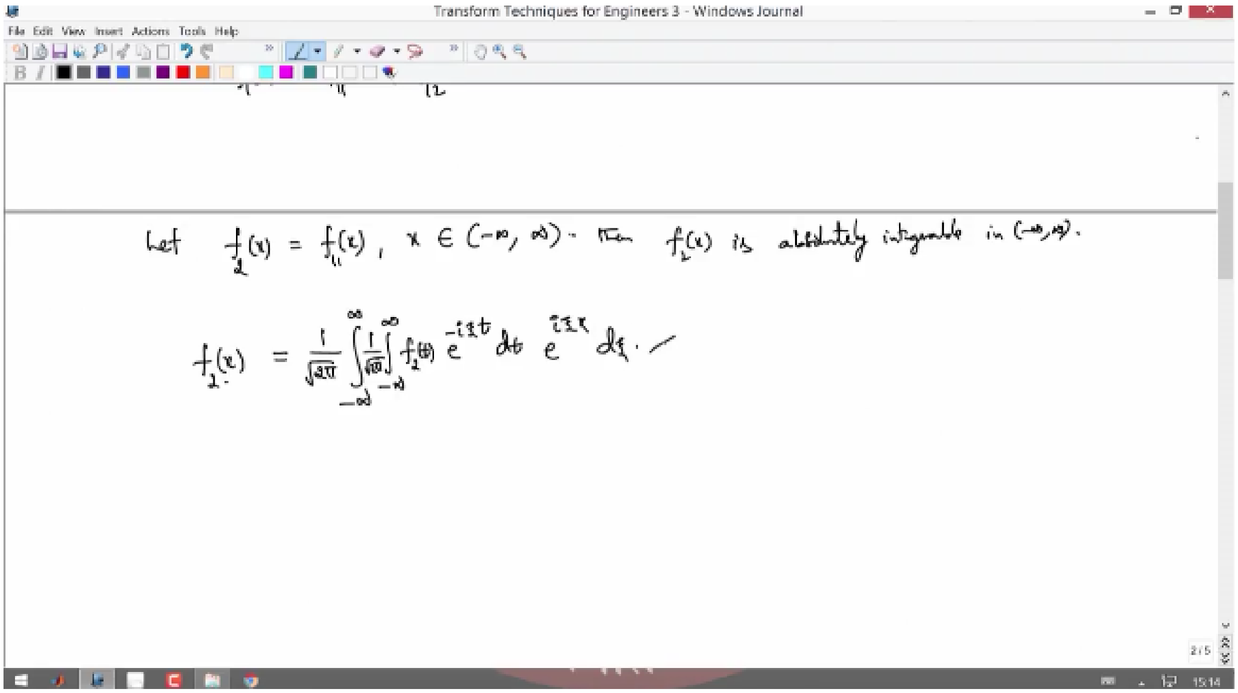
how you can split so that your F1(x) = F11(x) + F12(x), so we will consider only one such function, so let us choose only this part so this part you can do similarly whatever we can do, okay, so this is also absolutely integrable if F1 is integrable. If F1 is absolutely integrable F11 is also absolutely integrable, F12 is also absolutely integrable, so typically F11 is such a function, so F1 we can consider now, so let now, we can always do this breakup of the domains so that you can rewrite F1 as 2 such functions where positive side is there, negative side it is 0, so it's only defined positive side.



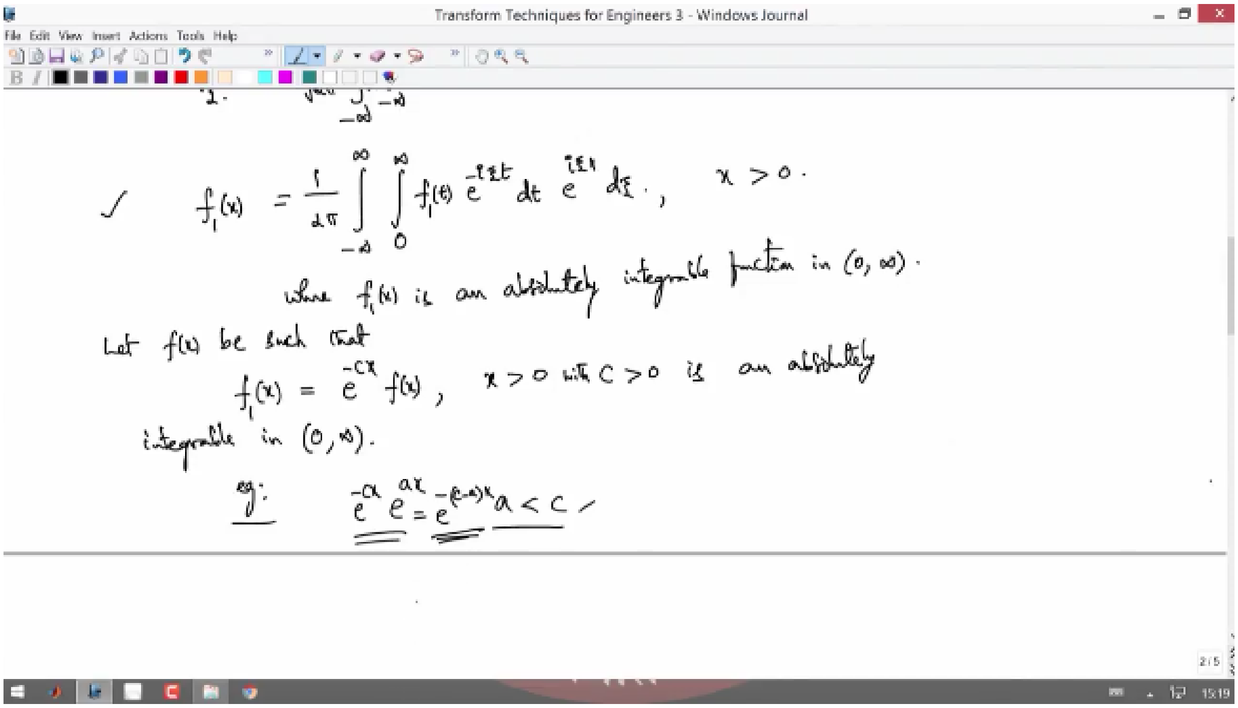
So now I start with this F1(x) as such function, so F1(x) is actually F11(x), I choose this typical such typical function for X positive, X belongs to 0 to infinity or minus infinity infinity, okay, so if I choose this one only one typical function of one I consider so you can also do the same ways for this one, okay, if I choose this if F1 is a, if F11 is since F, since this function F1 is absolutely integrable so you can see that this, let's call this F2 okay, F2 is this one, F2 is also absolutely integrable, F2(x) is absolutely integrable, integrable in minus infinity infinity, so what do you have from this Fourier integral theorem this your F2(x) = 1/2 pi integral minus infinity infinity E power or inside, inside you have this Fourier thing so that is minus infinity infinity, so 1/root 2 pi this you have root 2 pi actually, so this is together is 1/2 pi, so this one into F2(x) into E power -I xi X DX, then this is your function of xi and you have E power I xi X D xi, so this is your Fourier integral theorem, this is what you have, okay.



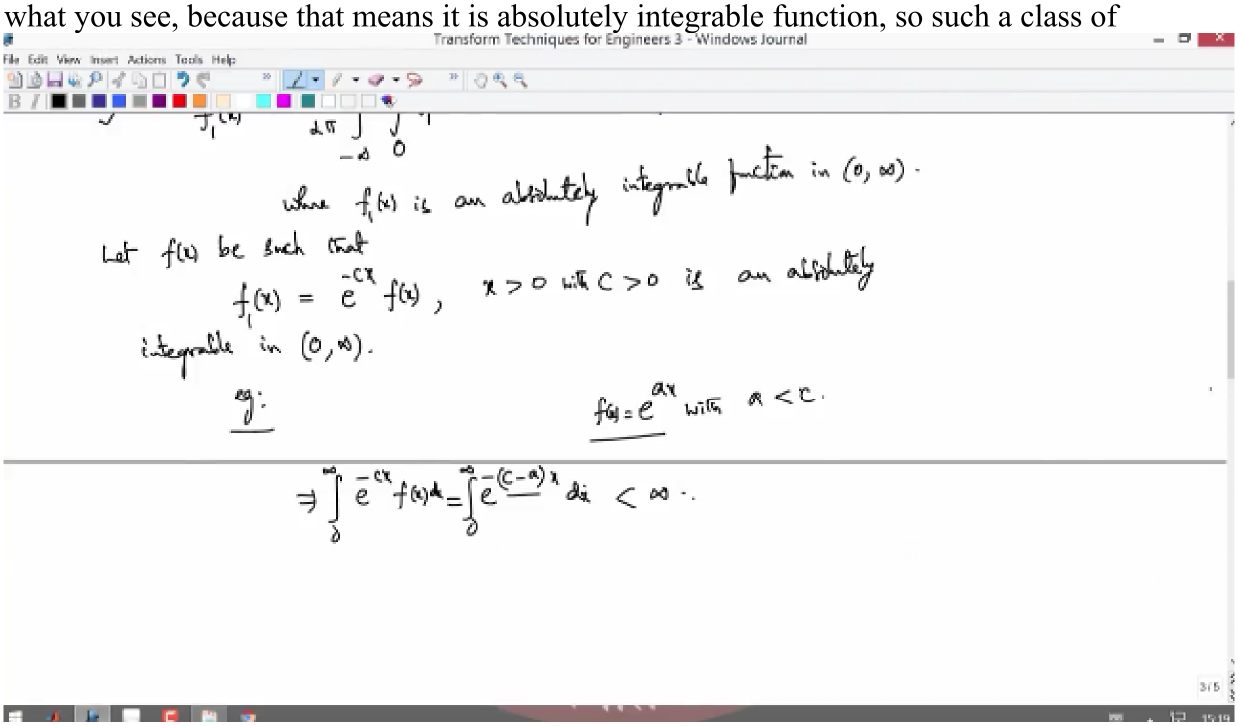
So just to avoid this X this dummy variable I will change to T, this is T so that you have this is the Fourier integral theorem, this is true as we have seen in the earlier videos because of Fourier integral theorem so this is true, always true.



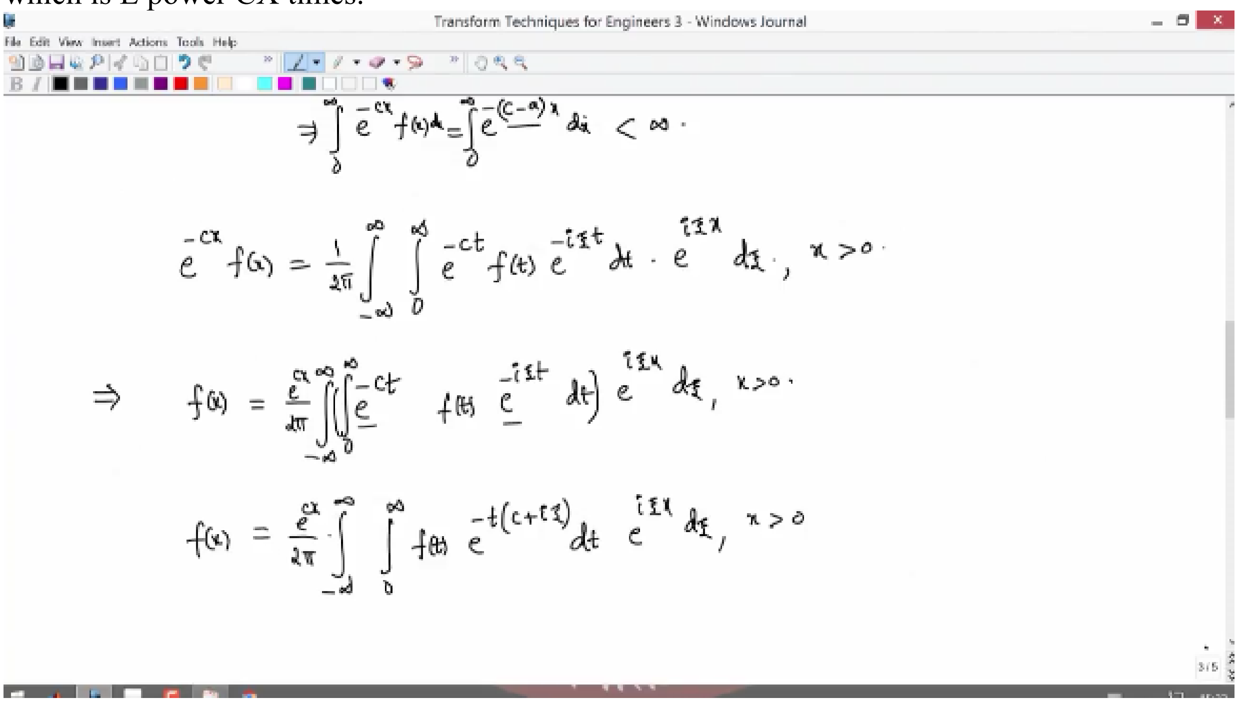
So I will do some, now you can put what is your F2(t) is F11(x), F11 is basically F1(x), F1(x) so let me choose F1(x), so what you get is 1/root 2 pi integral minus infinity infinity, 1/root 2 pi this you can put it together as 1/2 pi and you have this is from F2 which is F11, F11 is basically your F1, F1 is only X positive side, so it's from 0 to infinity F1(t) E power -I xi T DT times E power I xi X D xi, I just substitute what is F2, so if I choose X positive side I can write, what is F2? F2 is F2(x) X positive side which is same as F11(x), X positive side, F11 X positive side is F1(x), so F2 is actually nothing but F1(x) for X positive side, okay.
So here where F1(x) is an absolutely integrable function in minus infinity to infinity, okay, that's what is, F1(x) is basically from 0 to infinity, right, in 0 to infinity, so that is what you have, if you have this this is actually comes so far from Fourier integral theorem.
Now I choose some particular function for F1(x) which is some function which is not absolutely integrable function, but if I multiply with exponential with some -CX okay this is for X positive side, then F1(x) is, then once I multiply this for X positive side with E power -CX and C is positive, if I choose like this this is absolutely integrable function, okay, are you consider such functions F for which this is, this product of E power -CX with such F, such functions F(x) together assume that it is absolutely integrable, okay.
So how do I write this? Let F(x) be such that is this with C positive is an absolutely integrable function, integrable in 0 to infinity so such class of functions. For example, so what are such functions, for example you can take, you can think of E power AX okay, so F(x) behaves like E power AX at infinity, so that if you multiply with E power -CX with A is less than C, if A is less than C what happens? This is like exponential A-C, so what is this one? This is actually equal to E power –(C-A) because C is bigger than A into X this is absolutely integrable function, so from 0 to infinity it is integrable.



So one such function is, so I have to write example means E power AX, F(x) is E power AX with A is less than C, okay, so that you can easily see that this function, this E power -CX implies E power –CX times F(x) = E power –(C-A) times X is, so that this integrable, so this is infinity DX, infinity 0 to infinity DX is actually finite, okay you can easily integrate because this is positive quantity, E power negative minus of some positive quantity into X, so that is what you see, because that means it is absolutely integrable function, so such a class of

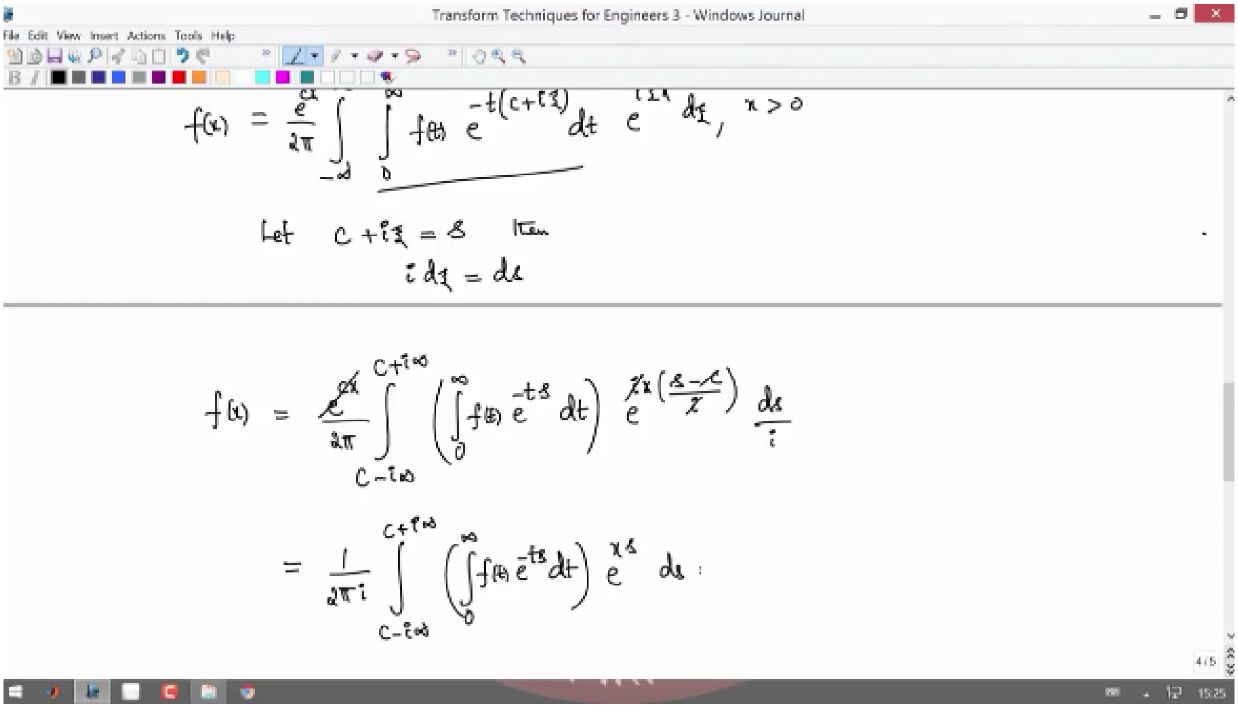


functions if you consider now substitute F1 in the place of, in the place of F1 in this Fourier integral theorem you substitute this E power -CX into F(x) okay, so if I substitute E power -CX into F(x) which is equal to, so 1/2 pi integral minus infinity to infinity and what you have so the integral 0 to infinity and what you have is F1(t) that is E power -CT times F(t) E power -I xi T DT times E power I xi X, right, so I xi X D xi, this is what I if I substitute this from this, okay. So where is this valid? X positive 2 side, so you take this, this implies you can rewrite this as F(x) take this exponential through the other side to see that minus infinity to infinity E power -C(T-X) okay, so I bring this to the other side F(t) of course you have 0 to infinity, F(t) E power -I xi T DT into E power I xi X DX, sorry D xi for X positive, so this is what you have, so instead of writing it here so what you do is, so let me put this E power -CT you write, you put it here this is the integral inside and you have when you bring it this side you have E power CX, so if you rewrite this as, let me write it properly -infinity infinity, integral 0 to infinity E power -CT F(t) into E power -I xi T, so this you can combine with this thing so E power I xi IT, IT you take it out, so if you take it out I xi or you take it so F(t) you can write this as F(t) times this one and this one you can put it together and write E power -T is common, and you can write C+I xi into DT, this is for DT and you have E power I xi X D xi X positive, so this is exactly your F(x) which is E power CX times.

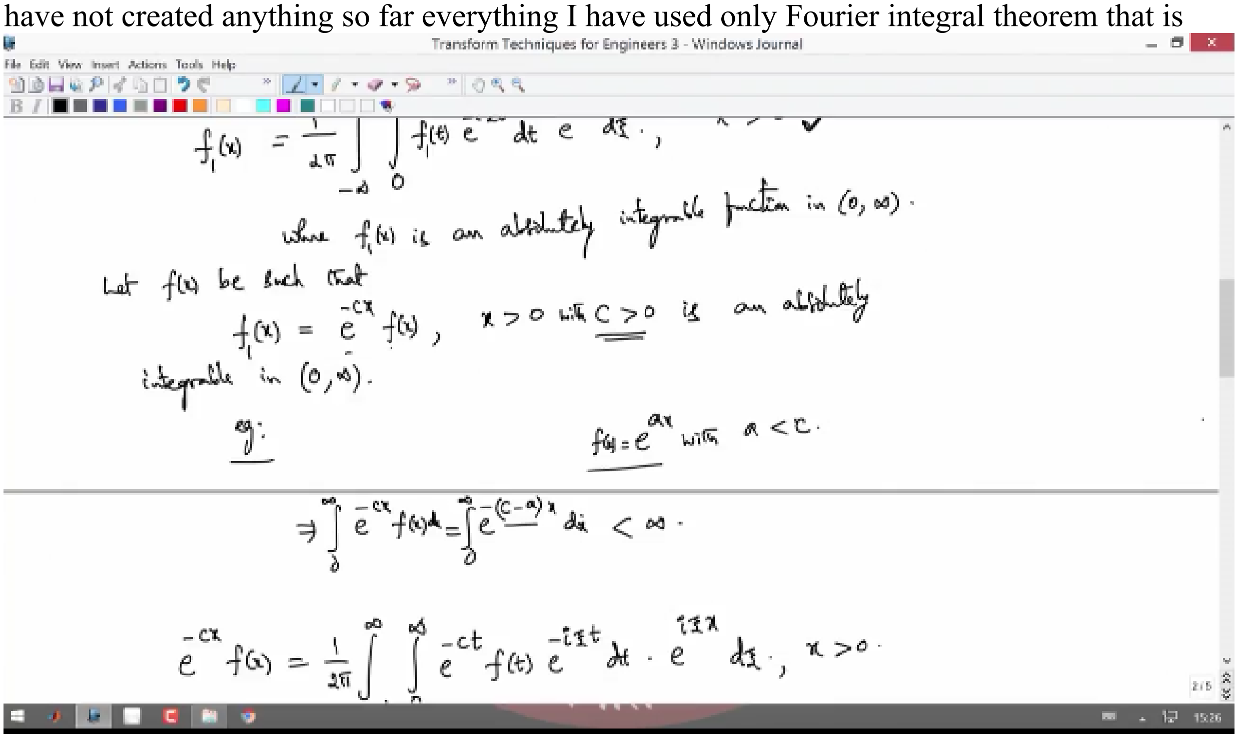


So what I do now here is I use a new variable C + I xi, C is a constant which is you have seen, C is some positive constant, okay, that's how I started, if I do this C+IG you take it as some variable S, let me use some S, okay, if I use S then what happens to, if I use this one D xi, I D xi will be DS, okay, so what happens to this? F(x) will be equal to E power CX divided by 2 pi integral -infinity infinity, so inside it will become this is like integrand for you, so you have F(t) E power -TS DT, so this is integral if I use this variable and E power IX times xi I replace with S-C divided by I, okay.

Then what is D xi? D xi is DS/I, so this is what you have, so you have this I I goes E power -CX and E power +CX that will go and then what you end up is 1 by, this I comes out you have 2 pi I this integral and you see when you substitute this, when I put xi = -infinity, you should write this as C-I infinity to this C+I infinity, so you have C-I infinity to C+I infinity and this integral inside what you have is the same F(t) E power -TS DT times, what you are left with is



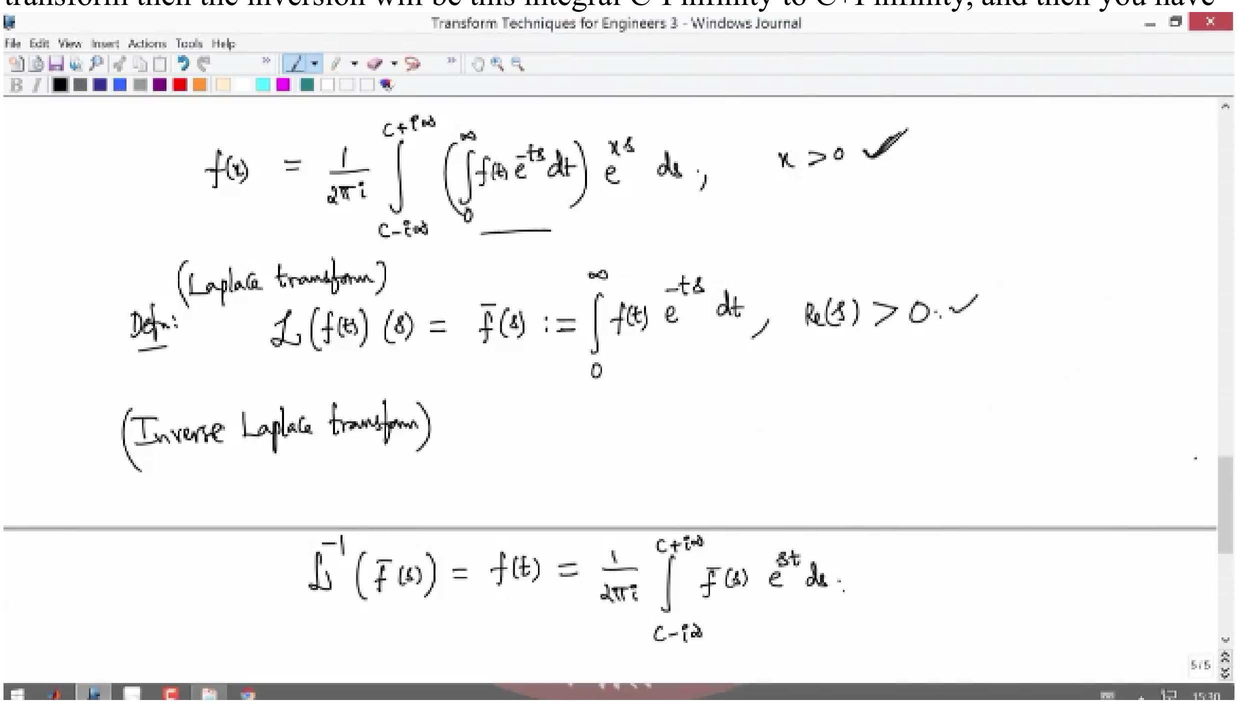
E power XS DS, okay, this is your F(x) X positive side, so this is I have not done anything, I have not created anything so far everything I have used only Fourier integral theorem that is



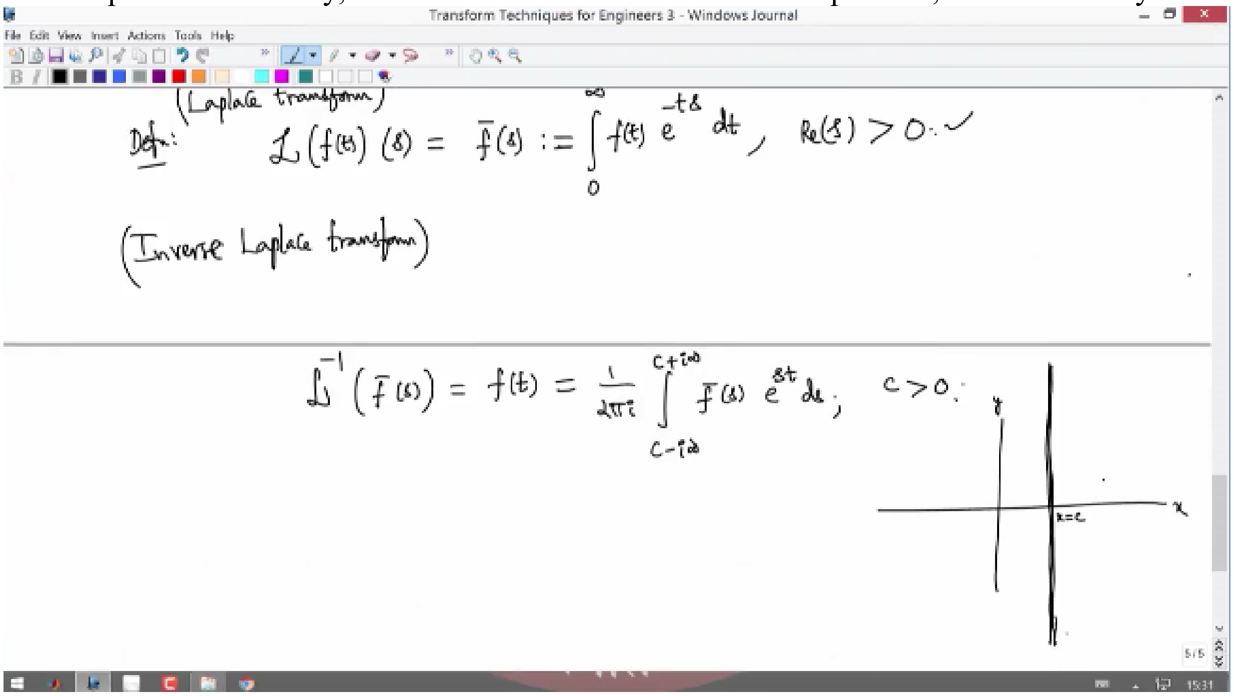
this one, this is the Fourier integral theorem I used or this okay, Fourier integral theorem I used and I only, in the place of absolutely integrable function I choose in this form, my absolutely integrable function I have chosen in this form so that F is such a class of functions, for example, one example of such function is exponential functions are also involved, you can also involve constant functions, you see that constant function is not absolutely integrable function in place you cannot define Fourier transform on it, but here you can define so in Laplace transform on it that is what you will see.

Now you have some kind of Fourier integral theorem for such class of functions F, so now we can define like in the Fourier transform I can define what is the Laplace transform, Laplace transform script L(F(t)) which is a function of S or we can write F bar(s) okay, F bar if you put that is a notation which is function of S, F is a function of T and when you apply the bar it becomes function of S, which is a definition is and put the semicolon equal, okay assignment, you take this as your definition of your Fourier transform if your Laplace transform F(t) times E power -TS DT, what is S? S is C+I xi, so what is C+I xi? If your F is type E power AX, if this is this is called exponential order function of, function of exponential order E power AX, then your S is basically I can say is actually S is a real, S is a complex number now, S is a complex number, real part of S is actually C okay, that C is actually bigger than if any, if your function is exponential function of order A for example like this then it is greater than A, otherwise it's anyway so you can think of 0, you can take it as 0 as of now, if your F is a exponential function of order A then you can put it as A, more than A okay, so at the moment let me write real part of S is positive so that this integral makes sense is well-defined, so this is your Laplace transform okay, so definition of Laplace transform.

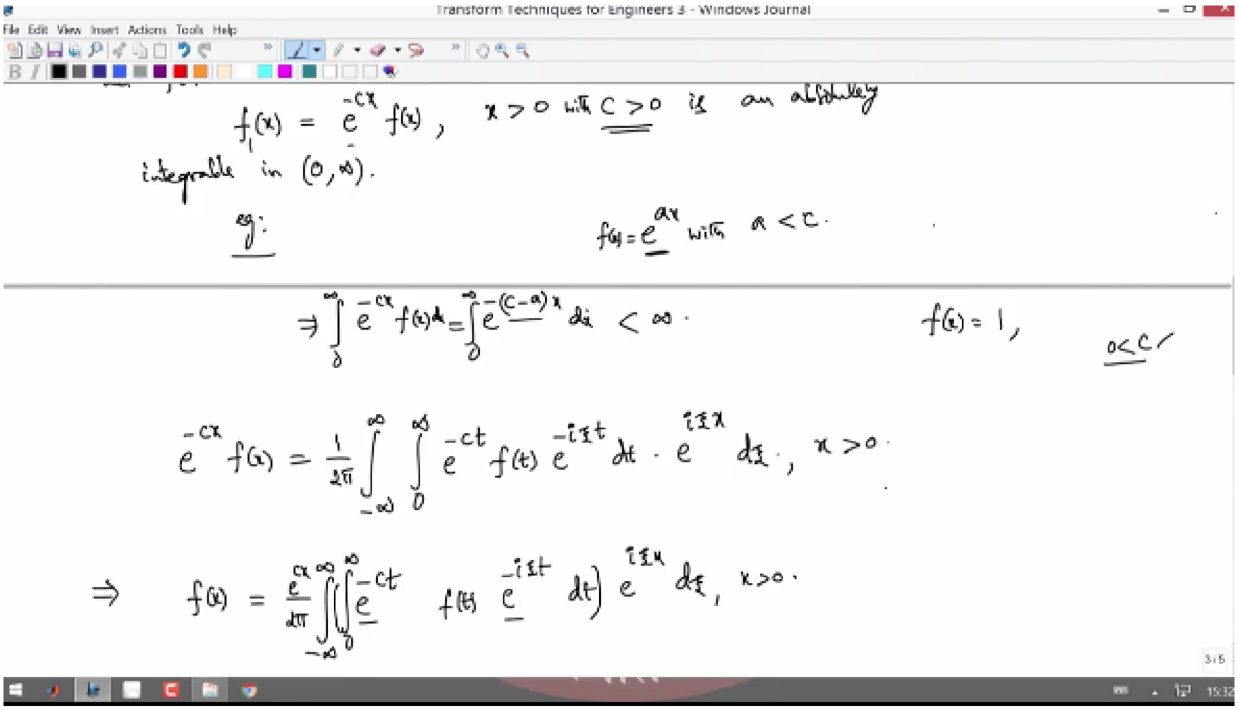
So inverse transform is inversion of inverse Laplace transform, so what is this inverse Laplace transform that comes from, like earlier from the integral theorem now this is your integral theorem you can write your Laplace transform, instead of F(x) let me use F(t) so if I use F(t) so that is the inverse Laplace transform of F bar(s) which is F(t), now in the place of X if you put F(t) 1/2 pi I, so definition is from this integral theorem this is the definition of the Laplace transform then the inversion will be this integral C-I infinity to C+I infinity, and then you have



F bar(s) and E power ST DS because it's a function of T, okay, and that's it, so you have C is, what is C? So if your function is absolute, if it is a constant function A is 0, C is always bigger than 0, so let me put it minimum C is positive, that way C is simply positive, so what is this integral inversion? The inversion integral is a contour integration so this is your X and Y axis, so if you take this as your C, X = C and on this line your integral is over on this line, so C-infinity to C + infinity okay, because S is always, real part of S is positive so you can, its defined positive side okay, so here it's well defined because C is positive, so over which you

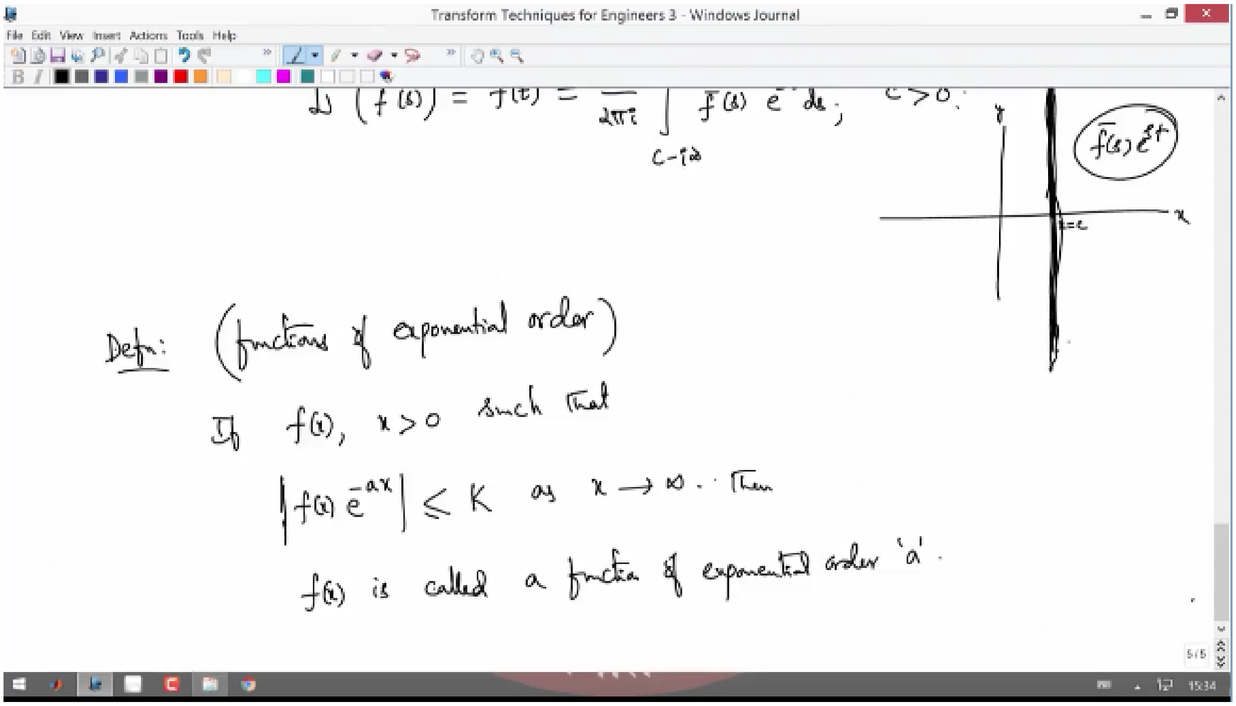


have if you, for this function F bar(s) times E power ST this is your integrand, integral of this over this line, that is complex integral, so that is what exactly you have to evaluate if you want to find the inverse, that is you can use your contour integration technique to evaluate this inversion, okay, that we will see later on, so this is your inverse Laplace transform, okay.
So again let me use if your function F is simply, if your function F(x) is, let us say A = 0 is a constant which is absolutely integrable, if it is not absolutely integrable function but F(x) into E power –CX, where C is positive okay it is absolutely integrable function then you have this, okay, C is positive, okay, so then you have same thing so you have this Fourier Laplace transform and it's inverse transform, okay.



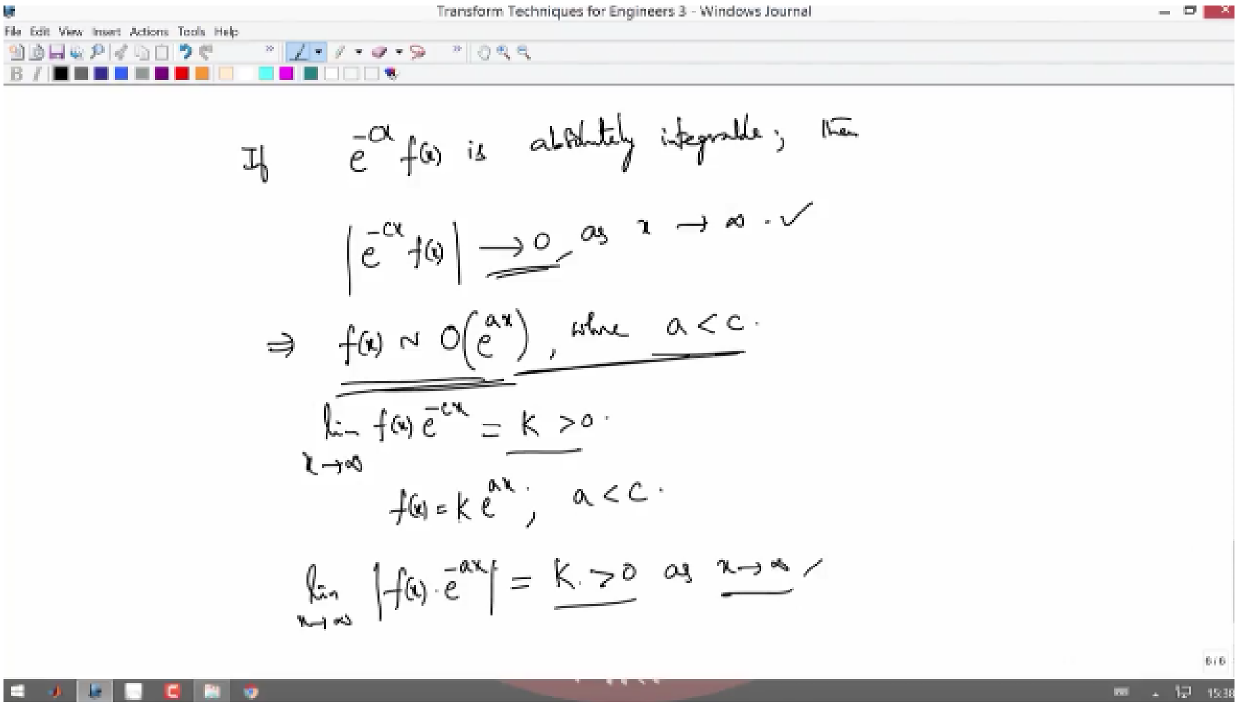
What I have done here is for this function F we have this exponential function of order A, this is the exponential function of order A this one, F(x) = E power AX is, is called functions of exponential order, or F(x) = 1 if you choose constant function or any constant function C or K then this is a exponential function of order 0, because K times E power 0X you can write, okay, so this into any E power -CX is absolutely integrable, so in that sense we can define, we actually consider F means, F is all such of, F of all such functions for which you have this you can define this Fourier Laplace transform and it's inverse transform.

So what are those functions let me define formally, they are called functions of exponential order, so I will define what exactly they mean, if F(x) is X positive side such that modulus of F(x) times E power -AX if you take this one if this is bounded with some constant K as X goes to infinity, okay so this is bounded as X goes to infinity, then what is this F(x)? F(x) is a, this is called a function of exponential order A, or exponential function of order A, okay, this is called



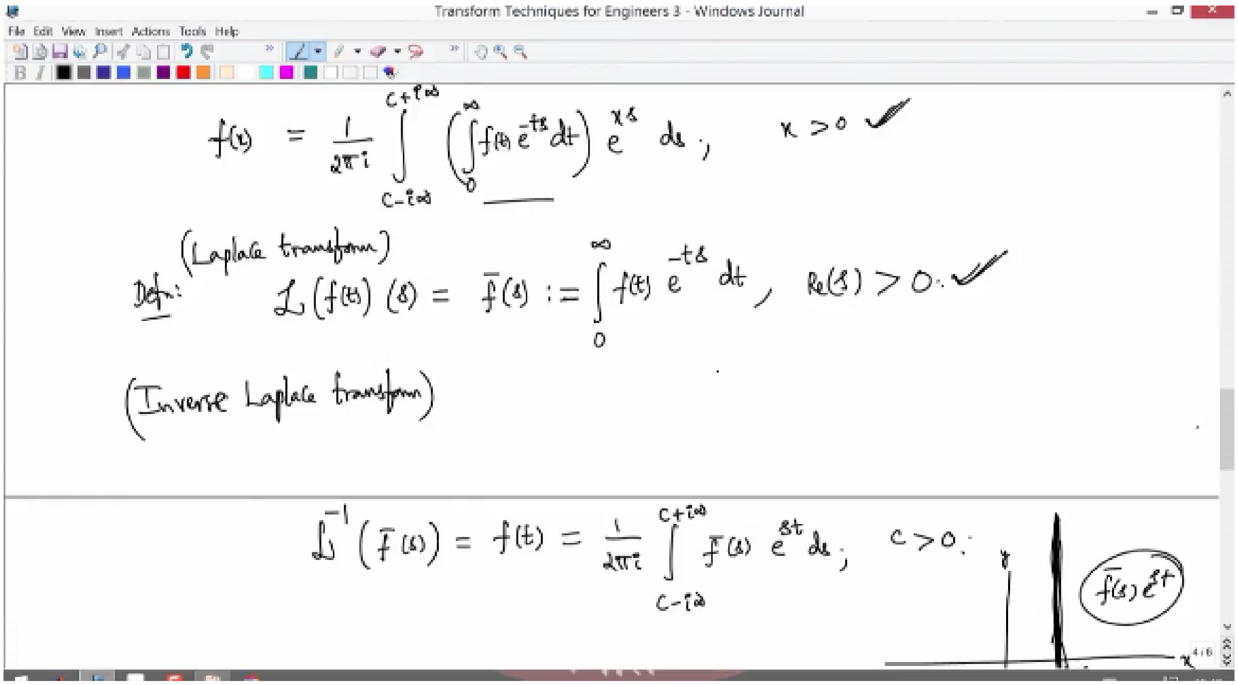
the function of exponential order A, so what does it mean? Is F(x) is behaving like E power AX that is F(x) behaves like Big O of E power AX, that is so what exactly this means? This means equivalently what you have is F(x) times E power AX this limit if you multiply as X goes to infinity or -AX okay divide this, so F(x) divided by E power AX that is F(x) into E power -AX as X goes to infinity this is actually because of big 0, this is a big constant, that we're calling K okay, such that for K positive, for a positive K then this is called exponential order, function of exponential order, so that is actually this is the meaning of this that means F(x) behaves like order of big O of E power AX, okay, so if your function is exponential order A then you have this is true okay, but now if you use this your earlier definition, so if F(x) equal to or rather if E power -CX times F(x) is absolutely integrable, means then what is the meaning of this? E power -CX into F(x) this goes to 0 as X goes to infinity that is the meaning, okay, it's not constant it is 0 so that means F(x), what does it mean? F(x) behaves like order off rather order of E power AX okay where A is less than C, right, that is the meaning, right.

So what does it mean? F(x) divided by E power –AX, okay, so this divided by this, what does it mean? If this behaves like E power AX, as X goes to 0, X goes to infinity this has to be constant, okay, so how do I convince you that this is, this means this where A is less than C. If A = C then this has to be constant, nonzero constant, okay, but that is not the case it's going to 0 that means so if this is the case F(x) is some constant times E power CX, okay, so if my F(x) is like constant times A, A is less than C then F(x) into E power -AX okay, this limit X goes to infinity is actually equal to K, which is positive as X goes to infinity, okay, if A = C is actually



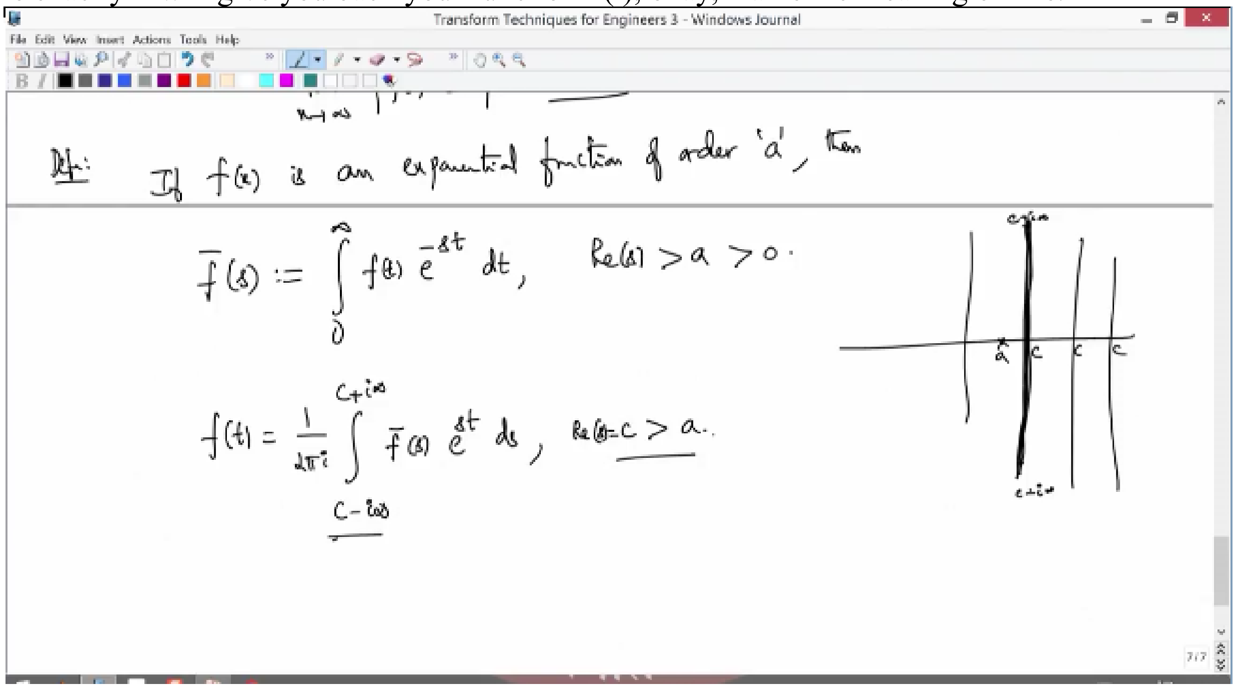
what do you have, if A = C then K into, if A = C E power -CX into K times E power CX that is going to be constant, but that is not true, so what you have is if you multiply like that it by taking F(x) as K times E power CX then it is converges to the positive constant, but that is not true is actually going to 0 that means F(x) has to be like this with A less than C, okay, so if you have what we have used is if you use this E power –CX F(x) is absolutely integrable, then this goes to 0, as X goes to infinity that means F(x) is an exponential of, function of exponential of order A where A is less than C, that is the meaning, okay, A is less than C, this is the argument with which you can see that.

The moment you see if it’s going to 0 and this function is exponential of order A, A is less than C that means it's of exponential order A, it's like E power AX, A is less than C, so that if you go

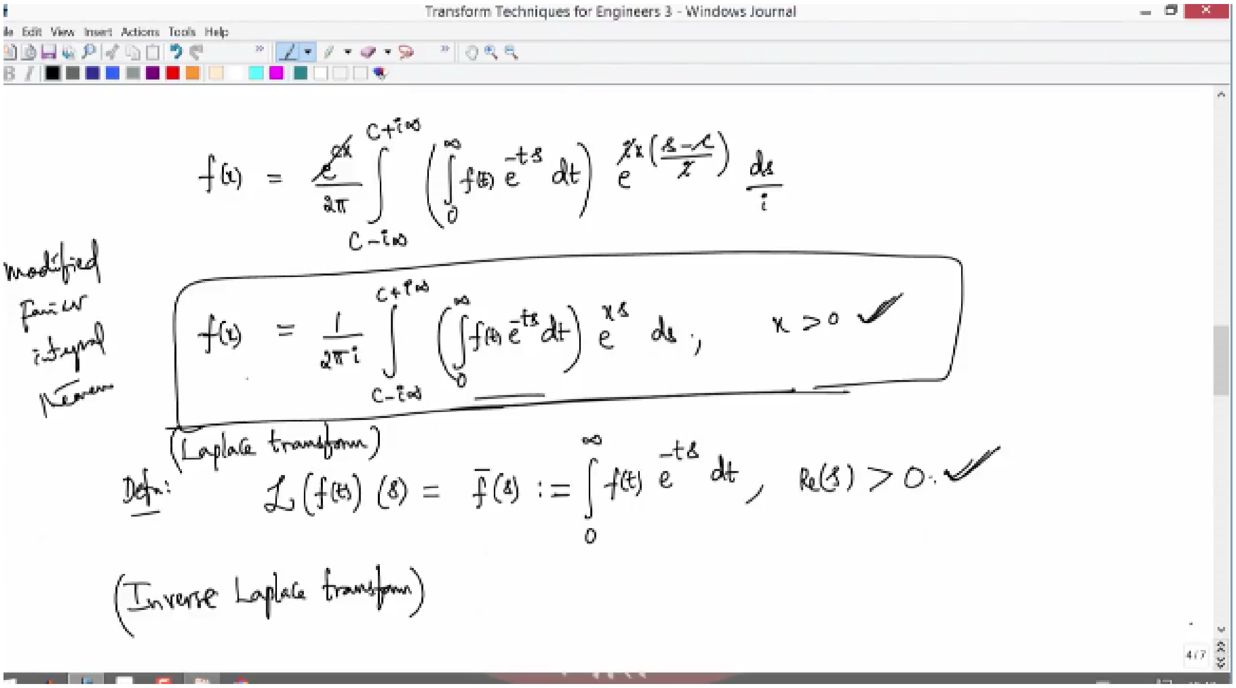


back if you look at here, so what happens here, so what is your C? You know what is your S? Real part of S, so if you look at the Laplace transform now with this definition, so once you know this is a function of Epsilon, it's a exponential function of order A which is less than C then what is your Laplace transform, F bar(s) which is equal to 0 to infinity, F(t) this is the definition okay E power -ST DT, so what is your S? S is, S you have seen is C+I xi, right, that is exactly what you have, right earlier what is your variables S, we have introduced a variable, S is where did we use that S, S is C+I xi, so C+ I xi, xi is between 0 to, -infinity infinity, so C+ I xi okay, so real part of S is, actually real part of C okay, that is real part of S is C because we have seen that if I choose F(x) into E power -CX goes to 0 as X goes to infinity, F(x) is of exponential of order A, A is less than C so you have C is bigger than A, right, so that means if now I can write definite definition, if F(x) is an exponential function, exponential function of order A, then exponential of order, then if that definitions are if your Fourier Laplace transform and it's inverse transform if they have to be valid, A has to be less than C that is what it means because order A it has to be less than C then only this is true, because S = C+I xi, so that means real part of S is C which is A, so put it together we can remove and write real part of S is greater than A, okay, so that is the meaning.
It is anyway so A which is positive which we know because it's exponential of order A which is positive, okay, there's nothing like exponential of negative order, okay, only positive order so with that definition you have this Fourier Laplace transform, and your inverse transform is 1/2 pi I C-, this is now your C, what is your C? C is let me write C is I infinity C + I infinity times F bar(s) E power ST DS okay, so this is your inverse transform, what happens to your C? C is in such a way that C is actually greater than A, so real part of S again what is C? Real part of S which is C, so that is actually S is actually real part so that is C is bigger than A.
If your function is of exponential of order A, A your C what you have to consider is if you are, the moment you have a exponential function of order A is positive here, A is here you have to choose, you can choose any C which is beyond, which is bigger than A that is your contour integration over this, anything you can choose here also, here also, anything you can choose as which is bigger than A, that's all, that is the meaning of inverse transform, so universe transform

you have this by the integral theorem but then you have to see that understand that C is actually bigger than A that means anything any line from that is C, C any line C either is here or here, here, so you have C-I infinity to C+ I infinity, so over which if you integrate this integrand that is exactly it will give you back your function F(t), okay, that is the meaning of this.



So you have extended the class of functions, initially Fourier transform and inverse transform is defined only for or rather Fourier integral theorem is defined only on absolutely integrable functions those class of functions you extend it to, extended now and you define the Laplace transform and it's inverse transform that means Fourier integral theorem we modified in such a way that you, that modified Fourier integral theorem is now valid for functions that are extended class of absolutely integrable functions that means other than X absolutely integrable functions you also have constant functions, functions of exponential order, order is some any A, okay, any exponential function of order A for example okay all these exponential so you have extended, all absolutely integrable functions are of order exponential functions of order 0 okay, so here these are also and these definition once you modify the Fourier integral theorem these definitions later on, then you define this Laplace transform and inverse transform for functions not just for absolutely integrable functions but also for extended class that is exponential functions of order, any order okay, so if you have a finite order exponential function of finite order you can define these Laplace transform and inverse transform, so that is the meaning, okay.



So what you have is finally if you look at this Fourier integral theorem, modified Fourier integral theorem that is this okay, so where is the, this is this, this is what it is, Fourier integral theorem, if you modified your Fourier integral theorem this is exactly what you have, this is the modified Fourier integral theorem, modified Fourier integral theorem, now it's valid not just for integrable functions, absolutely integrable functions F(x) but also for any functions, constant functions, exponential functions of order, some order, okay, so all these, so you basically extended the class of definition of F, so the domain of this integral theorem so for all functions, these functions which are valid here that class you extend it to bigger class, so that you can define from that modified Fourier integral theorem you have Laplace transform and it's inverse transform.

And what kind of functions you have? See there are bigger class not just exponential functions not just absolutely integrable functions but also exponential functions of order A let us say, okay, so if there's any finite order if you think exponential functions of finite order you can have this Fourier Laplace transform and it's inverse transform.

So what we have done so far is just considered Fourier integral theorem which we have proved rigorously for absolutely integrable functions, and then we modified it so that when you modify these functions F(x) has to be of exponential functions of order A, where A is less than C, okay, that's all, that's what exactly we have done that implies if you choose any such functions of exponential order A then you can define Laplace transform and it's inverse transform, Laplace transform once you define the real part that S is, S if you look, what is S? S is actually in your inverse transform if you look at it, the S is basically from C-I infinity to C+I infinity, what is that C? C has to be bigger than A, that is C is real part of S, so here both the case is real part of S is bigger than A, A is positive okay, this is how we define a Laplace transform and the inverse Laplace transform naturally, but you still don't know you may have to evaluate, you can evaluate this Laplace transform but it's not easy, straightforward to evaluate this inverse transform for a given F bar(s) okay, you may have to evaluate this contour integration.

So we will see basic elementary functions, exponential functions, constant functions, all these things let us first define, let us just calculate what is Fourier Laplace transform and then you can use, once you know its Laplace transform because you started with the function we already know inverse transform is the function itself, okay, so that way we will just try to get the gather the elementary functions for which you know a Laplace transform and inverse transform, so you can have them in the mind okay, and then even surely we will see the properties of this Laplace transform and then we will see its applications later on, in the later videos, okay. So we will see next video, we'll start making evaluating Laplace transform for elementary functions. Thank you very much.

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