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Transform Techniques for Engineers
Solution of Heat equation by Fourier
Transform
With
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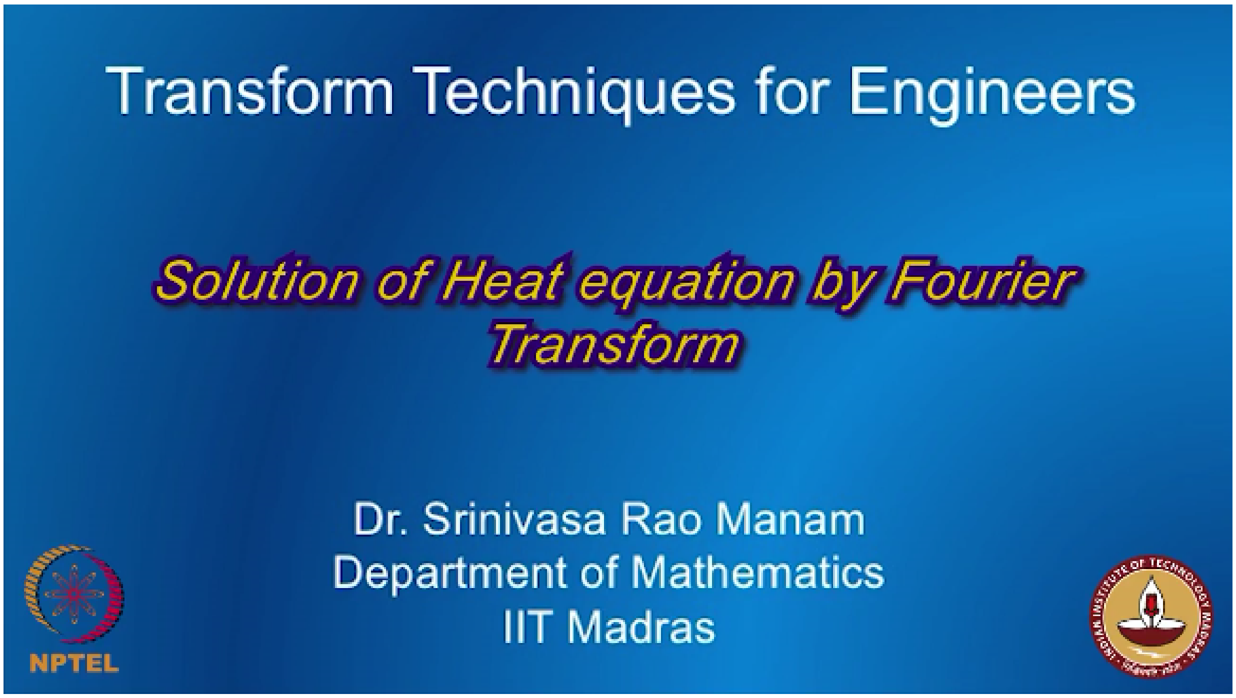


Introduction to lecture 27, welcome back in the last video we have seen how to solve initial value problem for a wave equation in the whole domain, that is full real line so initial value when you are providing so how to solve that problem, so we have seen that so what we derived is D’Alembert's solution because we have used Fourier transform, so basically the functions involved, the initial values and the solution itself we assume that it is absolutely integrable function and that means it goes to 0 at infinity.

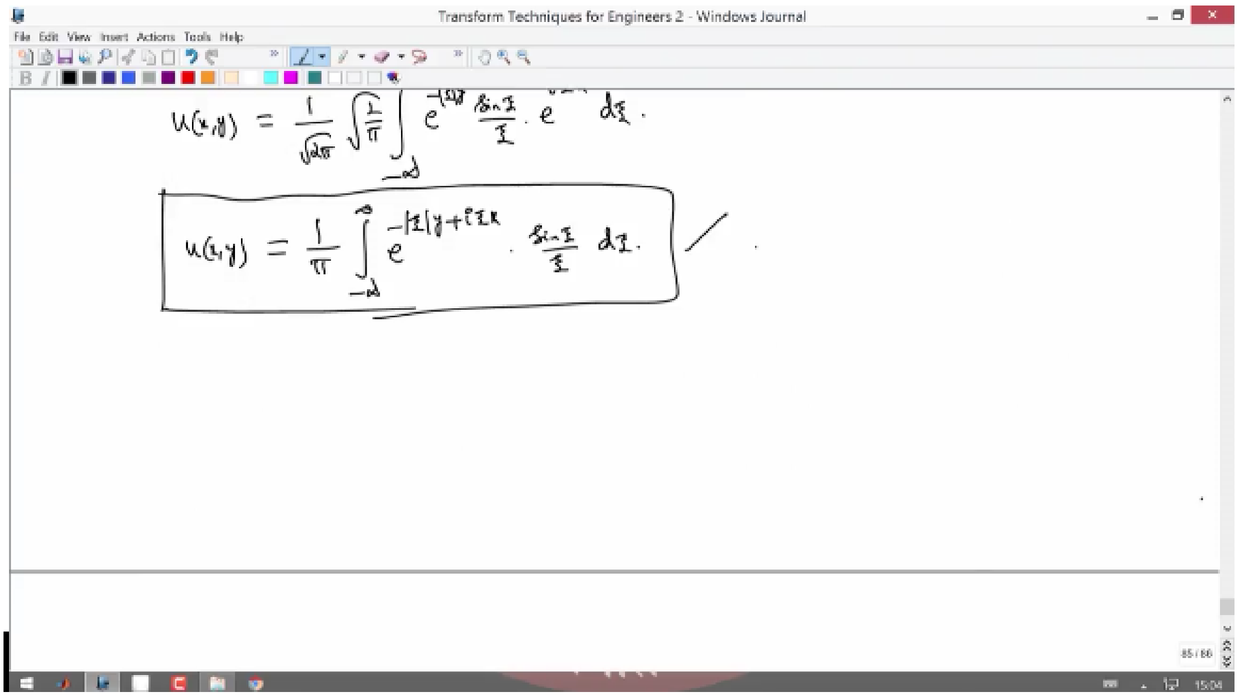
So but in reality but so when you look at the wave equation, so when you model the wave equations, the wave equation need not be going to be 0 at the solution or the displacement at infinity need not go to 0, but infinity there is no such infinity and reality, so at infinity you can safely assume that there is no wave, okay, particularly you can think of having no energy at infinity because there is no such, there is nothing like infinity so that's a notion, mathematical notion so if you assume that these initial functions and the solution function if they are absolutely integrable that means there at infinity they tend to 0, then you can assume, you can apply the Fourier transform, and that's what we have done and we derived the solution which is D Alembert's solution, but when you apply other methods for example if you directly can solve this wave equation, this initial value problem for the wave equation in the full real line we don't assume this solution or the initial values that are going to 0 at infinity, but still you derive the same D’Alembert’s solution okay, so in both the cases this is true.

So by Fourier transform you can solve this problem assuming that the waves at infinity, at both the infinities -infinity and +infinity they tend to 0 this wave energy will be tending to 0 at infinity, if that assumption you can simply apply the formally you can apply the, or legitimately you can apply this Fourier transform and derive the D'Alembert solution, and even when they are not actually in reality, so this wave energy need not be going to 0 at infinity still formally you can apply the Fourier transform and get the D'Alembert solution because that is just extension of what we have done, okay.

We have solved the problem assuming that energy at infinity are going to be 0, but it need not be true, so need not be 0 at infinity still you have the same solution that is D'Alembert solution this is what we have done in the last video.

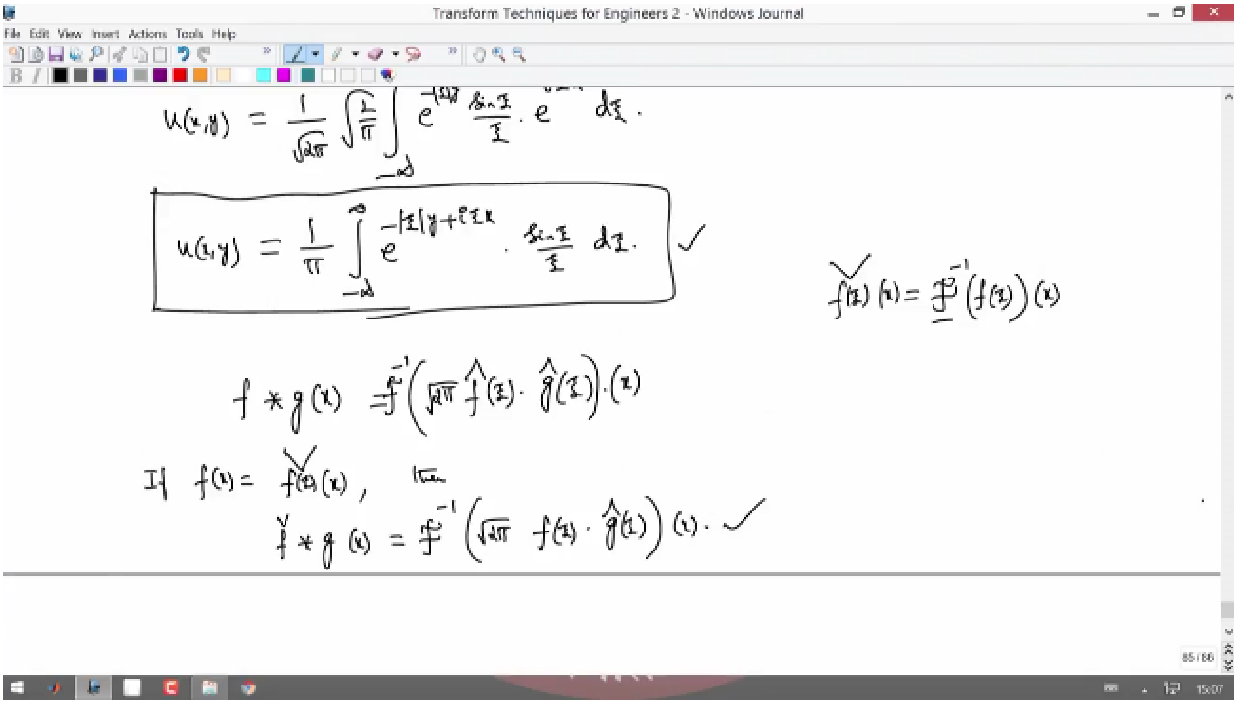
And we also have seen how to solve a Laplace equation which is actually a steady state heat equation, we have solved the Laplace equation in the upper half plane that means you can think of that domain that upper half plane as a plate, upper half plane as a semi-infinite, it's a kind of big half plane plate with boundary being Y = 0, that is X axis and which is at steady state that is at -1 to 1 on the X axis you keep the boundary data that is keep the temperature, constant temperature you maintain constant temperature say 1 between -1 to 1, and outside on the X axis you maintain temperature 0 so with that boundary solution, boundary condition for the Laplace equation which is the steady state heat equation, what you're looking for is when hot plate, when your plate is upper half plane and which is heated, and after some time if it reaches that, when it reaches that steady state and what happens is the heat is distributed all along this infinite plate, so the problem is to find the distribution of this heat, there is a temperature distribution of the plate by keeping that on the boundary you, while keeping the boundary at certain temperatures, so for example here between -1 to 1 you kept the temperature 1 and outside on the X axis that means at Y = 0 and X belongs to -infinity to -1, and 1 to infinity you keep the 0 temperature.

Then what is the temperature distribution of this plate in the upper half plane, that's what we have seen, but we find the solution being a kind of integral solution being integral, but that's oscillatory type of integral, so computationally is not good so this integral you cannot compute because of oscillatory terms in it, so we will try to simplify in this video and get you a nice form of solution that is in a computable form, okay, and then we will solve some other problems.

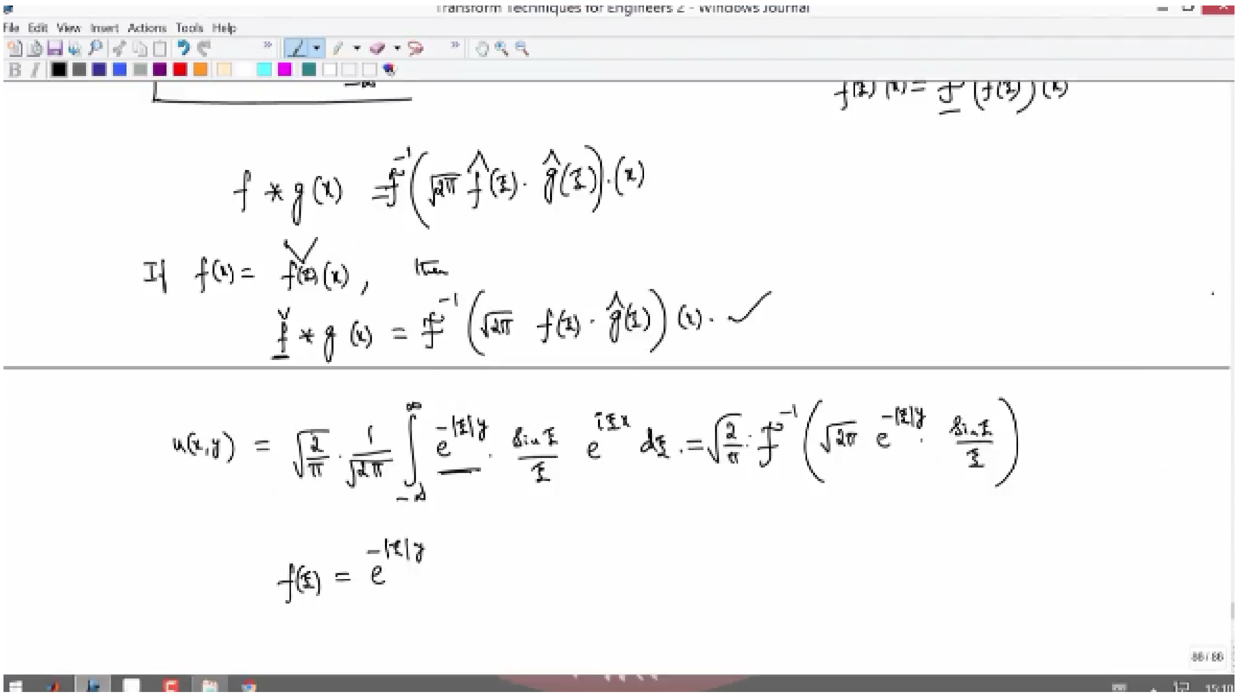


This is the formal solution as an integral, this is kind of oscillatory kind of integral because you are doing E power I xi X to avoid this oscillatory kind of thing we can derive another form of the solution, just by applying the technique that evaluation of the integral we have derived so Parseval's identity kind of technique if we use, so what we use is kind of before I do this we will try to use this technique, so let me use this, so you have this Fourier, you have this convolution product which is, if you take the Fourier transform of this on this okay, so then this will become this, and what you get is F cap(xi) times G cap(xi) right, so what we have is root 2 pi, root 2 pi is what you have, okay, so if you apply the inverse transform both sides what you get is this one, and you have this inverse transform for this, inverse transform of this full thing, so if I say F inverse, F inverse(x) now, this is a function of X now, okay, this is what you have and instead of F you start with F, F is a function of X, and you start with some function of xi whose inverse transform, Fourier inverse transform of F(xi) which is function of X, if I write like this if is the inverse cap, inversion let me write like F cap, F inverse cap(x) that means it's actually originally it's a function of xi for which if you apply inverse transform, what you'll end up is function of X, so if you choose this if F(x) is equal to some function, now if I replace F inversion which is a function of X, originally F(xi) on which you apply this, that is function of X.

And then what happens if you write this same thing what you'll end up is F inversion, convolution with G which is a function of X here, if you apply the inversion so though you have the inversion here, so F inverse, let me write here F inverse root 2 pi so that root 2 pi can always write outside, and here instead of F you have F inversion, so F inversion with the F cap that will be cancelled, what you end up is simply F(x) times here, right, so what you get is a F cap, F cap(xi) and you have inversion, so if you do the inversion so you end up getting F(xi), so you have F(xi) times G cap(xi) okay which is function of X, this is the identity we use to simplify this, okay, simplify that solution U(x,y) for X positive X belongs to full R and Y is positive, okay, so we'll see how this, how we use this.

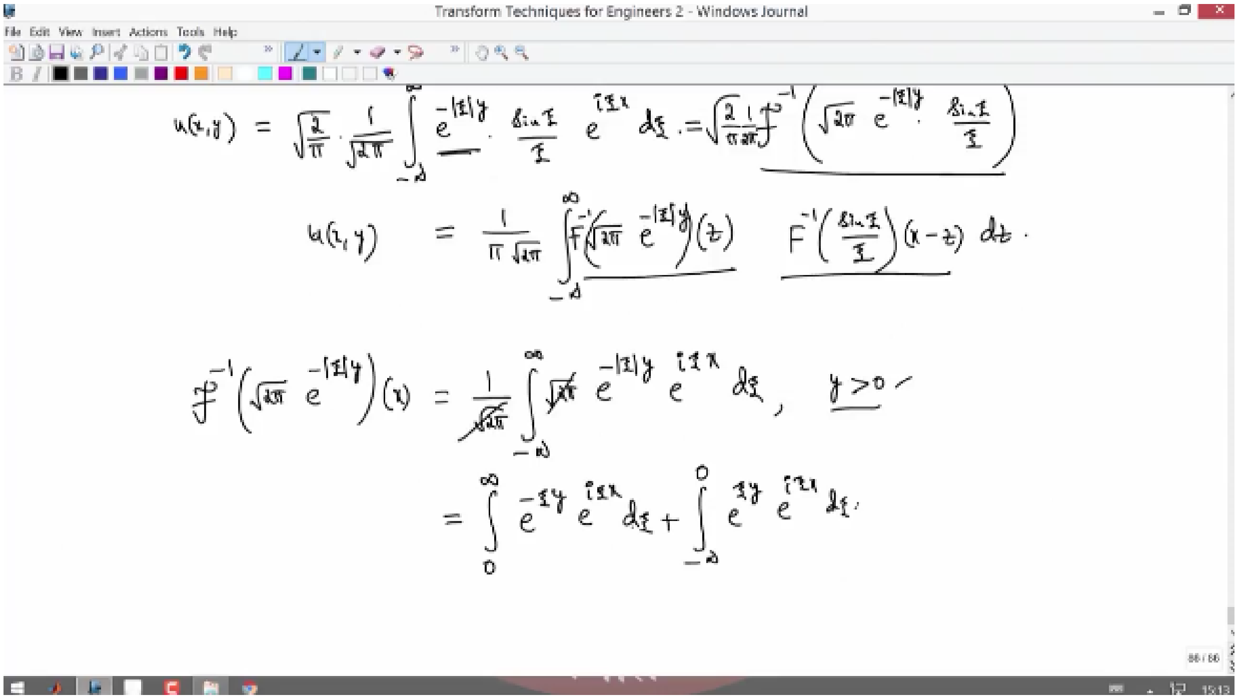


What I start with is inverse transform of, so here I use this inverse transform, so first thing is first thing I use is, this is the U(x,y) is 1/pi times rather if you write 1/root 2 pi, root 2/pi so that you have 1/root 2 pi that is actually root 2 root 2 goes, pi pi goes so 1/pi and this is -infinity infinity E power -mod xi Y time sine xi/xi E power I xi X D xi, so that is exactly your U(x,y), so this is your F(xi), F(xi) is E power -mod xi, Y, Y is a constant, okay, so if you take this inverse transform of this, okay, that means here you want here, so you end up so this is actually equal to root 2/pi, inverse transform of root 2 pi times, so you have root, so root 2 pi, root 2 pi one more root 2 pi right so you have, so root 2 pi F cap(xi) times G cap of, sorry, so this is F cap(xi), so this is F(x) and this is some G cap(xi) that is sine xi, y, okay, so let me write this as E power –mod xi, Y times sine xi/xi, okay, so that you have, to make this Fourier transform I need 1/2 pi, so that 1/root 2 pi, so what you get is 1/2 pi, root 2 pi root 2 pi goes and then that is 1/2 pi, okay.

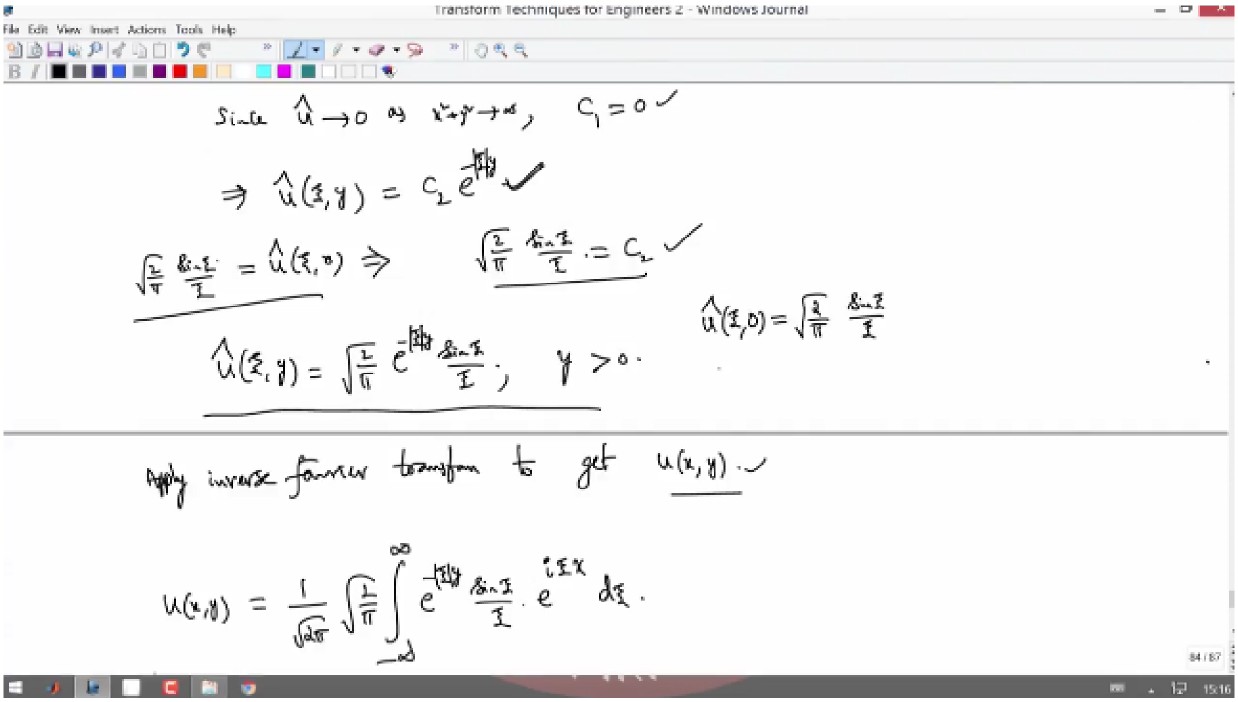


So you start with F(xi) is this, so but if I apply this technique, this equality what you end up is root 2 root 2 goes, root 2 so what you get is 1/pi times root 2 pi, okay, 1/pi times 1/root 2 pi this together and this is in the place of this I use this one, so that is -infinity to infinity, that is a convolution product of Fourier transform of this part, so root 2 pi times E power –mod xi, y, for which you take the inverse transform, which is the function of X, X,Y is there anyway, Y is a constant, so eventually this is the inverse transform of, so let me write instead of cap, inverse cap you write F inverse of this, okay, F inverse of this if you do which is a function of X, and then other one is G of, so that is so which is you have to see what is the inversion of this, inversion of, if you write this inverse F inverse of sine xi/xi which is a function of X, so X-Y right, so what is the convolution? Convolution is, this is F of, this is a function of, this you call as Y and some Z, Z is in the place of X, and this is going to be X-Z DZ, this is the convolution, convolution variable is simply Z variable I used, okay, Z is a real number.

So I need to calculate F inversion of this, and F inversion of this, that's all, then I put it together here, so that is the form you want to derive, so you want to simplify your solution. So to start with this F inverse of root 2 pi times E power -mod xi, y, y is a constant which is a function of X, if you look for what happens 1/root 2 pi by definition -infinity to infinity, root 2 pi that goes E power -mod xi, Y, Y is constant E power I xi X D xi, so this is exactly what you have, so you have –infinity, so you split this into 0 to infinity like we have done earlier, you can write 0 to infinity when it is 0 to infinity so it depends on Y is positive, Y is anyway positive, since Y is positive we know that is Y is positive, so xi is, when xi is positive you can write E power –xi,Y E power I xi X D xi + -infinity to 0 when xi is negative, so you have this is xi, Y is positive E

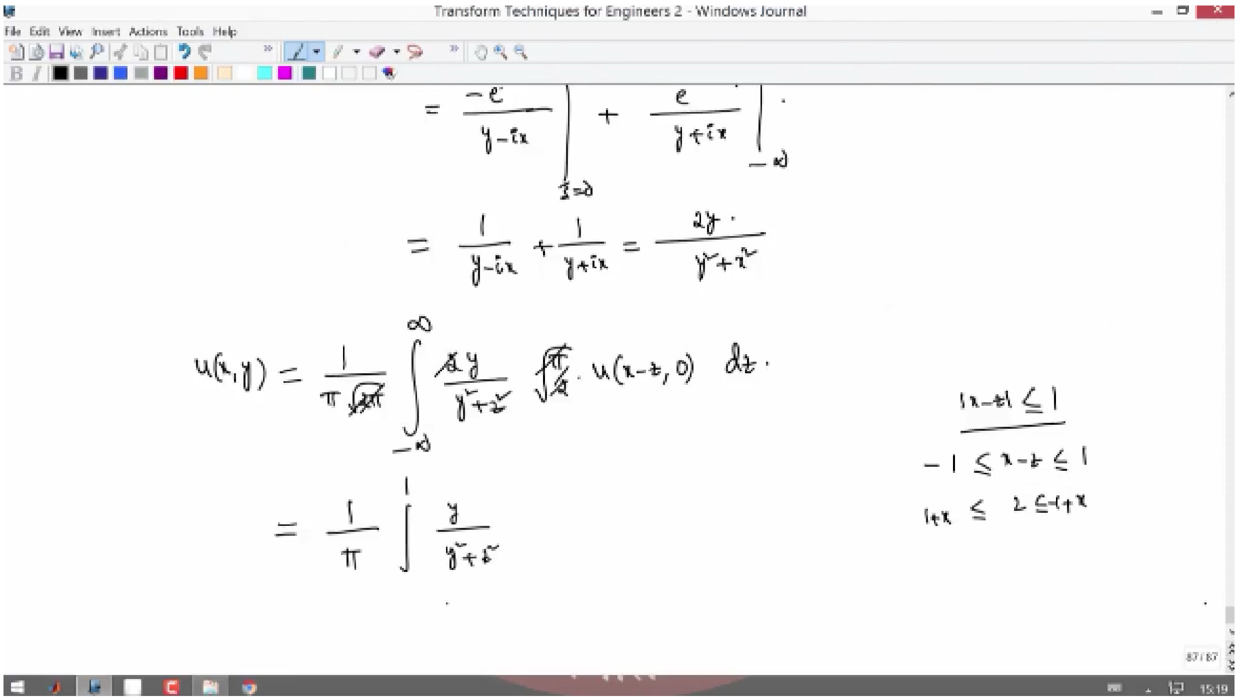


power I xi X D xi, so what you end up is, this is xi so if you write rewrite this is E power xi times –xi times, Y-IX D xi + -infinity to 0 E power xi times Y + IX D xi, simply integrate this to get 1/Y - IX E power –xi Y – IX, for this you apply the limits because xi is positive, Y is positive this is going to infinity, infinity this is going to be 0 and this is also similarly you can do this Y + IX and you have E power xi Y + IX this is from -infinity to 0, if you apply the limits infinity contribution is 0 minus minus plus you get end up getting Y-IX, okay, this is at what I'm putting xi = 0 to xi = infinity, and xi = 0 this is 1, minus minus plus so you have this one and here -infinity this contribution is 0 so you have ending up getting Y+IX, so together this is actually Y square - I square that is + X square, Y square + X square and Y + IX so you have 2Y, okay.

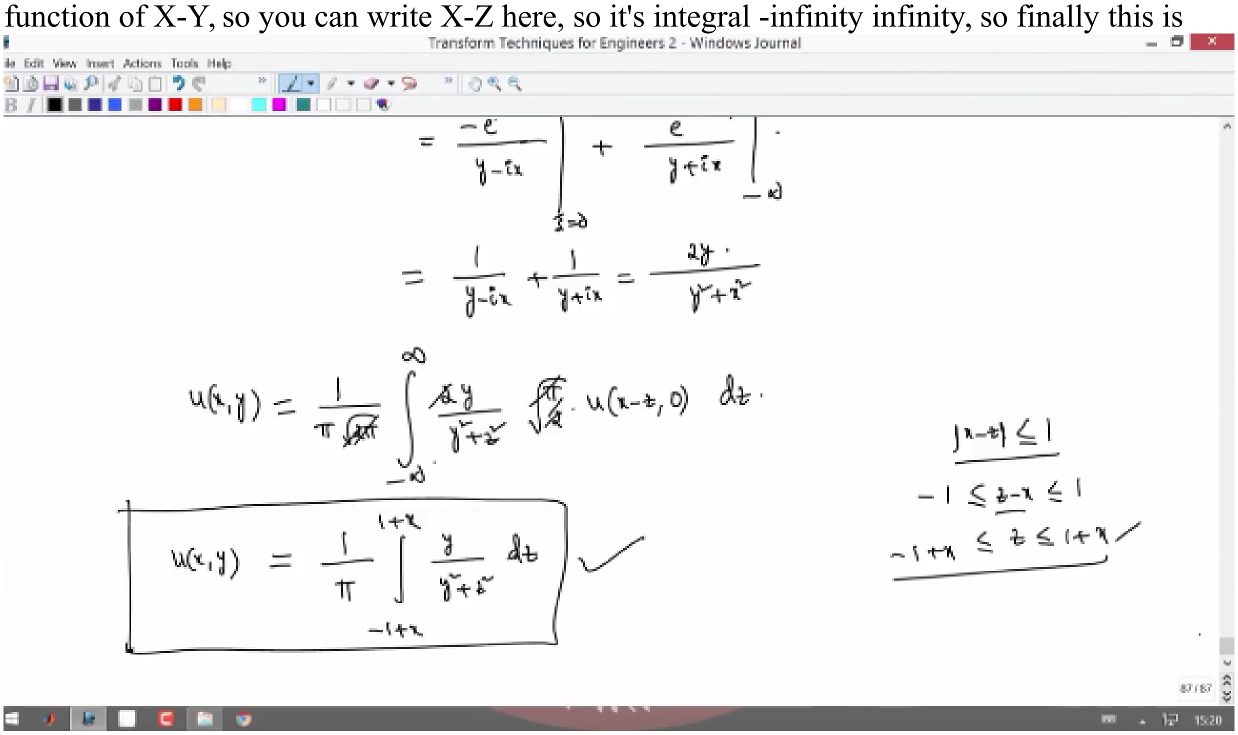


So this is what you have to replace here for this, and to see this one F inversion of sine xi/xi you don't have to calculate, we already have here, so this is your U(xi,y) is this, U cap(xi,0) from this U cap(xi,0) is equal to root 2/pi, Y = 0 if you put that is going to be sine xi/xi, sine xi/xi so what is U cap(xi,0) we know U cap(xi,0), U cap(xi,0) is this, that implies if you take both sides inverse transform what you end up is, you are getting U(x,0), U(x,0) is the inverse transform of root 2/pi times inverse transform of sine xi/xi, so this is what we use, so root 2/pi if you bring it pi root pi/2 comes out here, so pi inverse of that will replace here, so if you do that so what you end up is U(x,y) is equal to, let me write 1/pi root pi root 2 pi, 1/pi root 2/pi, 1/pi times 1/root 2 pi instead of this integral -infinity to infinity, instead of this you have 2Y/Y square + X square that is for F inversion of root 2 pi this one, okay.

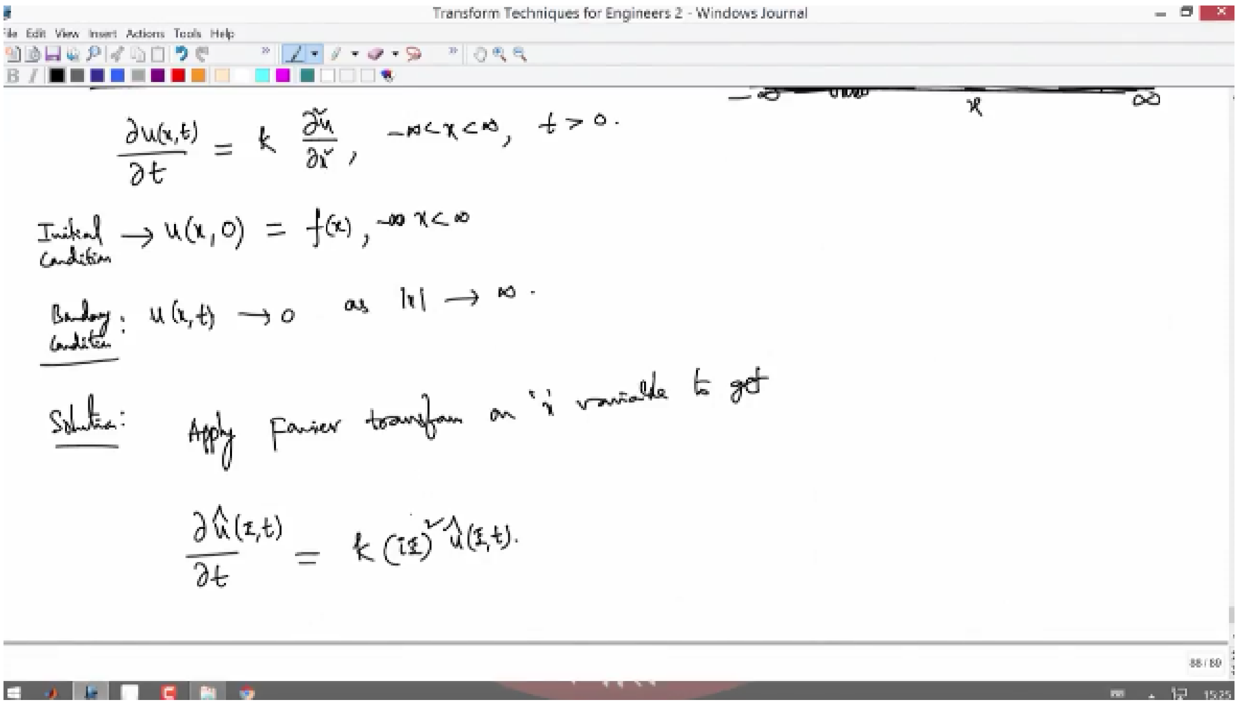
And for this you can replace U at root pi/2 times, U at U(x-y) so if you do that, this is a function of Z so I replace, instead of X I replace Z, Z square and here root pi/2 times U(x) that is U(z) X-Z you have to replace X-Z, so U(x-z, 0) DZ, this is exactly what you have, so this is actually equal to root pi root pi goes, so what you end up is and then this 2 this 2 goes, this 2 this root pi root pi goes so what you end up is -infinity infinity, and you have -1 to 1 Y/Y square + Z square and U(x-z), what is U(x-z)? What is U(x,0), is this one so this is X-Z, U(x-0), U(x-0) is you can use the initial data here that is 1 if mod X-Z is less than 1, less than or equal to 1, so that means Z is the independent variable so Z is between X-1 to, what exactly is this? X-Z is between 1 and -1, and you have Z is, -Z is between 1-X right, -X 1-X, and if you do it plus 1+X here and you



have a -1+X, so Z is that, so Z is between 1+X2 -, so -1 maybe this is not the way, so you can say Z-X is also same, so Z you want 1+X this side -1 + 6, so this is easier, okay, mod X-Z is same as mod Z-X, so if I use this there is no confusion, so you end up getting this one so Z is between -1 + X2, - 1+X and this value is 1, which is DZ, so this is exactly your function of XY, this is the solution what you get for the problem.
So you can also work out if U(x,y) XU(x,0) if this is as an initial condition, as a boundary condition if you have as a function of X you can simply replace F(x) there, okay some simply function of X-Y, so you can write X-Z here, so it's integral -infinity infinity, so finally this is



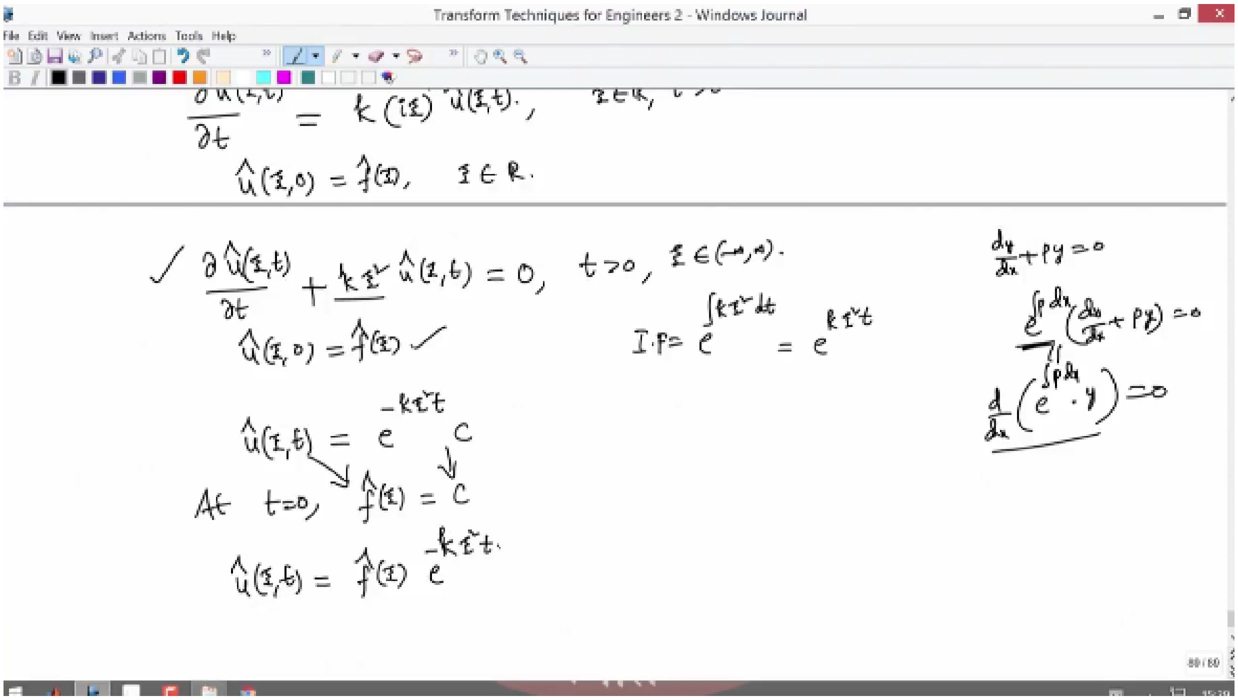
the form which is a nice form which computable form of solution okay, so this is the solution for this semi-infinite plate, sorry upper semi-infinite, a 2 dimensional plate which is at a steady state condition with the initial, with the boundary data is provided, for that boundary data this is the solution, this is how solution everywhere else in the upper half plane.
So let me do this, so far we have done wave equation and Laplace equation that means hyperbolic equation and L type of equation, so let's do parabolic type that is a heat equation we can also, we can also solve heat equation boundary value problem for the heat equation, solution of heat equation, one-dimensional heat equation let us say you consider a full rod, big rod from -infinity infinity, so long rod both sides infinity such a big, such a long rod one-dimensional so that this lateral heat is not insulated laterally, that means only going temperature heat flux is moving only in the one direction, so that is one-dimensional wave equation, so along this this is your X, X is from -infinity infinity, you have this, this is governed by the heat of that temperature in this rod is governed by, for all times is governed by U(x,t) is the temperature that is governed by dou U/dou T which is equal to K kappa, sorry, that is depending on the material constant specific heat constant that is depending on the material of this rod K, and you have a one-dimensional rod so you have only dou square U/dou X square, X is -infinity to infinity and T is positive, so you don't have any boundary here so you have to provide only except, you have to provide it only initial condition that is U(x,0) and that is equal to, that you can provide as some function of X that means initially this rod is at some temperature, what happens at all times, okay, at different times how it diffuses in this, under these conditions how the heat diffuses in this rod initially let us say if it is like this X between -infinity infinity some function you can give, this is given initial condition, initial condition and you also have a boundary data because the boundary is -infinity infinity, if you want to apply again, so we want to apply Fourier transform you can expect at infinity it's natural that naturally can expect at infinity there is no temperature at in the rod, okay, so you can say that U(x,t) goes to 0 as mod X goes to infinity, that is a boundary kind of boundary condition.
So how do we solve this boundary value problem together it's called initial boundary value problem, there is not actually boundary, so you want to apply the Fourier transform you need this which is physically is expected, so that's why're saying this boundary condition, so this is basically initial value problem for the heat equation for an infinite rod, so a solution we apply the Fourier transform if you apply the Fourier transform because domain of X is -infinity infinity you can apply the full Fourier transform, if you apply full Fourier transform, apply Fourier transform on X variable, to get what you see is if you apply on this you get dou U cap of, U(x,t) become U cap(xi,t) dou T, this is equal to kappa times I xi whole square U cap(xi,t),



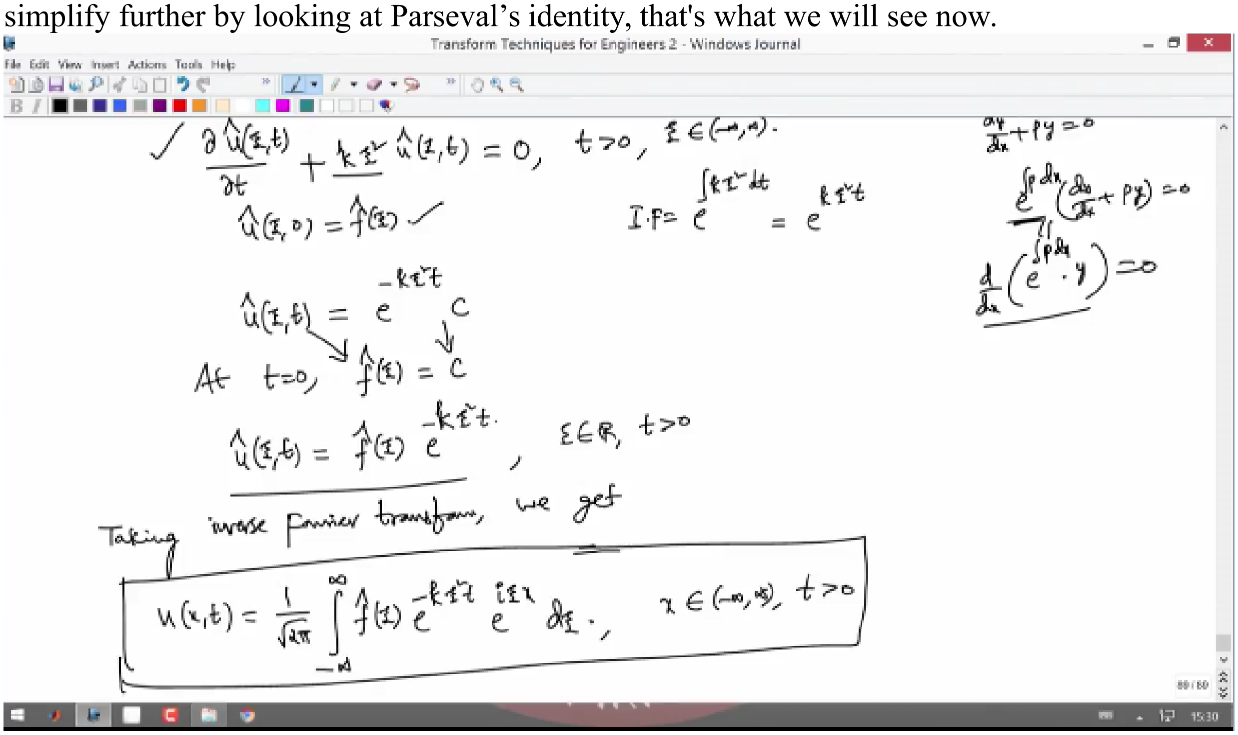
if you apply twice 2 derivatives where I, xi comes out square, so this is what you get and xi is anyway full real line and T is positive, so this is what is the equation becomes.

What happens to the initial condition? Initial condition is U cap(xi,0) is F cap(xi) that is straightforward, xi belongs to full real line and you can here also U cap(xi,t) goes to 0, so boundary means if you apply the boundary so really don't apply, so we don't use this one, this one simply because of this we are able to apply this Fourier transform because it has to go to 0 at both Infinity then only this Fourier transform as an integral makes sense, okay, so otherwise in general we don't need to provide this we are writing this because that is physically expected and also so that we can use the Fourier transform, okay, this boundary condition so this is what we want to solve the equation now becomes U cap(xi,t) dou U/dou T and this becomes minus minus plus this side you have K, kappa xi square U cap(xi,t) = 0 for T positive, and xi belongs to -infinity infinity.

And U cap(xi,0) is F cap(xi), this problem, this is ODE, in single variable ODE so you can solve it, if you solve this first order ODE with respect, so time derivative so you can see that U cap(xi,t) so this is your P, this is like a linear equation DY/DX + PY = 0, so if you apply E power integral PDX, X is the independent variable, so what we have is T is independent variable here, you multiply this both sides DY/DX + PY = 0, so this is actually equal to E power integral PDX times Y for which full derivative, so this is exactly same as this, so if you do apply that, what you end up is getting U cap(xi) = E power, so when you apply this E power integral K xi square, that's a constant so you have an instead of DX you have DT, so simply you have E power K xi square T that is your integrating factor, okay, so this is called integrating factor, so that's what it is, so when it comes the opposite side you have E power -kappa xi square T times C the constant, okay, this is the general solution of this ordinary differential equation, so if you apply this condition U cap(xi,0) = F cap(xi) which is equal to, when you put T = 0 this is what you get C, okay, so this becomes this and this becomes simply this, so that means at T = 0, so that means C you find, so now your solution is F cap(xi) E power – kappa xi square T, so that's all so you have xi belongs to full R and T positive.



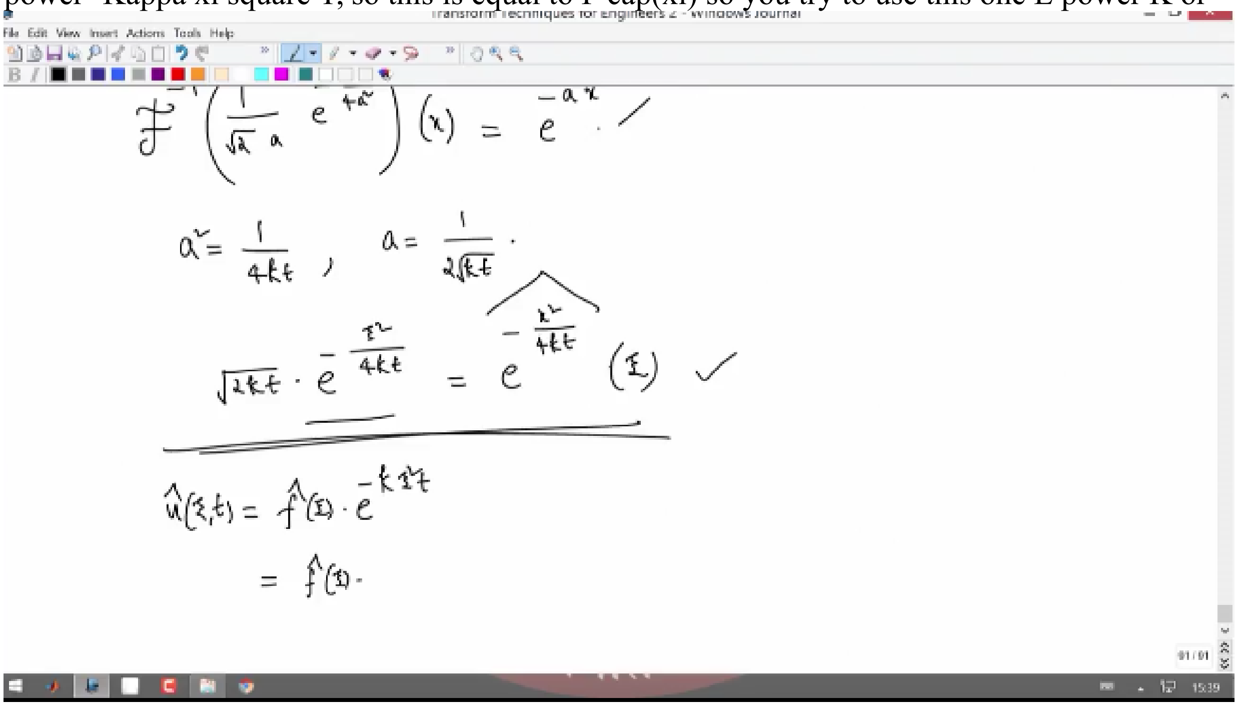
So if you want to find how the temperature behave in this rod for all times, initially at temperature F(x) and you simply take this inverse transform, taking inverse transform, inverse Fourier transform we get, what we get is U(x,t) = 1/root 2 pi integral -infinity infinity, F cap(xi) E power – kappa xi square T, E power I xi X D xi, this is exactly what you have X belongs to full real line and T is positive, so this is actually your solution this is again like earlier we can simplify further by looking at Parseval's identity, that's what we will see now.



So to see this integral to simplify this integral this is actually solution, but that is oscillatory type of things which is not in a computable form, the solution is not in a computable form is an integral, so what you do is we will try to, we have already seen that I think we have seen earlier

in otherwise even otherwise let's work out E power -A square X square, if you take this as a function, if you take the Fourier transform for this what you get is 1/root 2 pi Fourier transform of this function is -infinity infinity E power -A square X square, so into E power – I xi X D xi, no DX, okay that's what you have, so this is equal to let me put it 1/root 2 pi -infinity infinity, so what I do is I try to make it as a square here, so E power -, you have AX, A square X square AX + something square, so what exactly is that? AX, so if you do that two times AX times what you need to have is I xi/2A, if you have suppose you have then end up getting E power I xi X that is exactly what I have, okay, so if I have to replace this something as I xi/2A, then A square X square I have here, 2AB that will be this, so you also end up getting E power -I square, that is minus minus plus xi square/4A square, this is already you have here so you have to compensate with minus, so that is exactly what you have to add here, -xi square/4A square DX, okay, so this is equal to E power –xi square/4A square times 1/root 2 pi and you have integral -infinity infinity, and like earlier you simply, so we have used actually earlier so what you end up is E power -AX + I xi/2A whole square DX.
So like earlier you can construct, this is your X axis, this is your X axis from -infinity infinity you consider this rectangle, so E power AX +, so if you construct this, if you take this as a contour integration, complex integration if you consider this integral is same as this integral from this to this, okay, so that is same as saying if I choose AX + I xi/2A as some T ADX = DT as this is a constant, so you have E power –xi square/4A square 1/root 2 pi this is actually if you do that I have to change this to minus, I have to write I infinity minus, at infinity as an X, at -infinity +I xi/2A to +infinity + I xi/2A times E power -T square DX is DT by A, so A comes out which is a constant, so this is what you have, okay, but then if you use the contour integration like I've seen in the earlier videos you end up saying that this integral over this is same as this one, so this will be you can easily see that this is same as this by contour integration, okay, so this is actually equal to E power – xi square/4A square 1/root 2 pi times A and this value is root pi, it's from calculus okay, make it a double integral or double integral from -infinity infinity, -infinity infinity E power -X square + Y square DX DY, use the polar coordinates you calculate, and then but this is same as I square, if you call this as I this is you can split this is, because X variable, Y variables are independent, you can either separated so you have this is like this one E power -Y square DY, so these are same integrals so you have I square here, so this whole thing is I square is equal to what you end up is pi by polar coordinates if you substitute X square + Y square as R square, so E power -R square, R DR D theta and this is from 0 to infinity, and 0 to 2 pi, if you do this you end up getting this value is pi, that is what you have to see, so by seeing that I value will be root pi, so that's how I calculate it.
So I think I have explained earlier, I think I have already explained in the earlier video this idea, so if you cancel this you end up getting E power –xi square/4A square/root 2 A okay, so the inverse transform of this, inverse transform of this is E power -A square, from this you can write inverse transform of, if you take both sides inverse transform you end up getting 1/root 2 times A E power –xi square/4A square take the inverse transform F inverse transform which is a function of X is nothing but E power -A square X square, this is exactly what you have.
Now what I do is I try to choose, I choose A square as 1/4 kappa T, if I do that A will be 1/2 root kappa T, so put it here in this so that you have F inverse of 1/root 2 times A times, A will be is 1 over 2, 1 over 2 means you have 2 goes up, 1/A is root 2, so 2 times root kappa T times E power –xi square by, A square is 4 kappa T = E power -X square/4 kappa T, this is what you have, okay, so you have what you had is root 2 root 2 goes you can put it here, so 2 under root

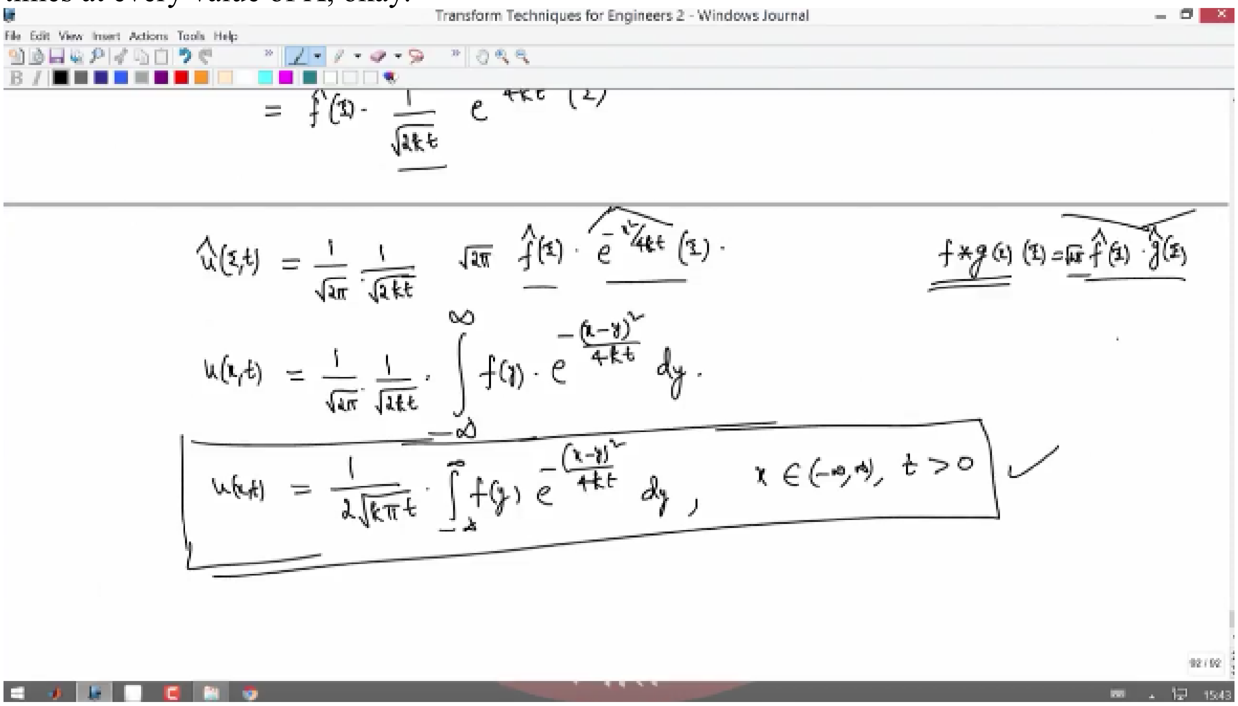
okay, root kappa times exponential of minus this is this, okay, so Fourier transformer if I bring this Fourier transform this side if I apply Fourier transform both sides this goes you end up getting function of xi is this, okay, so this is what we use, this one so if you use what happens to your U cap(xi)? U cap(xi) is this, U cap(xi,t) what you have is U cap(xi,t) = F cap(xi) times E power -Kappa xi square T, so this is equal to F cap(xi) so you try to use this one E power K or



A square, xi square by, so I think I made a mistake here so xi square/4A square, 4A square is, I made a mistake here, so this is okay, this is a root 2, 1/A is 2, so that is root 2K this is okay and this E power –xi square by 1/4 times xi square/4 times, A square is 1/4 kappa T, so 4 4 goes, kappa T goes up xi square kappa T, so this is exactly what we need, okay.

And this is okay, this is fine, so you end up writing this one with this, so times so E power kappa T this I replace with this 1/root 2 KT this is anyway constant, T is constant and this Fourier transform of this one X square/4 kappa T for which you take the Fourier transform, so this by this is this, okay, so if you use now both sides Fourier inversion so this is equal to 1/root 2 pi times, 1/root 2 kappa T so I multiplied this so you have a root 2 pi F cap(xi) times Fourier transform of -X square/4 kappa T which is function of xi, okay, this is exactly your U cap(xi,t) so you take the Fourier inversion on this U(x,t) left hand side and this is going to be root 2 pi times 1/root 2 KT this Fourier inversion of this one you have seen this F convolution of G(x) equal to, if you the Fourier transform of this is equal to Fourier F transform of this with root 2 pi times G cap(xi), that is exactly what I have, F and G this one, root 2 pi you have, so if you take the inverse transform for the whole thing so you have this inverse transform of this is exactly this one, so you try to write, so this into this is the convolution you have to write for this F is F, F(x) or F(y) times convolution E power minus, in the place of this you have X -Y whole square divided by 4 kappa T DY, that is the convolution, okay.

So if you write this as 1/2, 2 comes out root K pi T times integral -infinity infinity, U(x,t) is F of let me use a new variable, okay, you can use Y, Y is not at all there the dummy variable F(y) E power -X-Y whole square/4 KT DY, this is for X belongs to -infinity infinity and T for all T positive, so this is exactly form of solution you're looking for, which is in a computable form there is no oscillatory integral, so this solution you can easily compute for all X and, for all times at every value of X, okay.



So this is how you can solve heat equation, one-dimensional heat equation with the initial data that means you have infinite rod with initially some certain temperature what happens after at later stage, later time so what happens how the temperature distribution in the rod, so that is what exactly you see here as a solution, okay, if you solve by this is solved by Fourier transform, so this is how we solve this boundary value problem which is basically a temperature distribution in an infinite rod, and initially a temperature initial at certain temperature, let us say F(x) okay, we solved purely based on this Fourier transform we solved it, we can also solve other boundary value problems depending on the domain, so if your domain is only semi-infinite domains, for example 0 to infinity X domain is, for example if you consider Laplace equation in the quarter plane or in the upper half plane or semi-infinite rod of temperature distribution in the semi-infinite rod that means special domain is 0 to infinity, you can also apply the order X domain is 0 to infinity so that you can apply Fourier sine or Fourier cosine transform, so we can also see such problems where we apply only, we will see the application of Fourier sine and Fourier cosine transforms.

We will see that in the next video, time permits we will move on to Laplace transform, so we will define what is Laplace transform, will give you the motivation in the next video. Thank you very much.

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