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TRANSFORM TECHNIQUES FOR ENGINEERS

Fourier Series -Examples

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Welcome back. In the last video we have seen a time signal. How it is split into different signals of different frequencies and we represent that time signal has a series, sum of all these linear combination of these signals of different frequencies which are discrete frequencies. This is called Fourier series and we will see how this Fourier series converges to the time signal point-wise under what conditions on f under what conditions of the function f we have this convergence. We'll see before we do that we prove the result. We'll do some more examples. Last we have seen mod x with period minus 1 period 2 mod x that is defined on the closed interval minus 1 to 1. If you repeat it extend it over full real line as a periodic function and we can have a representation for a periodic function f(x) which is mod x as series, as a Fourier series. We have seen. We just found the Fourier coefficients. We calculate the Fourier coefficients and put it as a series. That series we are calling as a Fourier series. We will just start with in this video, we'll have one more, few more examples. The advantage of this Fourier series results is that we can also get some value of some series you can find from the Fourier series. e will see how we start, we will take this example there is a function the time signal is that f of x is x times x times let us say 2L minus x. So this function let us take a function like this and where is it's defined? It's defined everywhere actually. If you see it is just a polynomial it's actually defined on a full real line but we will take it as a periodic function with period 2L so you can say that is defined over 0 to 2L. So if you view this as a function which is periodic over a real line. So it's going to be 0 to 2L and 2L to 4L and similarly 0 to minus 2L and so on you can have a representation. So you have whatever be the function that's going to be defined here. This function f is here. F is over here so it's going to be repeated everywhere here and here everywhere. Okay.

So you can extend like any function, any function defined on a finite interval you can extend it over full real line as a periodic function. So if you consider this so what is Fourier series? So this consists this as a time signal. You can break this into different frequencies and we have defined in the last video what are the Fourier coefficients with period a Fourier coefficient of a time signal f of x with period L. So here we have a function that's with – that is a function with the period 2L. So what are the Fourier coefficients? Fourier coefficients in this case so let's define so they are 2 divided by so it's period is L so here so now we have a period is 2L. So you have 2L. This is from 0 to 2L and this function f(x) cos n Omega naught x, dx. What is Omega naught? Omega naught is 2 Pi by L which is here 2L a period is 2L so which is Pi by L. So you have 1 by L 0 to 2L f(x) cos n Pi x by L dx. this is my area which is running from n is from 0, 1, 2, 3, and so on. So that has a0 as well. So similarly other coefficient bn which is 1 by L 0 to 2 L f(x) instead of cosine now you have sine, sine n Pi x by L dx and is running from so 1, 2, 3, onwards.

So we will see we will just calculate these functions these Fourier coefficients for the function x into 2L minus x. So what we start with an you can have a0 first a0 is 1 by L integral 0 to 2L f(x) is x 2L minus x n equal to 0 cosine value is 1 so you have dx this is going to be x square L by 2, so that's 2-2 goes x square minus x cube by 3. That's a antiderivative and you have 1 by L and if you substitute these limits what you get is well L-L goes x square so 4L square and then this 1 minus L square actually 8L cube that is and if it goes with 1L so you have 8L square by 3. So it is equal to 4 times 1 minus 3, what is this value? So 4L square comes out you have 1 minus 2 by 3. This is equal to 4 by 3L square.

So a naught we found as this. Now we can calculate any an and it's running from 1 to 1 onwards. So 1 by L 0 to 2L and f(x) cos n Pi x by Ldx and it's running from 1, 2, 3, onwards. We do integration by parts here so 1 by L you have sine n Pi x by L divided by n Pi by L this and then for which you have 0 to 2L limits minus L by n Pi integral 0 to 2L sine n Pi x by L and then you have to differentiate this f(x) which is x into 2L that is 2L minus 2x so you have 2L minus 2x is the derivative of f which is dx. So if you calculate L-L goes here and also goes here. So what you get is sine function when you substitute x by 2 L L-L goes 2 n Pi sine n Pi 0 and sine 0 is 0. So this is completely 0. So you have again if you do one more time integration by parts 1 divided by n Pi. Now L by n square Pi square into, now you have cos n Pi x by L for which you have 0 to 2L minus minus plus, and you have cosine is minus so here it's minus here so minus cos n Pi x by L divided by n Pi by L so L divided by n Pi comes out this is already plus here, and this is going to be minus minus plus and again one more minus comes here. So you have minus L by n square Pi square integral 0 to 2L cos n Pi x by L of course I think I missed f(x) here so f(x) here and you have here f dash x which is L into 2L minus 2x. So for which you have this limits. Okay here if you differentiate this one this is your f(x) now so you get a minus 2. So it's going to be plus 2L dx. So here if you can see here 2L minus x L minus 2L minus 2L is going to be 0 of cosine and also what do you have and you put x equal to 0 so you get 2L square divided by n square Pi square. Cos 0 is 1. okay. So here 2L minus 2x. That is not 0 so 2L minus 2x 2L L minus 2L this is going to be minus L. so minus L square divided by n square Pi square cos 2n Pi that's 1 minus cos 0 is 1 so here 2L square divided by n square Pi square.

So together it's - 4L square by n square Pi square. That's you have this term and here you will get the 2L by n square Pi square we have antiderivative is sine n Pi x by L divided by L by n Pi. So divided by n Pi by L that is L by n Pi for which you put the limits 2Pi 2L so this is the final result for an and so this is equal to minus 4L square by n square Pi square plus 4L square by n square Pi square. So here n cube by cube and sine L-L goes 2n Pi so that is 0 so actually this is 0, so nothing, no contribution here. So this is exactly what you get for ans for ans is nonzero here. Similarly you can find bns. So bns are so instead of cosine you have sine here. That's the only replacement. So let's so work out what is this bn. Bn is 1 by L 0 to 2L f(x) times sine n Pi x by L dx. N it's only from 1, 2, 3, onwards.

So if we do the integration by parts twice you have 1 by L f(x) minus of this cos n Pi x by L divided by n Pi by L this L-L goes minus minus plus integral 0 to 2L and you have 1 by n Pi L by n Pi that L-L goes again with outside L and this is sine n Pi x by L into f dash that is 2L minus 2x with sine. It's what it is. So this is equal to now here when you put a 2L cos function will give you 1, cos 2L cos this function cos function will give you 2L that is 1. So you have minus x into so basically what you have what is f 2L x minus x square. So 2L x minus x square so 2L x so you can write here 2L x minus x square is my f(x). So f(x) 2L is actually equal to 4L square minus 4L square which is 0. So f at 0 2L 0 and f at 0 is what if you put 0 there is also 0. So you have this is going to be zero. So finally what you end up is 1 by n Pi again for this integral if you do the integration by parts 2L minus 2 x so this is cos right. So it is cos. So you have a sine n Pi x by L by n Pi by L into for which you have 0 to 2L and you have minus L by n Pi integral 0 to 2L sine n Pi x by L into differentiate that is 2L by 2 x that will give you minus 2 so it's going to be plus 2 dx.

So together again now sine sine will give you a 2L and 0 at 0 so this will be 0 and you have here so 2L by n square Pi square this will be minus cos n Pi x by L into L by n Pi for which if you apply limits what you get is 2L by n square Pi square 2L cos n Pi x by L will be 1 minus L by n Pi take it out so you get minus L square by n cube by cube. And we are left with only for cosine. Cosine will be minus. That is also 0 okay 1 minus 1 which is 0. So bns are 0 you got only ans and a naught nonzero. 4 by 3L square a naught and an. So [Indiscernible] [0:15:51] if we assume the Fourier series if you assume result that in any time signal sufficiently good time signal if we can split it into time signal of different discrete frequencies you can take a linear combination of all these and then you can say that this is actually equal to your time signal. So we will see that is a Fourier series of this function f(x) is so we can write this so which is 2L x 2L minus 2 x 2L minus x into x so this is your same signal which is a naught by 2. So what is the value of a naught which we found 4 by 3L square so 2 by 3L square is a naught by 2, 2 by 3L square that is a naught by 2 plus b naught is 0 and you have a Sigma n is from 1 to infinity an is minus 4L square by n square Pi square. Cosine cos n what is your Omega naught fundamental frequency which is Pi by L. So Pi by L n Pi by L x Omega x. Okay. So this is exactly your Fourier series.

so minus you take it and out it outside. so this is your Fourier series. So you actually assume that this Fourier series converges to the signal so that's why we equate it here. So this is still actually question mark unless we prove that it actually converges point-wise. You fix your x this series converges to this function to 2L x minus x – 2L x minus x into x if this converges see so we're only assuming as of now if you calculate like this you will get your Fourier series or actually converges to the sum function f when this function is differentiable smooth piecewise differentiable function you can easily say that okay that will actually prove the result.

So we're just calculating this right side and we say that this is actually still in the question mark actually true and see once you assume that is the Fourier series so you have this is a Fourier series of this function now what can we say. So what is this Fourier series so this is equal to you can write 2 by 3L square minus 4L square by Pi square if you take it out and what you get is a series n is from 1 to infinity 1 by n square. So cosine cos n Pi by L x divided by n Pi x by L divided by n square. So this is your – this is what it is 2L minus x into x. So what do you get? So if I put x equal to 0 so where is is x? x is between 0 to 2L. So we will put x equal to 0. So putting x equal to 0 will give me left hand side is 0 so you'll get 2 by 3L square which is equal to minus 4 by 4L square by Pi square into sum n is from 1 to infinity cosine function at 0 is 1, so 1 by n square cosine cos 0. So it is this. So this will give me Sigma 1 by n square a well-known result 1 to infinity which is equal to 2 by 3L square into Pi square by 4. So 2-2 to go so what you get is Pi square by 6. So you get this Pi square by 6 this is a well-known result. Okay so Fourier series once you calculate the Fourier series you try to put the – fix your x value that will give you some series if the series so happens to be a well known series you will know the sum of its sum of that number series that is that series value you will get it as from the Fourier series. It's not the other way so you try you give your own number series and it's difficult to find a suitable function f for which the Fourier series will give you once you fix the x value the series, the number series you have given. Okay. So that's not the other way. So you do the or Fourier series so that will give you, that will generate once you fix x value it will give you some numbers it is that number series value sum of the number series you know the value.

So here so this is the well-known result and also you can also get one more here. So by putting x equal to L if you put x equal to L what happens? So you get left hand side L square which is equal to 2 by 3L square minus 4L square by Pi square and you have n is from 1 to infinity cosine of n Pi put x equal to L cosine of n Pi which is -1 power n divided by n square. So put n equal to 1 cos Pi that is cos Pi is minus 1 and so on yeah. So if you remove this one so you have a minus 1 and you put 1 here. So you will get just leave it here. So this will give me L square L square cancels and you get this series, the value of this series n is from 1 to infinity minus 1 power n by n square is equal to 2 by 3 times Pi square by 4. This is correct. Cos n Pi yeah so this is also a value same. So you get Pi square by 6 minus 1 power n by n square which is equal to Pi square by 4 that's correct. So you get Pi square by 6 something is wrong. L square oh yeah this is a mistake. So this mistake is here. So this is equal to minus of this Okay so let me do it carefully.

So 4 by Pi square minus 4 by Pi square equal to 1 minus 2 by 3 this is 3, 1 by 3 so this will give me a Sigma minus Sigma n is from 1 to infinity minus 1 power n by n square which is equal to Pi square by 3 into 4 okay. So this is equal to Pi square by 12. Okay. So this is well-known result when you put x equal to zero and you have this result now. So this will give it is n equal to 1 this is minus 1 by 1 square so it's going to be 1 so it's going to be 1-1 minus minus plus so 1 minus or divided by 1 by 1 square plus minus 1 by 2 square plus 1 by 3 square minus 1 by 4 square and so on. Those value is actually Pi square by 12. So this is the result you get. This is also well known. Okay. So this is how you can get results. You can get the sum of the number series from the Fourier series. We will do example one more example. We will try to calculate the Fourier coefficients and the Fourier series for a given function and then out of that we will try to get the sum of the series. So here we will choose the another example which is a function is something like this. Some constant k and a function x. k is actually between minus 1 to 0 and then here this is between 0 x less than 1. So if I have like this this is a piecewise function you can see from minus 1 to 1, minus 1 to 0 this is a constant function k. That is 1. If k is 1 is like this and if it is between 0 to 1 it's going to be like this.

So you can always extend this function over a full real line like this and go on like this. You can go on. So you can extend both sides like that as a function of function over a full real line but it's with the period. So this is same thing, pattern is repeated over interval of length 2 so the period is 2. So this is a piecewise periodic function f(x) is a piecewise function. Piecewise continuous function with period 2. Okay. So let's see the coefficients, Fourier coefficients which is what are the Fourier coefficients in this case? Period is 2 so you have 2 by L, L is the period so an, which is a 2 by L is 2, L equal to 2, so integral this is from minus 1 to 1 because this is from minus 1 to 0, minus 1 to 1 the function is defined. f(x) this function cos n Omega naught that is the fundamental frequency is 2 Pi by L, L is here 2 so 5 Pi. So cos n Pi x, dx. So this is your ans. Bns are similarly so 2 by 2 is 1. So bns are similarly you can define, f(x) instead of cosine you have sine there, n Pi x, dx. So this is running from n is from 1, 2, 3, onwards so n is also 0 here. So both if you define at n is equal to 0, 1, 2, 3, and b0 anyway 0.

So let's calculate what is this an. So an equal to so if you calculate so you have minus 1 to 1, k, k is a constant and you have cos n Pi x, dx minus 1 to 0, minus 1 to 0 is this value plus 0 to 1 and you have x cos n Pi x, dx. That's my an and it's running from 0, 1, 2 ,3, onwards. If it is 0 so an equal to 0 if you put so let's start with a0, a0 is cos 0 put na equal to 0 cos 0 is 1 so it's 1 integral 0 so k times 0 minus 1 so it's 1, k1, k plus and here is simply when you put n equal to 0 this is x, x square by 2 so it's ½. So this is your a naught. So an, n is running from 1, 2 onwards that we calculate here . So if you do this sine function n Pi x divided by n Pi that is antiderivative. K is outside so it's between minus 1 to 0 so this is anyway 0 for n is equal to 1, 2, 3, onwards because 1 by n we are dividing we have to make sure that is nonzero.

So this is 0 anyway. So this contribution is 0 minus that's gone so the other integral plus now we do the integration by parts here. So again so sine n Pi x into sine n Pi x divided by n Pi between 0 to 1. So again contribution is 0 minus now you have this integral 1 by n Pi integral 0 to 1 sine n Pi x the derivative of x is 1 so that is what it is. So you have minus 1 by n Pi now this is minus, so anti derivative is minus cos n Pi so n square Pi square minus cos n Pi x by n Pi so that n and Pi this outside and Pi will become plus, this minus minus plus so you have cos n Pi x this is between 0 to 1.

So you see that this is cos n Pi minus cos 0 so what you get is 1 by n square Pi square minus 1 power n that is cos n Pi minus 1. So this is what you have n is running from 1, 2, 3 onwards. So this is my an. Similarly you can calculate your bn which is k times minus 1 to 0 instead of cosines we will take the sine. And then plus 0 to 1 x times sine n Pi x, dx. n is from 1, 2, 3 onwards. So this is equal to k times minus cos n Pi x by n Pi that is the antiderivative for which you apply minus 1 0 and you have a minus minus plus. Okay. That is a simple integral so that is the derivative that is the first integral. Here you will do the integration by parts for this integral second part, second integral. So you have again minus x minus x cos n Pi x by n Pi for which you apply minus 1 to 0 limits and this minus minus plus you have this integral 0 to 1 cos n Pi x by n Pi and the derivative of x is 1 so you have this. So this will be cos 0 is minus 1 k by n Pi you get cos 0 is minus 1 and you have cos minus minus plus cos minus n Pi that is cos n Pi that is -1 power n. This is what you get, plus and here again minus x so x is 0 that is 0 and a cos and Pi so what you get is minus minus plus 1 by n Pi and evaluate cos n Pi minus 1 so that is cos n Pi that is going to be -1 power n, okay, plus this is sine n Pi x by n square Pi square between 0 to 1. So this is because of sine sine n Pi minus sine 0 so this is anyway 0, contribution is 0. So you get finally what you get is minus k by n Pi plus -1 power n by n Pi into k here plus 1. This is what is your bn.

So an is this and our bn is this and with a naught being k plus 1/2. So what is your Fourier series? Function f which is piecewise function which is K between minus 1 to 0 and x between 0 to 1. So this function write now as a Fourier series. k plus 1/2 a naught by 2 is k plus 1/2 k plus 1/2 is 2k plus 1 by 2, that is a naught into 2 divided by 2 so you have 4. Okay plus Sigma n is from 1 to infinity ans now I use this ans 1 by n square Pi square -1 power n minus 1 that is what is my an into cos n Pi x fundamental signals, cos n Pi x plus. Now bn. Bn is this. Bns are -1 power n k plus in k plus 1 by n Pi minus k by n Pi. This is my bn into sine n Pi x. So this is my Fourier series. This is the Fourier series for a piecewise continuous function f(x) which is like this which looks like this if you draw which you can break. This hf signal if you have a signal like this if it repeats like this you can break this into frequencies. Fundamental signals of frequency, discrete frequency 1, 2, 3, and so on. Okay. As you see frequency, fundamental frequency is here simply w Pi. So the fundamental time period is fundamental frequency is basically this is a fundamental frequency. It is just Pi, okay. So Pi ns, you will get Pi ns so n Pi are fundamental frequencies, different frequencies so you have a discrete signal signal. Okay.

So this also if once you fix x equal to 0 for example here okay. So you can fix x equal to 0 or some other value or maybe x equal to half. Okay. So I just want to give you some remark here. So this function is defined over minus 1 to 1, of course here. So you can include the endpoints minus 1 and plus 1. Okay. So here and here so this is a closed interval and at zero this function is actually discontinuous okay. So one is a k value other one is actually zero. It starts at zero. Okay, if you take the limit as x goes to zero from the left side it is actually zero here, if you take the limit this side as x goes to zero function value is actually a k. If K is zero it's continuous function. If k is zero what you get is this is the function, continuous function. Of course, it's not differentiable. And it is only continuous function. If k is nonzero you will have piecewise continuous function. Continuous function up to here and up to here a continuous function x. So it's not exactly defined. So we will see exactly where this point at this value what is the Fourier? Fourier series is this okay. So we can say as of now x is between minus 1/2. So I have not said this is the Fourier series at zero okay. So we are only just as of now we are only believing the Fourier series by constructing – by construction from the Fourier coefficients this whatever you get that series that converges to the function f(x) wherever it is continuous as of now where the discontinuous points is actually converges to the value of function, average value of the function from both sides. So if you actually see this function once you fix x equal to zero we will prove this as a theorem and there may be in the next video and you put x equal to 0 what is the value of f(x) at left side f0 minus that is actually a k, f at 0 plus which is anyway x so that is 0. So it's average value f of 0 plus plus f of 0 minus divided by 2. So this is k by 2. So this is the value you will – this Fourier series converges when you put x equal to 0. Okay. The Fourier series constructed by the Fourier coefficients it converges to the function whenever it is continuous at a point of continuity it actually converges to the function f(x) at their piecewise discontinuities, jump discontinuities at which average value from both sides you limit exist so the average value it converges. This is what we will see it will prove it in the next video.

Thank you very much.

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