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Transform Techniques for Engineers
Fourier Integral Theorem – Proof
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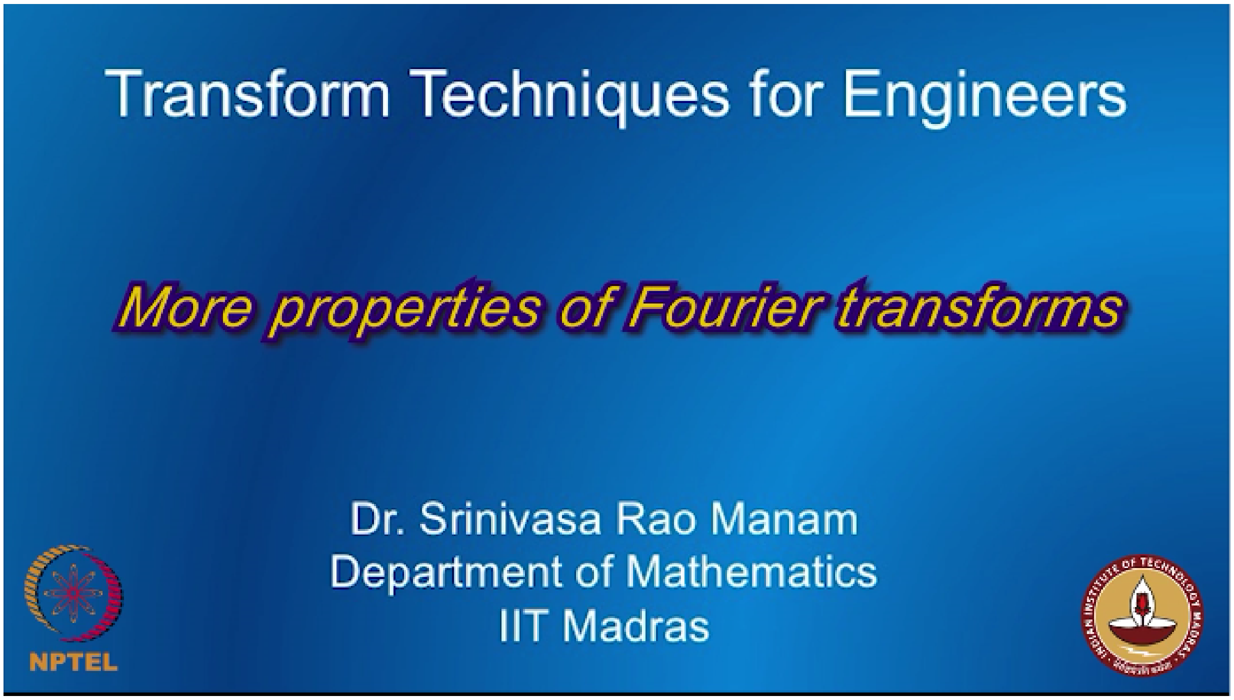
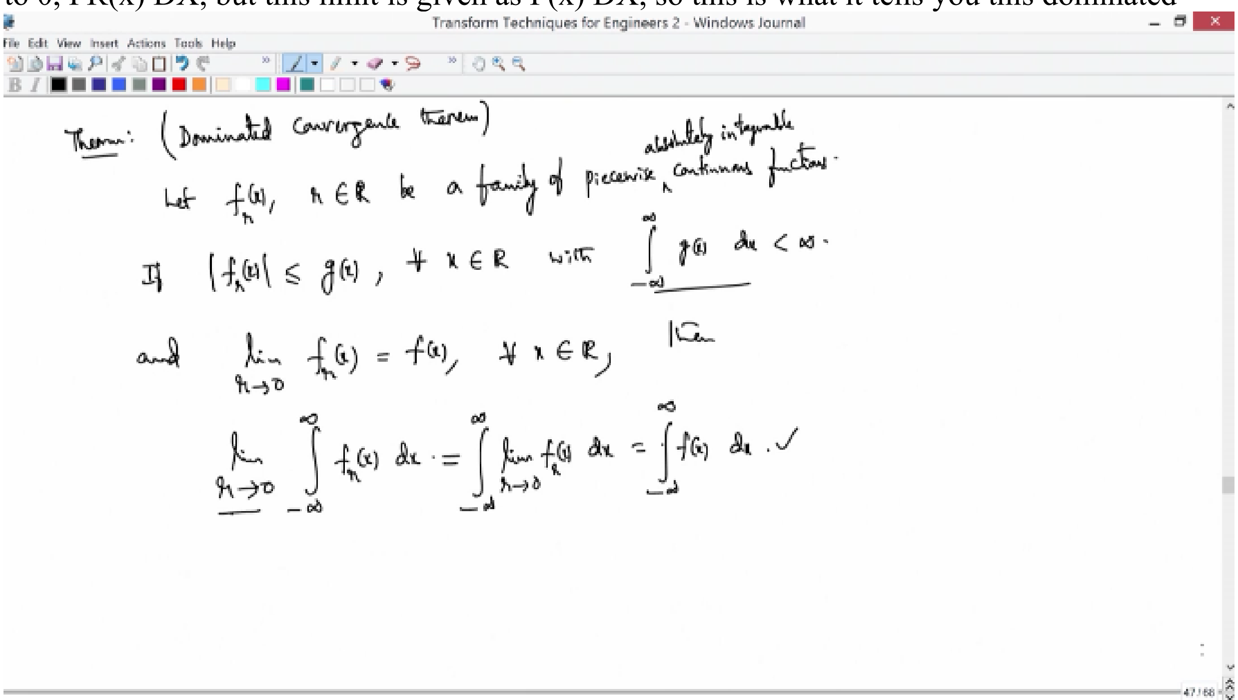


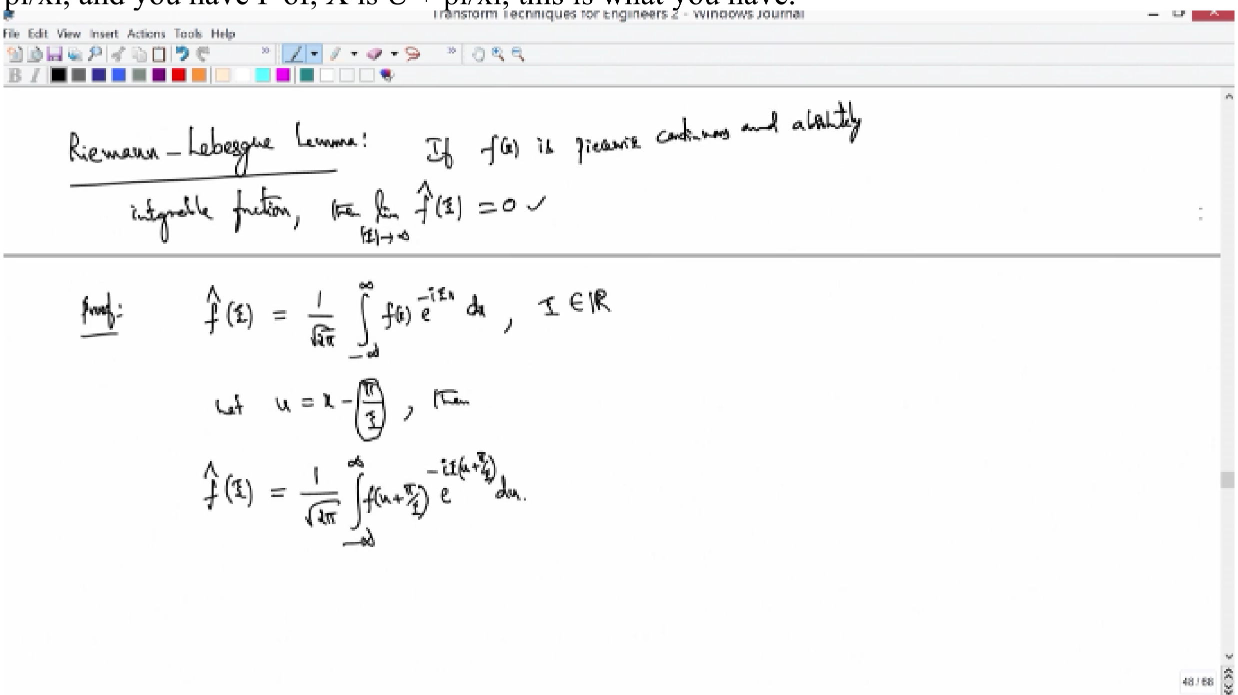
Welcome back, in the last video we have seen Riemann Lebesgue Lemma, the property of a Fourier transform the Riemann Lebesgue Lemma for periodic signal, signal over finite time period.
So what we are looking for the Riemann Lebesgue Lemma for a non-periodic signal that is the signal over a infinite time, so that is our full real line F(x), X belongs to the full real line, you can also prove this by dominated convergence sum theorem so which is a another version of Weierstrass M-test which if you know, I think you know from calculus Weierstrass M-test sequence of functions converges to something uniformly, so you have, you try to look for this Weierstrass M-test this is a similar version, so I will try to give you that theorem without proof and then we make use of that theorem and we'll prove this Riemann Lebesgue Lemma that is more convincing you, okay, proof will be more convincing if I try to use this theorem rather than giving earlier proof that is actually evaluate but if you don't follow this is much easier, so I'll try to give you this theorem without proof or dominated convergence theorem or it tells you is, I choose some functions, okay with parameter R, for every R you have some sequence of functions be a family of functions, okay, be a family of functions, family of how do what kind of functions we choose a piecewise continuous functions, okay, and also absolutely integrable functions, piecewise absolutely integrable, absolutely integrable continuous functions that is exactly, what are those functions in your Fourier transform? What we do is now if I have this condition FR(x) this quantity is less than or equal to some function, this is dominated by this function, all these functions for every R is dominated by this function G(x) that belongs for

every X with the real number, and what you have is, G is also absolutely integrable function okay.
So DX is positive it's finite, okay, with okay, with this and what we have is this limit FR(x), as R goes to 0 this is actually F(x) point-wise convergence for every X belongs to the real. Suppose you have this limit then what you have is a limit H, limit and let me choose, so this limit exists R goes to 0, okay.
So now I try to integrate FR(x) DX, okay, so I think I should choose this G(x), it's dominated by this which is anyway positive, so modulus of G(x) doesn't matter, though G(x) means modulus or without modulus, because G is always positive this means this G is absolutely integrable function.
So in this case so once you have this limit, the conclusion is this -infinity to infinity, these are absolutely integrable functions, okay, if this is actually equal to, what this theorem tells you is you can allow this limit to go inside this integral, so if I do this, this is what is the case, R goes to 0, FR(x) DX, but this limit is given as F(x) DX, so this is what it tells you this dominated



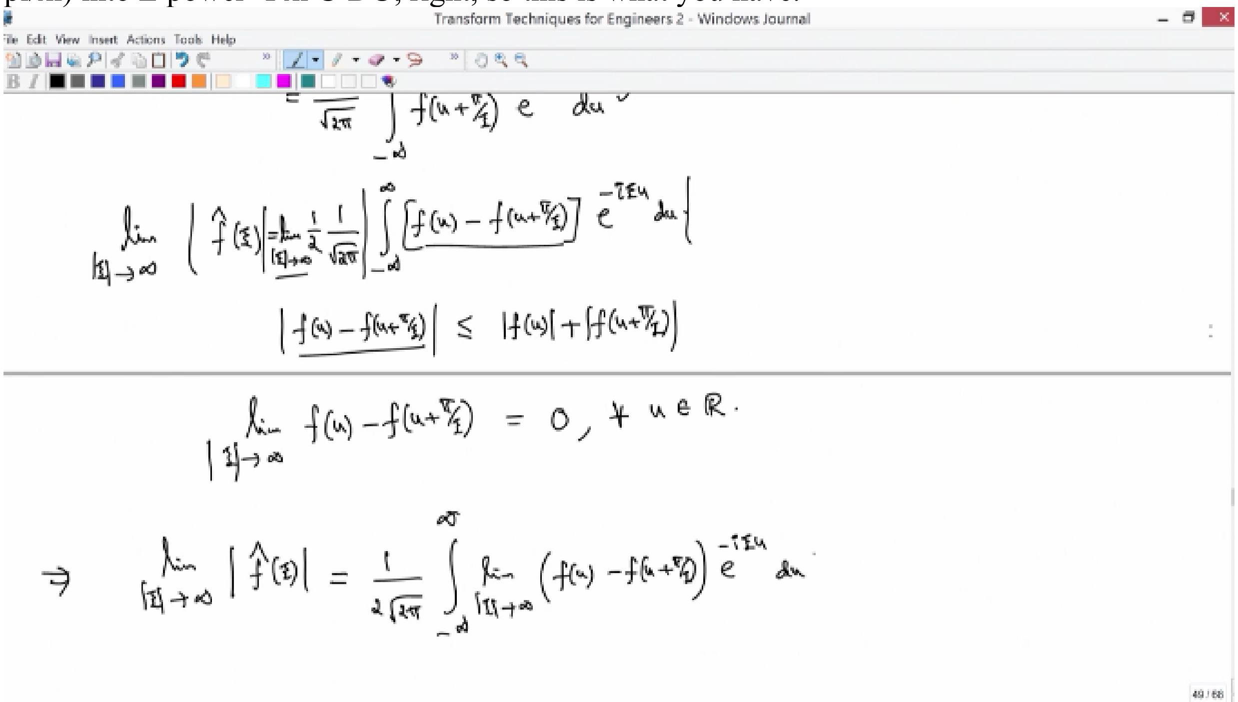
converges, so whenever you have, you want to have a limit you have a limit outside some integral you want to pass this limit inside the integral what you have to do is you just need to check whether your sequence of functions, those sequence of functions whether they are dominated by some function which is absolutely integrable, if such a case you can just pass this limit inside okay, we will laugh and use, is very useful result so you try to, if you don't understand this, if you have not seen this theorem you try to understand this statement you can use very often and throughout, okay, throughout the course, and many other courses is useful so try to use this, try to understand this statement and follow this, okay, use this whenever it's required.
So now we'll try to use Riemann Lebesgue Lemma again, another proof using this theorem we'll try to prove this Riemann Lebesgue Lemma, what we have proved in the last video, so if F is F(x) is absolutely piecewise, absolutely piecewise continuous and absolutely integrable functions, then the F cap(xi) Fourier transform mod xi goes to infinity has to be 0, so this is we will prove this way, very easy way, so this proof will be more convincing for you, so let me

choose start with this F cap(xi) which is 1/root 2 pi and you have -infinity to infinity, F(x) E power -I xi X DX, so what I use right now is a small trick let use, let me use U = X – pi/xi, let me choose such a variable then what happens F cap(xi) I use this change of variable into this integral, so left hand side will not change so I have this 1/root 2 pi for every fixed xi, xi a real number so once you have this you see X, and when you put -infinity this is a fixed number, so you have -infinity to infinity, DX is DU now, okay and you have E power minus I xi, X is U+ pi/xi, and you have F of, X is U + pi/xi, this is what you have.



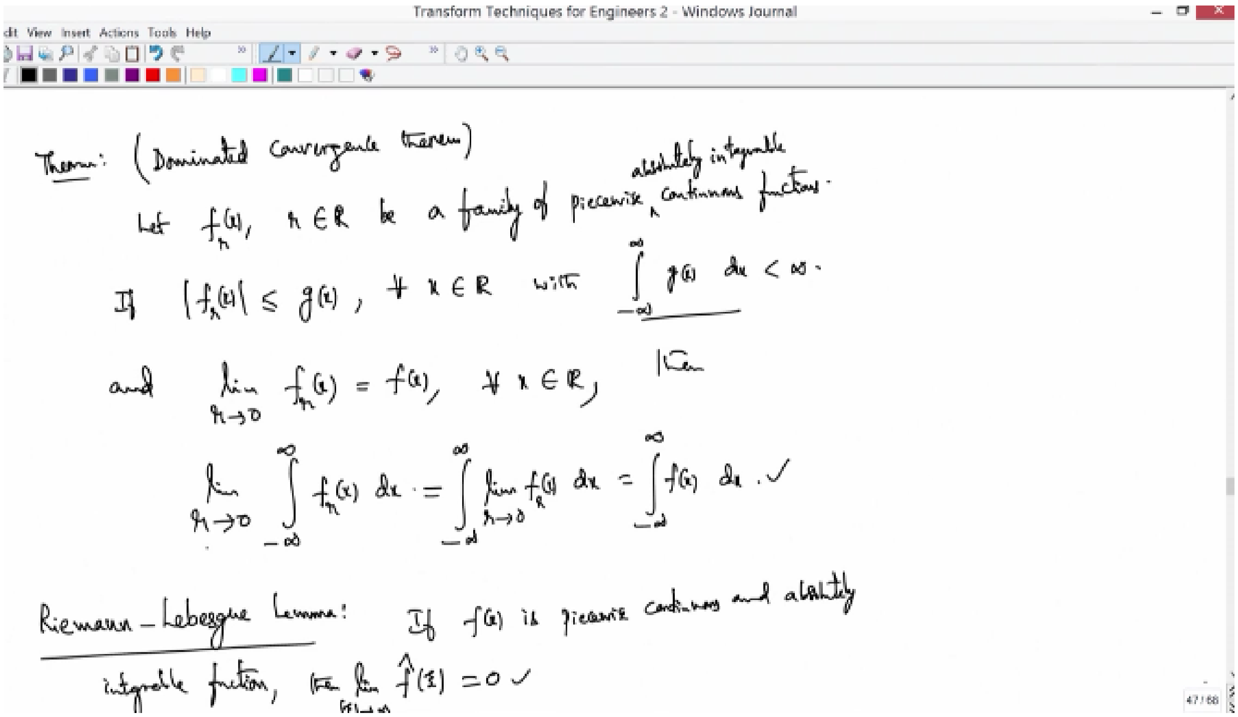
So what is this one? This is actually equal to 1/root 2 pi, so E power -I pi, so what is this one? So integral -infinity to infinity, F(u+pi/xi) this is E power -I xi U, and what you have is E power -I xi into pi/xi, that is E power -I pi that is cos pi is -1, so you have a -1, so -1 comes out, so you try to take the average of these two, some of this and take half of that so you have F cap(xi) that is the trick you have, so you have 1/root 2 pi -infinity to infinity, and you have 1/2, 1/2 this quantity plus this quantity divided by 2, so and if you do that F(u) - and you are adding because already – sign, so you have F(u + pi/xi) okay, and you have this, this is same E power -I xi U DU, this is exactly what you have.

Now you try to take the modulus for this, you have a modulus and this is less than equal to 1 by, so now for this you try to apply both sides limit mod xi goes to infinity, if I apply this mod xi goes to infinity, I want to pass this limit inside this integral for that I have to choose for different xi values, xi is the parametric like earlier in your theorem, R is a parameter, here xi is the parameter, so if you look at this mod F(u) – F(u + pi/xi) this quantity first of all, F is absolutely integrable function, so this quantity is also absolutely integrable function, and this is dominated with, I choose this mod F(u) + mod F(u + pi/xi) this simple inequality, so this quantity is absolutely integrable function implies this quantity itself is absolutely integrable, and now you have a function which is sum of this absolute values that is also new function, this is your G, this is also absolutely integrable function, and what you have is a limit F(u) – F(u + pi/xi) as xi goes to infinity, both sides mod xi goes to infinity this quantity is actually going to 0 for every U belongs to real line, it’s clear right, so you can pass this limit, you can pass this limit inside this integral, so you have that implies what you get is mod xi goes to infinity, F cap(xi) is equal to 1/2 times square root of 2 pi you can take -infinity to infinity, F(u) -, so I can this is the limit, limit mod xi goes to infinity, I put this goes inside and you have F(u) – F(u+ pi/xi) into E power -I xi U DU, right, so this is what you have.



But this goes to, this quantity goes to 0, okay, so once you have the modulus you take this modulus inside okay, and now you have the modulus here, so take this modulus inside what you get is 2 root 2 pi -infinity to infinity, limit mod xi goes to infinity the modulus I'm putting inside of F(u) – F(u + pi/xi) and then this into this modulus, this modulus thing inside E power –I xi U DU, so because of this modulus on into this you can put this as 2 modulus and this modulus is 1, so you have 1 modulus here, and this quantity is actually 0 whatever it so, this is less than or equal to 0 that means positive quantity less than or equal to 0 means it has to be equal to 0 okay, so that is exactly what you want to show.

Limit mod xi goes to infinity, Fourier transform at extremes, both the extremes is going to 0 that is the meaning of this, okay, this is a Riemann Lebesgue Lemma using this important theorem in calculus that is dominated convergence theorem, so if you have a sequence of functions and you want to, you know its limit it converges to some function and if you want to integrate these functions and pass the limit and you can do this provided this on sufficient



condition is this denominated convergence theorem that tells you that these function, sequence of functions are dominated by certain thing which is whose integral value is finite such a thing it is there then you have this you can pass this limit inside this integral.

So using that you prove this Riemann Lebesgue Lemma, we will try to prove one more small property, before we'll try to prove this I think it's a, this is the useful thing, so let me do this one, so let if F(x) is absolutely integrable function and -infinity to infinity, and piecewise and piecewise function, and piecewise continuous function and this, okay, in this in full -infinity to infinity, then F cap(xi) is a continuous function, is a continuous function again this property also you can prove this Fourier transform is a continuous function always as a function of xi, xi belongs to the full real line, so what do you do to prove this thing we have to prove F cap(xi + some H) you take -F cap(xi), now if you allow this H goes to 0 both sides, this you want to show that this is equal to 0, okay, so by definition this is actually equal to H goes to 0 this is the proof of this property, okay, so H goes to 0, F cap is 1/root 2 pi and this will be -infinity to infinity, F(x) which is common in both the things and what you get is E power –I xi + H here, X- E power minus I xi X DX, this is what you have, this is equal to limit H goes to 0, 1/2 pi this E power -I HX you can take it out, okay, E power -F(x) into E power -I xi X that is common, what you have is E power –I HX -1 into DX.

Now we can pass this limit because this is the limit H goes to 0 you have this function, so clearly E power -I HX -1 limit H goes to 0 is equal to 0, and these are dominated by, so what

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12. If $f(x)$ is absolutely integrable function and piecewise continuous function in $(-\infty, \infty)$, then $\hat{f}(\xi)$ is a continuous function. ✓

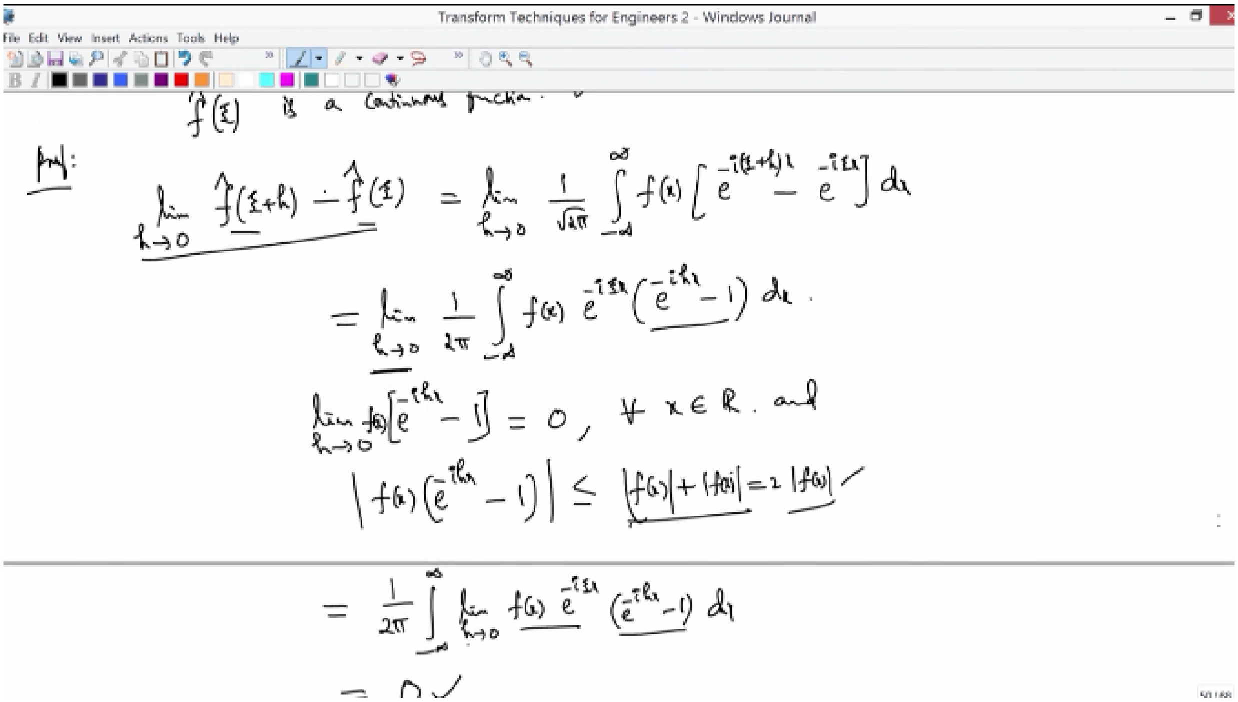
Proof:

$$\lim_{h \to 0} \hat{f}(\xi+h) - \hat{f}(\xi) = \lim_{h \to 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[e^{-i(\xi+h)x} - e^{-i\xi x} \right] dx$$

$$= \lim_{h \to 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{-i\xi x} \left(e^{-ihx} - 1 \right) dx .$$

$$\lim_{h \to 0} e^{-ihx} - 1 = 0$$

we do is if you choose like this you have to show that this is dominated by some function which is maybe modulus of this placement, if you choose 2 which is not integrable what you do is you try to choose F(x), any F(x) this is what is the case, this is true for every X belongs to R, then and what you have is now F(x) times E power -I HX -1 this is dominated by F(x) +F(x) okay, modulus of this, this is because F is absolutely integrable function so you have integral of this is also, this is a 2 times F which is 2 times mod F(x) which is absolutely integrable function, so you have dominated by this, now I can pass this limit inside so what you get is a limit H goes to 0, so that is 1/2 pi -infinity to infinity, if you pass this limit, limit H goes to 0, F(x) E power –I xi X, E power -I HX -1 DX, this is nothing to do with H, this limit you pass it here that is 0, okay, so that means what you approve is this limit equal to 0, that means F

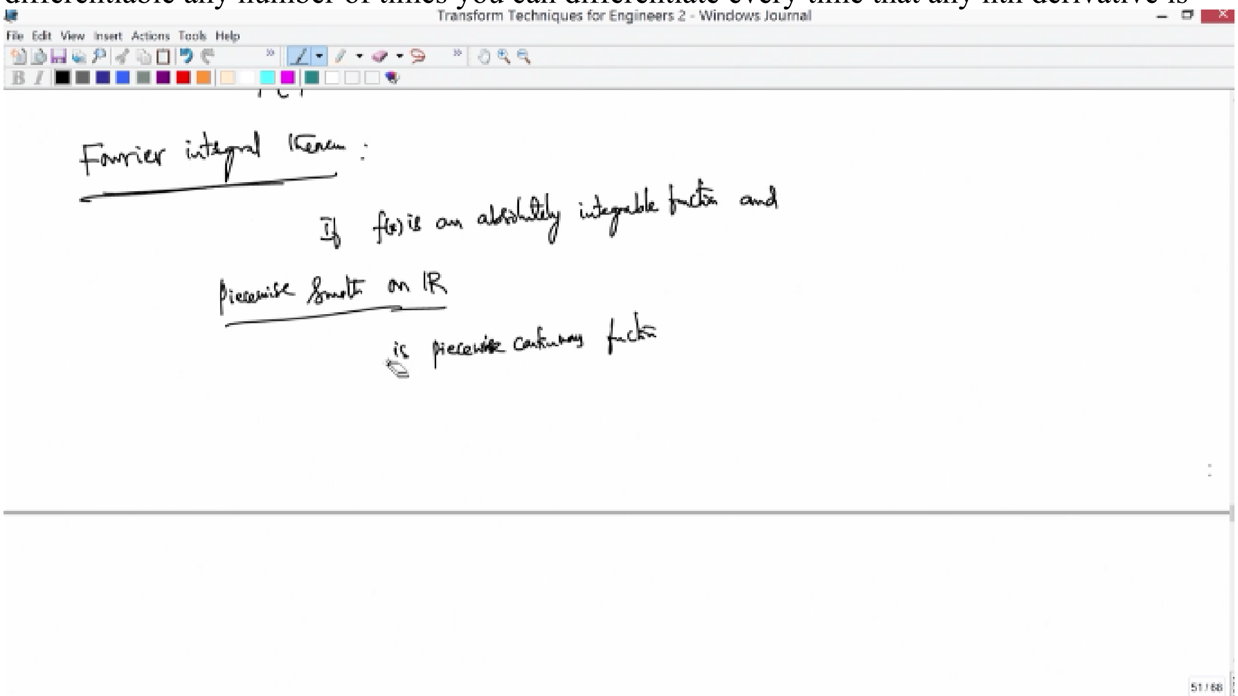


cap(xi) is continuous function implies F cap(xi) is a continuous function in -infinity to infinity, we may use this some places so it's always better to know this good result, so if F is F cap, Fourier transform is always continuous function, F need not be a continuous function but it is a, but this Fourier transform its smooth and so basically it makes it continuous transformation, Fourier transformation makes it a continuous function, non-continuous function to a continuous function.

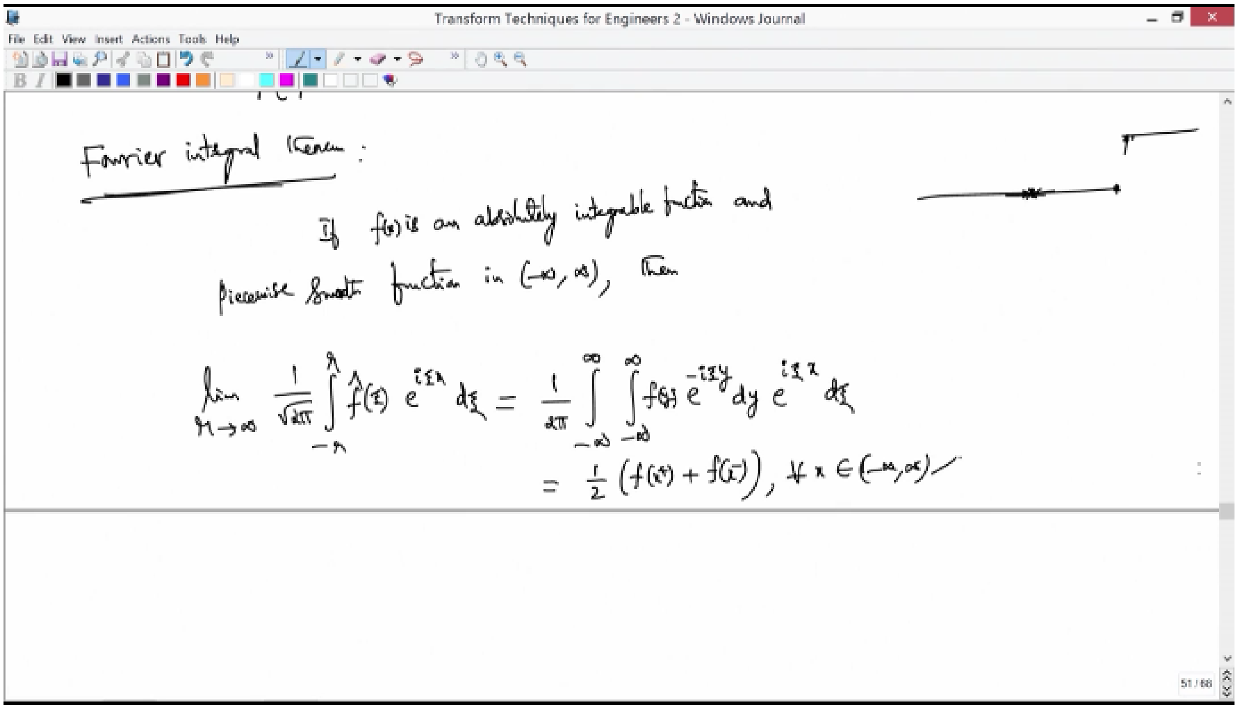
Now I will try to prove this basic result that is Fourier integral theorem which we will prove rigorously with this once, once we prove this result Fourier integral theorem whatever we have defined as a Fourier transform and its inverse transform are legitimate.

Earlier we have proved this Fourier integral theorem in an intuitive way that is to explain, that tells you that Fourier series that is a Fourier transform of a periodic signal is connected with the Fourier transform of non-periodic signal, so what we have done there is we have taken the Fourier transform of a periodic signal that is a Fourier series on its inverse transformation is Fourier series, okay that Fourier series using through Fourier series we try to using the Fourier coefficients and the Fourier series we try to show that it converges that the final Fourier integral theorem we have proved by allowing, but finite interval that is from –L/2 to L/2 as I allowed L goes to infinity finally, so periodic signal becomes non-periodic signal and then that is your full real line and also you have seen what is the limit of that as L goes to infinity in an intuitive way in more formal way we have derived this Fourier integral theorem, so that is it's not rigorous proof but that gives you the idea that is the Fourier series the finite signal Fourier transform and this infinite Fourier signal over infinite time, it is -infinity to infinity non-periodic signal Fourier transform is also they are connected, but this we will try to prove mathematically in a more rigorous way, so let us choose if F is, F(x) is an absolutely integrable function and piecewise smooth function, piecewise smooth means N derivatives you take, it is N derivatives are it is N times different, any number of times it is differentiable except at finitely many points it is, the derivative is piecewise continuous function that means you have only finitely many only jump discontinuity that allowed for any N derivatives of such function F(x), piecewise smooth on are, what we require here is only peaceful smooth only, F double dash is piecewise

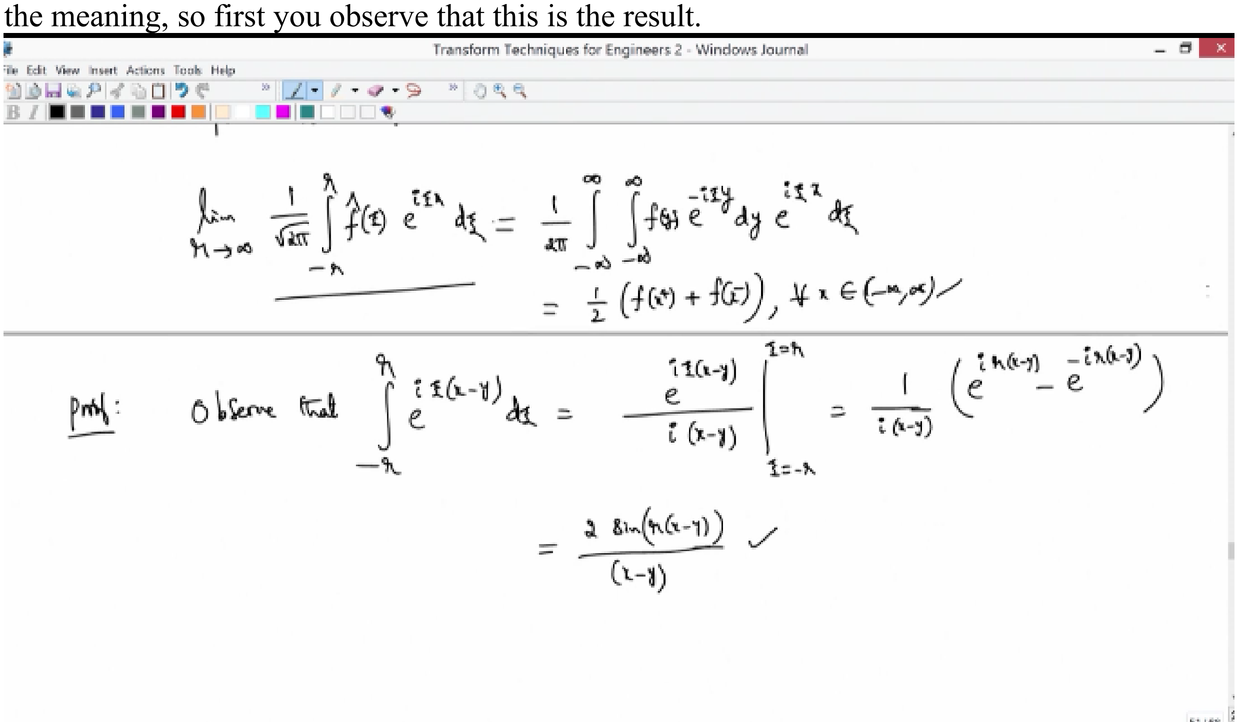
smooth function, is piecewise continuous function, that is the meaning of smooth here, we use only this one F2 derivatives.
In general definition of piecewise smooth if I recall any derivative, any derivative it's infinitely differentiable any number of times you can differentiate every time that any nth derivative is



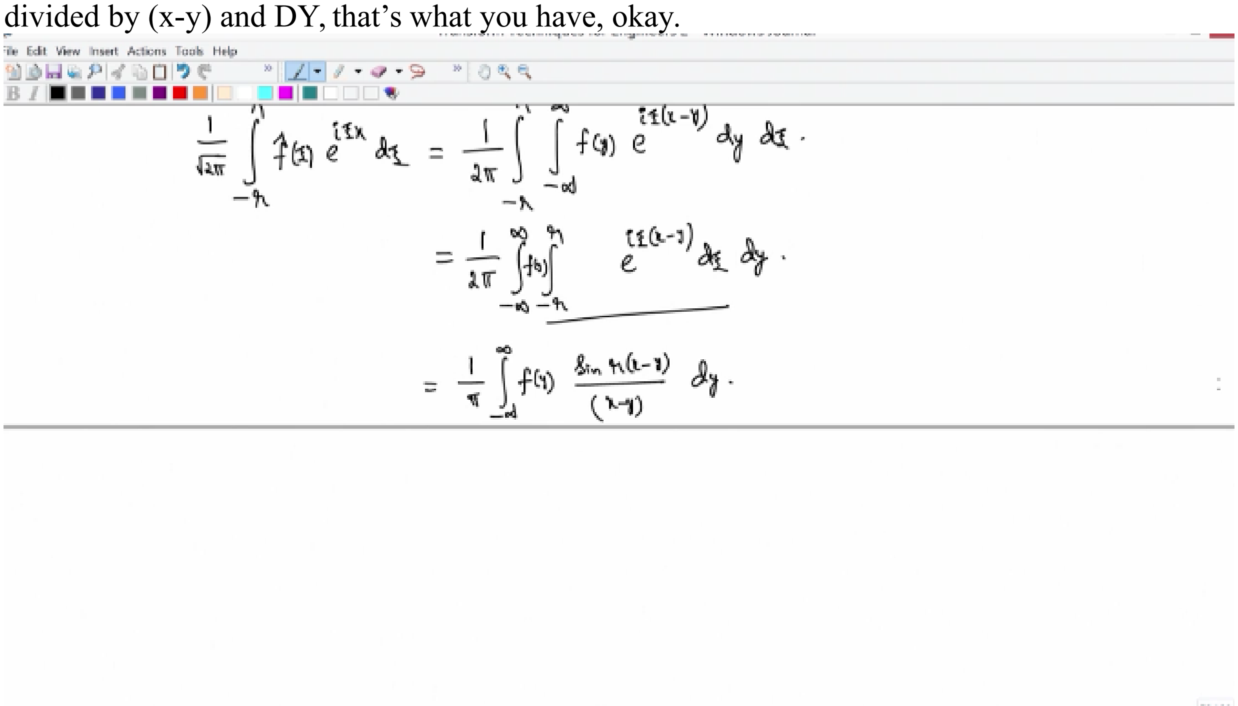
piecewise continuous function, so that is the meaning of, so let us this we choose the smooth function, piecewise smooth on or in -infinity to infinity, then what you have is this limit I'll write like this, 1/root 2 pi -infinity to -R to R, F cap(xi) this we know already, okay, Fourier transform that we defined DX which is equal to 1/2 pi, now this is as if you allow R goes to infinity -infinity to infinity, root 2 pi I have actually and inside this is also 1/root 2 pi that if I bring it outside 1/2 pi and this is a definition of Fourier transform if I write here you have F(x) E power -I xi X, okay DX, this is D xi this is what you want.
And you have E power +I xi X D xi, so I will try to use this different notation so let us use DY so that you have DY here, so this is your DY, this is a dummy variable, so this is exactly your Fourier integral theorem this is equal to 1/2 of F(x+) + F(x-), if it is discontinuous there is a jump, if it's a discontinuous there is a jump here so this half of that jump that is the meaning of this, if it is here both sides limits are same so half of that is actually the value, so in any case this is what is true for every X belongs to R, R is minus infinity to infinity that is the meaning, so this is exactly we will try to prove this in a more rigorous way now.



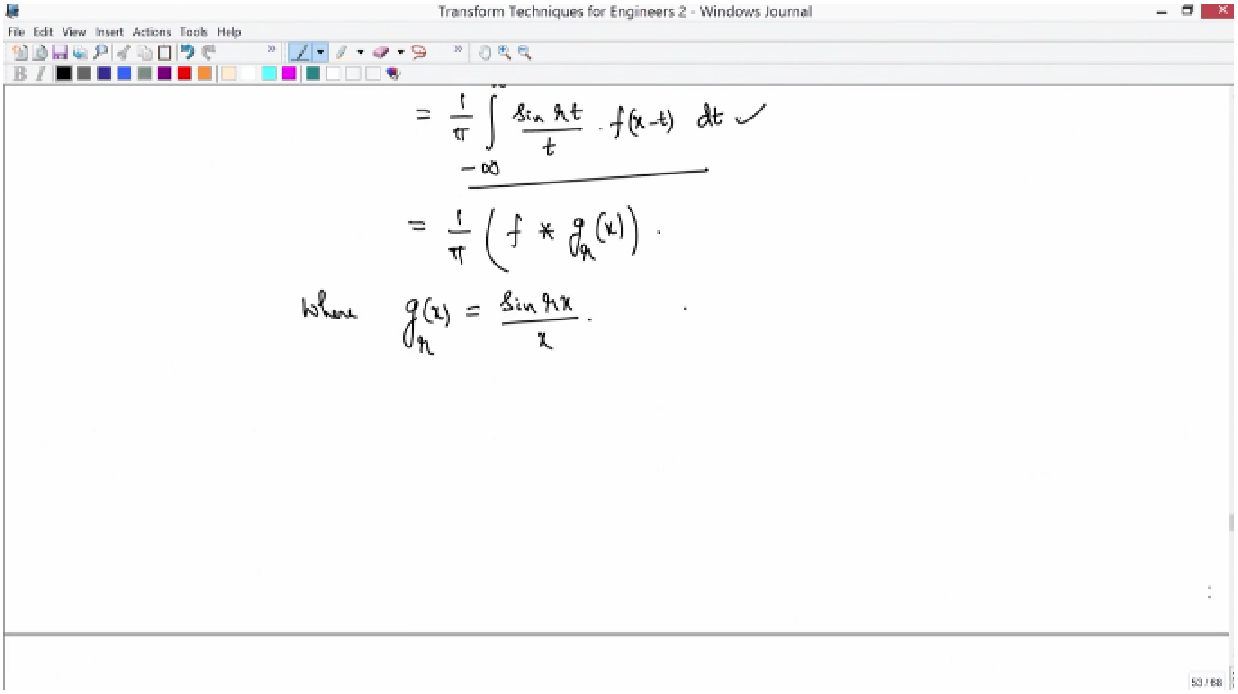
So proof, let me start the proof, so we first observe that integral -R to R E power I xi X-Y D xi, what is this integral? What is this one? So this you try to integrate you'll get divided by I (x-y) and E power I xi (x-y) and then xi = -R to xi = R, if you apply this you have 1 by, so -I by or rather keep I times (x-y) E power I xi is our R, (x-y) -E power -IR (x-y) divided by 1 over this, that is exactly what you have, okay, so this is same as what is this one? This is 2 times sine R (x-y) divided by 2I, so 2 I I goes, so what you get is 2 times of that divided by (x-y), so this is the meaning, so first you observe that this is the result.



Now what I do is I try to take this part in your statement 1/root 2 pi -R to R F cap(xi) E power I xi X D xi, so this is equal to and I'll replace F cap by this Fourier transform definition, so that you get 2 pi -R to R, and this is going to be -infinity to infinity, F(x) or let me write F(y) E power I xi (x-y), E power – I xi Y, I combine with this so to write that into D xi, DY and then D xi, so you can, because this quantity is absolutely integrable, integrand is absolutely integrable and you have already seen, so this is as a function of XY, Y and xi is a function of Y and xi, xi is anyway between -R to R which is a finite interval, this is a integrable function, this is the double integral is finite so all you can do the integration, you can do order of integration you can change and you can write this as -R to R, and minus or rather if you change the order of integration because there are two different integrals domains, so if we change the order of integration it's going to be –R to R F(y) E power I xi (x-y) D xi and DY, so this you do integration F(y) you can write here, so you remove here, and if you try to do this, this integral that is exactly what we use here, so you substitute there so you get 1/2 pi -infinity to infinity, F(y) 2 times, so 2 comes out, so here 2 comes out, so 2 2 goes, 1/pi to sine R times (x-y) divided by (x-y) and DY, that's what you have, okay.

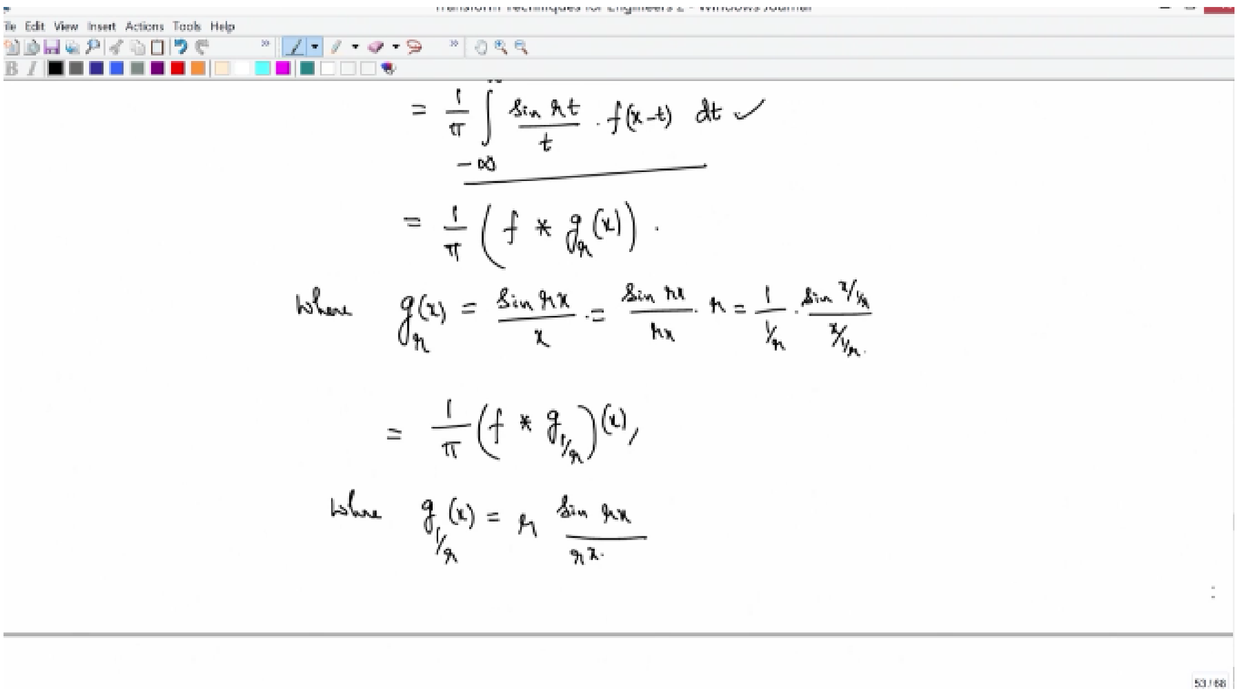


So if I use X-Y =T, okay, then what happens to Y? Y is X-T, okay, so 1/pi you just do the change of variable in this, if I do this sine RT/T and you get F(y)is (x-y) DT, DY is DT, you have a –DT, DY is –DT, X is fixed, okay, so you have X is fixed here. So and then what happens to your limits? When Y = -infinity it's going to be +infinity, T is +infinity and this is going to be minus infinity, this minus minus if you remove you can interchange, okay, can interchange limits with that minus, so this is what you have.

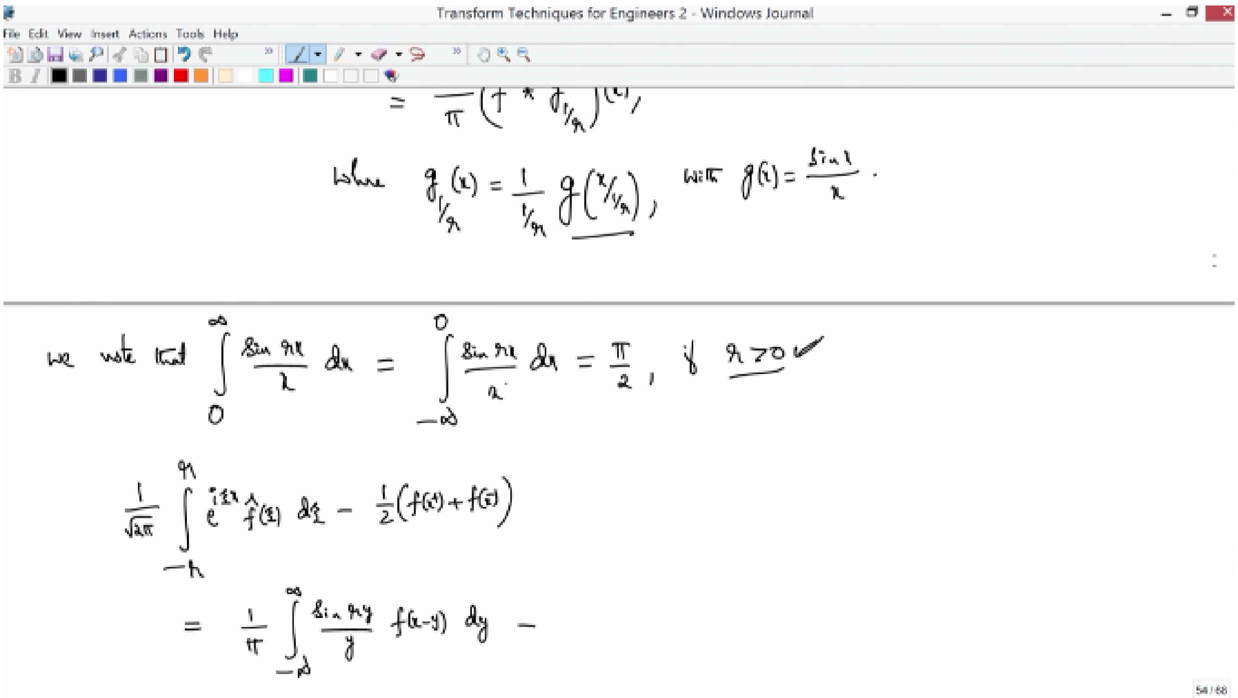


And now I can rewrite this as, this is exactly definition of my 1/pi times, what is this? This is if you look at it this is exactly were convolution product of, let me use 1/pi times convolution product of F star, G is some G(x) where G is, where G(x) is equal to, what is your G(x)? Is sine R, so G depending on R let me call this GR, okay, GR T/T this is a function of X I wrote, so I have G RX/X, okay, so you have like this, this is exactly your convolution product.

I will rewrite this as 1/pi times F star of, instead of writing this way what I write is, I write G of 1/R (x) where G of 1/R (x) is 1 by 1/R times G of, that is a rather sine, sine R times, sine RX I write X divided by 1/R divided by X, okay, X into 1/R so with this 1/R if I write this is exactly what you have, right, this is RX so you try to write sine RX/RX into R, okay, so if you want this one you can rewrite here also, so I can write sine RX divided by RX, so I have to multiply R so this will give me I can rewrite 1 divided by 1 by 1/R times sine X divided by 1/R divided by, X divided by 1/R that's how you're writing, so that way if you rewrite so you can write G,

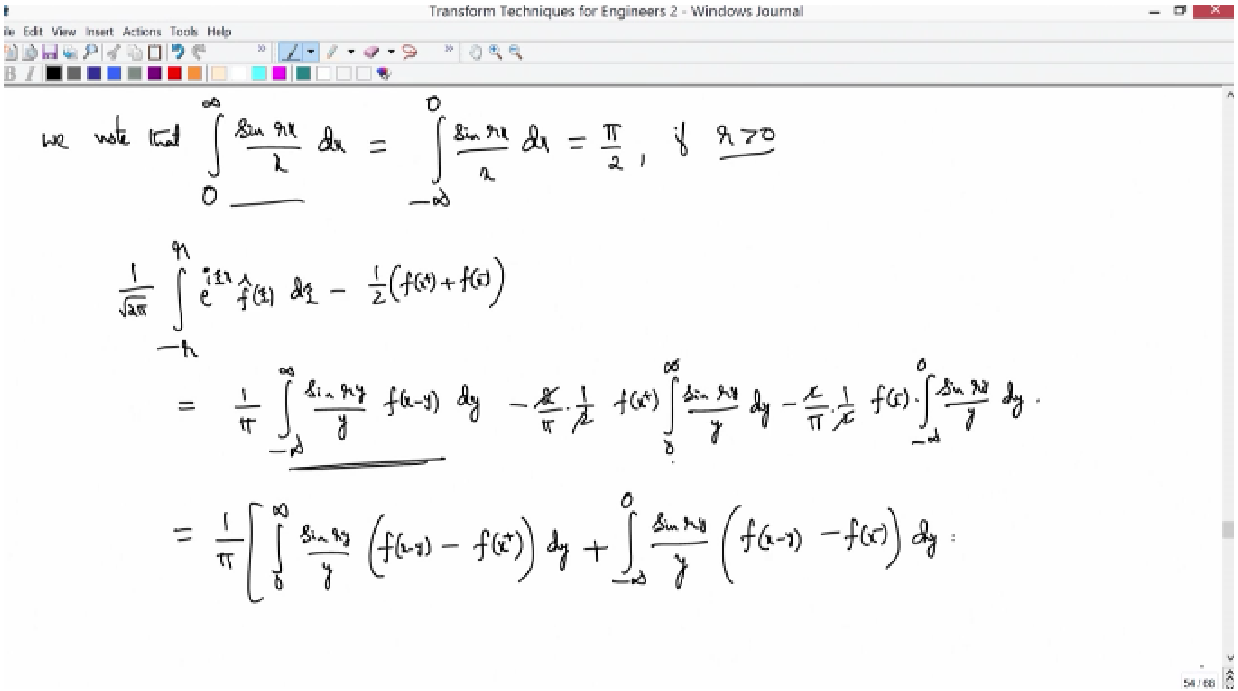


what is your G 1/R is? G1/R is 1 divided by 1/R times this is X divided by 1/R, so that is so you can rewrite this as some function if you define G(x) is sine X/X so you can write G(x divided by 1/R) with G(x) sine X/X, so this quantity I replace sine X/X, so in the place of X I put X divided by 1/R, G(1/R), okay, so in this way of writing you have some advantage, so we'll use that one and we'll try to prove, so eventually what we do is, I allow R goes to infinity so that's going to be G(0), I allow eventually R goes to infinity.
Another important thing that we need is this integral 0 to infinity sine RX/X DX this is, because this is an even function this is even function, because of this even function -infinity to infinity of this is actually 2 times 0 to infinity sine RX/X DX or 2 times -infinity to 0 so that means to say both are same, so 0 to infinity this is same as -infinity to 0, okay, whose value is actually equal to pi/2 if R is positive, right, which is anyway which is the case, R is anyway positive, so what is R? So R I choose -R to R, R is going to, so R is always positive, so because R is positive this quantity is pi/2, so this we note this that these two integrals we may use eventually okay.
So having done this now I can take the left hand side minus the limit, so what we have this is the left hand side, and this minus and what you look for is this one, so this minus this if I take I look at its estimate, and I show that there is going to be 0 as R goes to infinity. So let me use 1/root 2 pi and -R to R, E power I xi X, F cap(xi) D xi -1/2 times F(x+) + F(x-) this is actually equal to, so if you now put this first one is 1/pi, first one is this, this one is 1/pi, right, so 1/pi first term is 1/pi -infinity to infinity, sine RY/Y F(x-y) DY, the other one is minus, because I



use this, I try to use this one, these two integrals I use it here, so how do I use this? I multiply pi/2 or rather 2/pi into 1/2 F(x+), for this X+ I will try to use, and because I multiplied this 2/pi I have to multiply, I multiply pi/2, so 2 pi pi into pi/2, so that pi/2 I replace this integral, so 0 to infinity sine RY/Y DY, for the other one again you do the same way 2/pi times 1/2 F(x-) into -infinity to 0, sine RY/Y DY, so this is what is, 2 2 goes here both the places, and what you get is 1/pi.

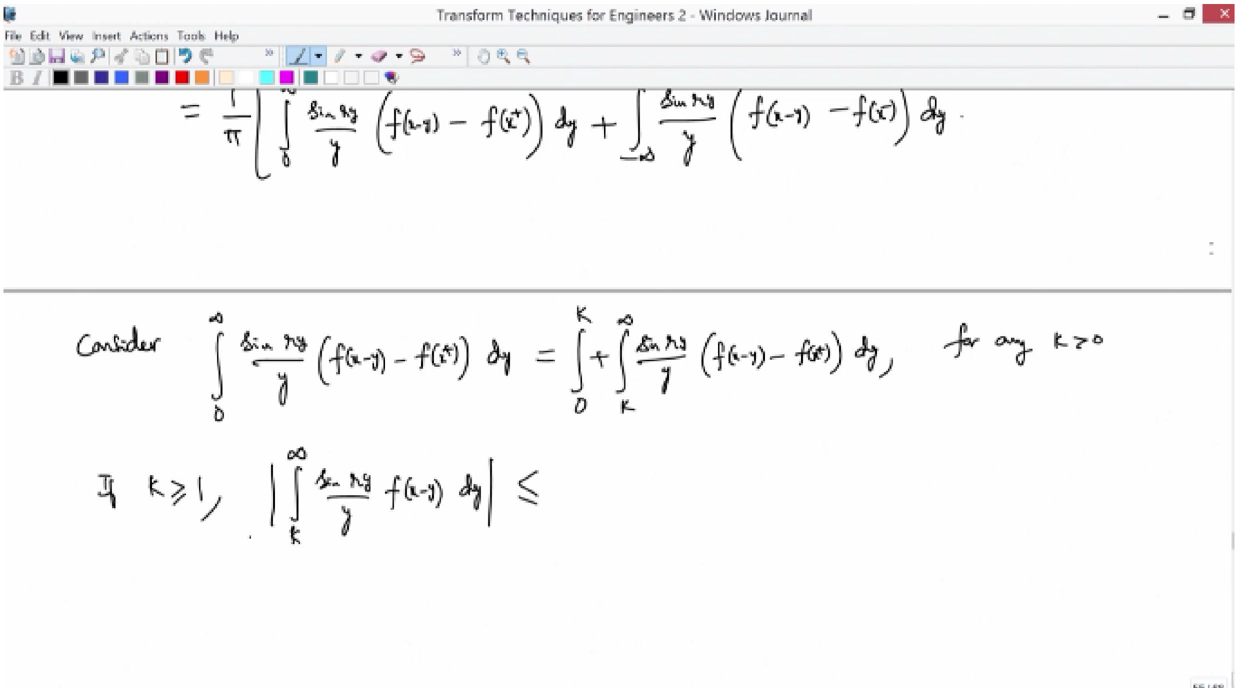
Now what you have is, now you try to split this into two parts, 0 to infinity, first I tried to use 0 to, you rewrite this as a 2 part, 0 to infinity, and -infinity to 0, 0 to infinity if you write and this sine RY is common, sine RY/Y is common, and what you have is F(x-y) from this and here -F(x+), from 0 to infinity into DY, I combine these two parts first 0 to infinity part here and this integral, if you do this is what is the result + -infinity to 0 again you do the same thing sine RY/Y, Y is common, and what you get is here F(x-y) and here - what you get is F(x-) DY this is what you have, so this is exactly you have.



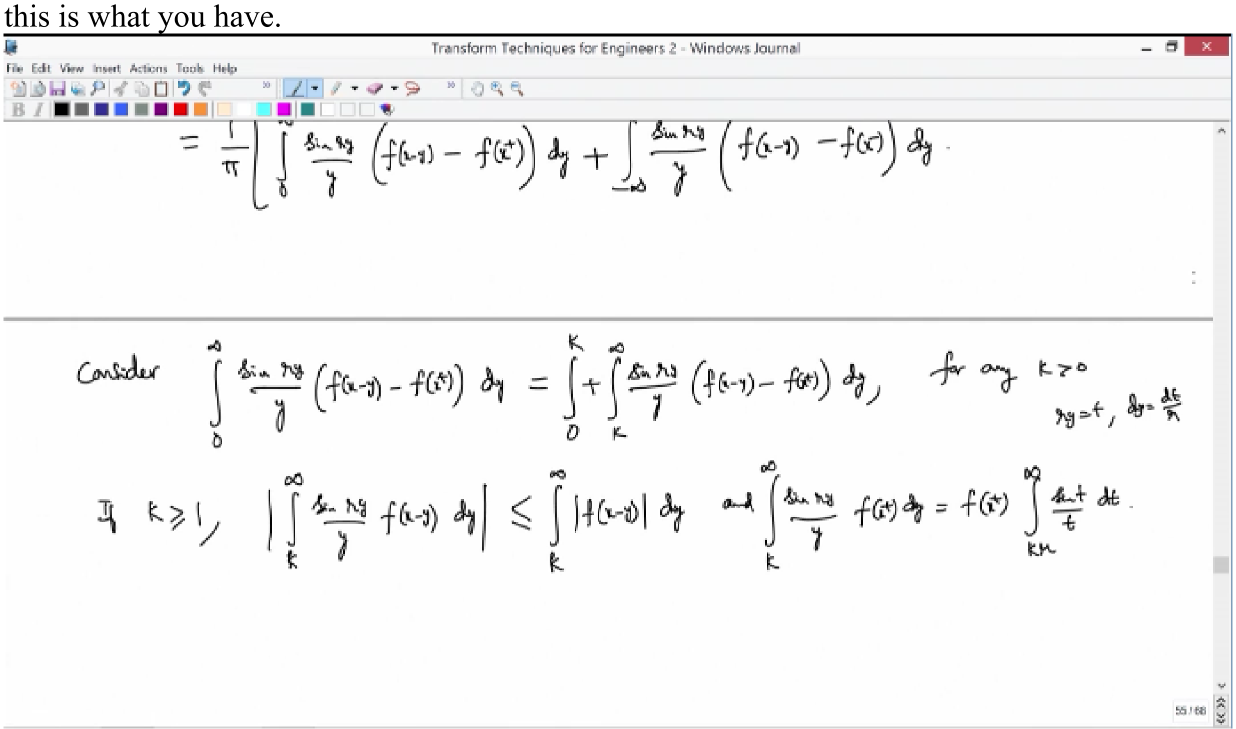
Now what I do, I will try to take this one, so we will look at this one this integral and we will try to estimate this integrals as R goes to infinity what happens to these two integrals, we will try to guess (Indiscernible 0:38:50) okay, okay.

So we will try to look at this integral and I also try to look at this integral, and we will see eventually as R goes to infinity what happens to, I am trying to prove that this R goes to infinity each of this will go to 0, okay, so how do I see this? We look at this integral first and this is a similar way you can do this, okay, so let me do only one, so we'll consider 0 to infinity sine RY/Y F(x-y) - F(x+) DY this is equal to, so I split this into again some big number some number you take for any K, for any K positive I can split this into 0 to K because this is a 0 to K + K to infinity, I can always do this sine RY/Y F(x-y) - F(x+) DY, okay, this I can always do.

If I choose K is greater than or equal to 1, what you have is this is from K to infinity because this is more than 1, 1 to infinity, see if K = 1, 1 to infinity, sine RY/Y F(x-y) DY this quantity, this is always less than or equal to, because if K is bigger than 1 sine is, sine of RY is always less than or equal to 1, 1/Y if Y is in K to infinity, if K is bigger than 1, 1/Y is always less than



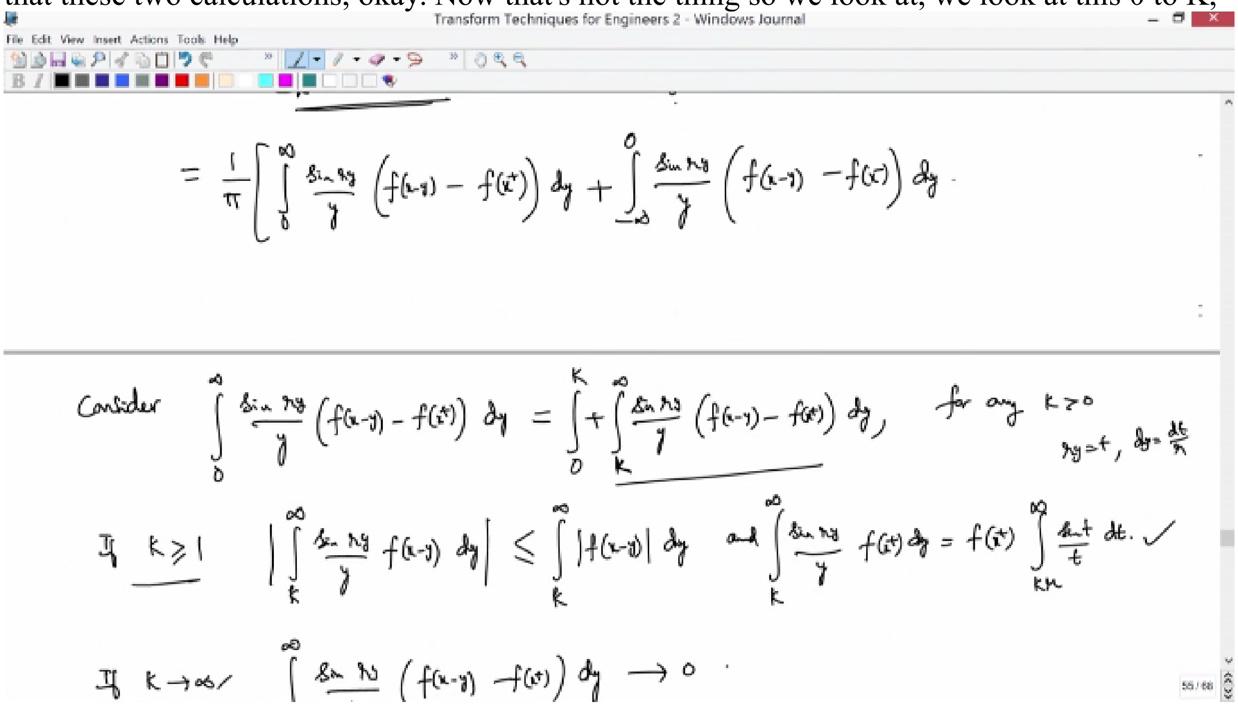
1, so in that way you have K to infinity modulus of F(x-y) DY, and what you have is the other one, other one is simply what happens to your K to infinity sine RY/Y F(x+) DY this is actually equal to F(x+) comes out, and this is from K to infinity, so try to put RY as T variable, okay, so then what you get is sine T/T so what I do is, RY=T, so you get R DY, DY is DT/R, so you have 1/R DT/R, and Y is T/R, T/R is R comes up, okay, so R R goes, so you don't have to write anything here, sine T/T DT and what happens when K, so you get T is from KR to infinity, so this is what you have.



So if you have K is greater than or equal to 1, this is actually true for any K this is true, this quantity this integral, but this integral is valid only for K greater than 1, so for K greater than or equal to 1 both the things we use, okay.

Now if you can see if I allow K goes to infinity, K goes to infinity what happens? If I allow K goes to infinity this is going to be 0, and this is also going to be 0, so that means this is also going to be 0, okay, so as K goes to infinity clearly this is going to be, as if I take K bigger this quantity again that's actually get kind of trivial thing, okay, so this quantity you can ignore for bigger K values this can be made as small as possible for any R, okay, so that is the reason we have done this two, so these two quantities you can easily see separately that these two, this K to infinity integral is actually, you can make as small as possible for bigger K values as long as R is positive, okay, which is anyway positive.

And we look at the other part, this K to infinity, so let me write as K to infinity sine RY/Y F(x-y) – F(x+) this quantity DY, this goes to 0, as K goes to infinity, that's what is the meaning of that these two calculations, okay. Now that's not the thing so we look at, we look at this 0 to K,



0 to K so such a big K value if I choose and second integral is nullified and the only first integral is the actual value, so if I choose such a big K value, so for such a big K you what you have is sine RY/Y, F(x-y) – F(x+), is it X+ or X-, I think it's X-, right? I combine it with, I think, I think I have chosen, doesn't matter I take X+ only, X+, okay, this is X+ DY, what happens to this? This is equal to 0 to K, now I can rewrite the sine RY/Y, earlier we have defined as, okay, so 0 to K you write this as sine RY times, this you rewrite those so this F(x) where F(x-y) – F(x+) divided by Y as some G(y) you try to define it like that, if you do this you

Consider $\displaystyle\int_0^\infty \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy = \int_0^K + \int_K^\infty \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy$, for any $K>0$

$\lambda y = t$, $dy = \frac{dt}{\lambda}$

If $K \geq 1$ $\displaystyle\left|\int_K^\infty \frac{\sin \lambda y}{y} f(x-y)\,dy\right| \leq \int_K^\infty |f(x-y)|\,dy$ and $\displaystyle\int_K^\infty \frac{\sin \lambda y}{y} f(x^+)\,dy = f(x^+)\int_{K\lambda}^\infty \frac{\sin t}{t}\,dt.$

If $K \to \infty$, $\displaystyle\int_K^\infty \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy \to 0$

$$\int_0^K \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy = \int_0^K \sin \lambda y \cdot$$

where $\dfrac{f(x-y)-f(x^+)}{y} =: g(y)$

have this G(y) DY, okay, so what is this one? This is actually equal to, and this is between now think of G as a function which is a, which is function, Y belongs to 0 to K, so which is a periodic signal, and because F is absolutely integrable function and 1/Y, Y is between 0 to K, okay, and you can see at 0 it is well-defined, G is a function you have, you have to worry about see G is, F is a piecewise continuous function, so this numerator is piecewise continuous function except at Y = 0, so in this 0 to K interval G(y) is piecewise continuous, and it is piecewise smooth function rather, okay, piecewise continuous, or piecewise smooth function you can think of because that is what we have chosen in the high policies, F is a piecewise smooth function.

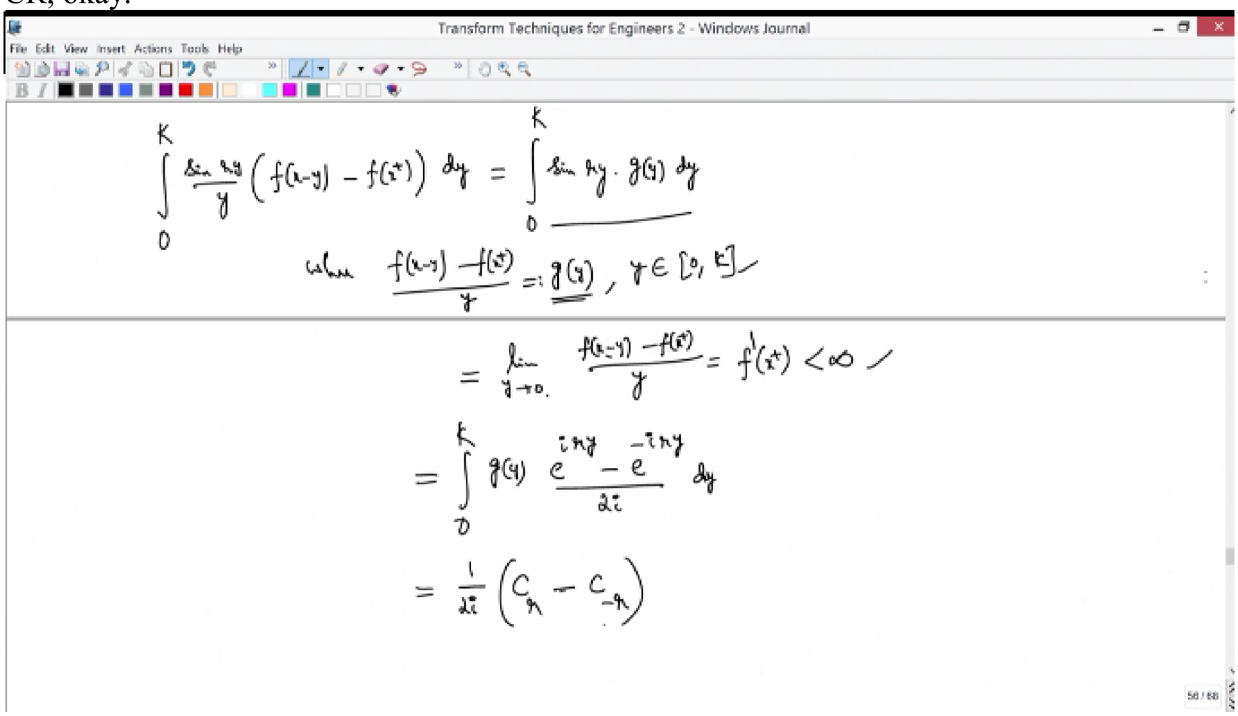
If $K \to \infty$, $\displaystyle\int_K^\infty \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy \to 0$

$$\int_0^K \frac{\sin \lambda y}{y}\left(f(x-y)-f(x^+)\right)dy = \int_0^K \sin \lambda y \cdot g(y)\,dy$$

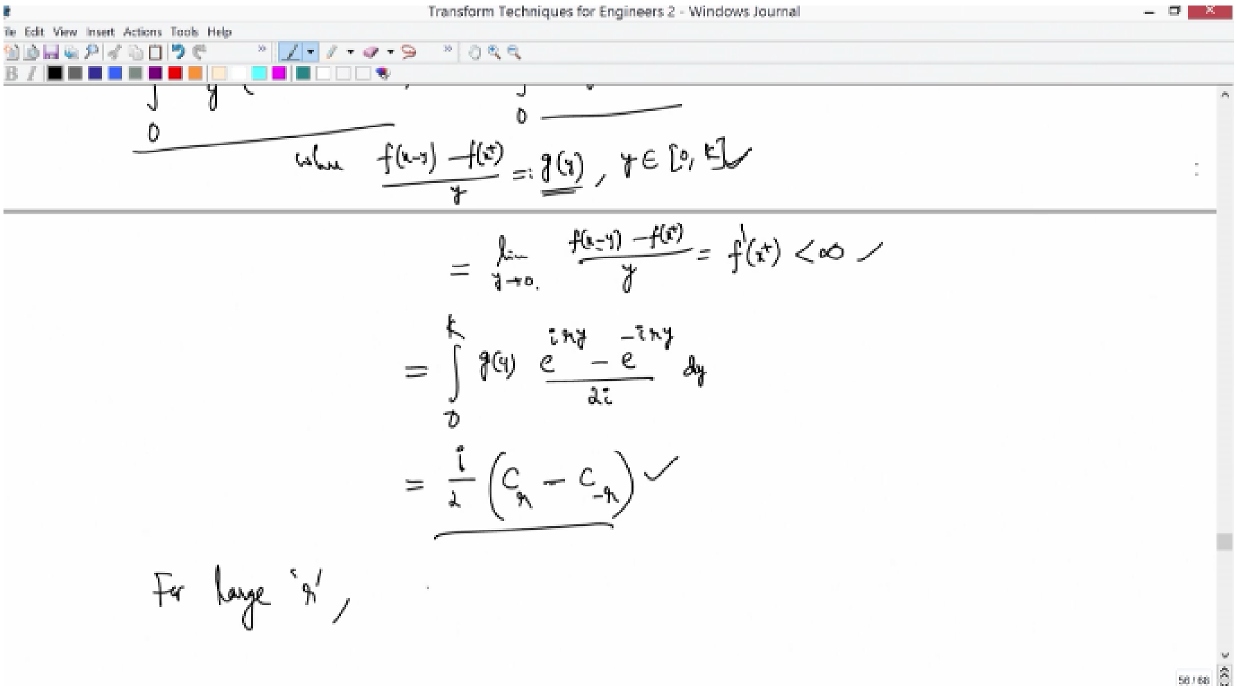
where $\dfrac{f(x-y)-f(x^+)}{y} =: g(y)$, $y \in [0, K]$

$=$

So on this interval, finite interval F is a smooth wise, smooth piecewise function accepted Y = 0, at Y = 0 what happens? So we'll look at this limit, limit Y goes to 0 what happens to this (x-y) – F(x+) divided by Y, so this is actually equal to, what is this value? This value is actually going to, F(x) as Y goes to 0 what happens? G(y), this means this is X+ as Y goes to 0 this is actually is equal to the derivative of DF of DX, that is F dash(x+) plus side, you can think of that, okay, so that we know that this is F dash is piecewise smooth, and piecewise smooth means, F dash is a piecewise smooth means it has only jumps even at 0, at 0 it should, it cannot have infinite jump that means this is a finite, this limit exists as a function derivative, so this is finite implies G(y) is piecewise smooth function that means you can write this, you can think of this as integral 0 to K, G(y) now I can rewrite this as sine to Y as E power I xi, IR, some so rather some IRY – E power –IRY divided by 2I into DY, so this one, first term if you use 1/2I comes out, first term is actually G cap of, that is Fourier transform, so Fourier coefficient, so here let me use the same notation G cap RCN, okay CN(-c) rather how do I put it? CR of, CR of, this is CR rather, CR is the coefficient, here - C-R, okay, this is C-R and this is CR - this is CR, okay.

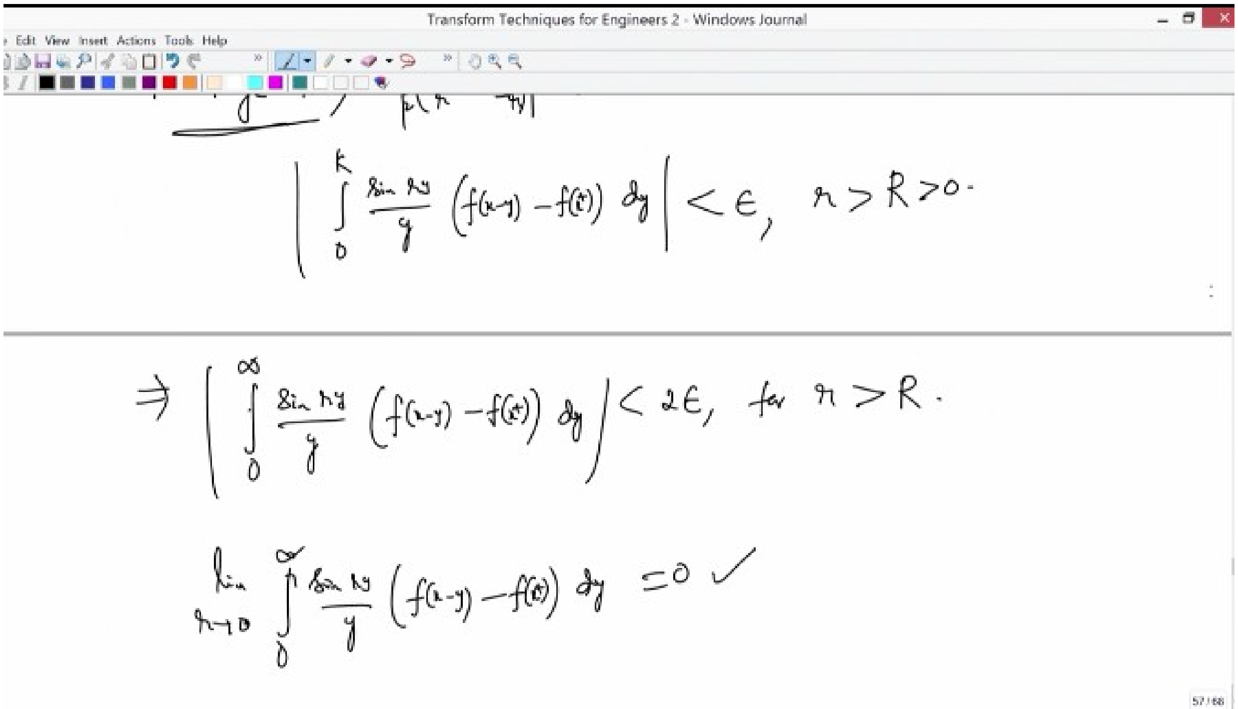


So if I take this up –I, so you make it plus so you have at this CR and this is C-R, this is how it is, okay both are same. So you can easily see that this quantity is 0 to K integral is actually equal to this. Now you can easily see, now what we have to show is we have seen that G is a piecewise smooth function, G is bounded at 0, so that piecewise smooth function in this interval, K is sufficiently big so that I can make this quantity, this quantity make less than epsilon, what happens 0 to K this part, 0 to K integral will be this one, now if I allow as for sufficiently for large R, that is what you want, R goes to infinity this quantity can be made less than epsilon, okay, so this can be made less than epsilon, okay, so CR – C-R by a Riemann

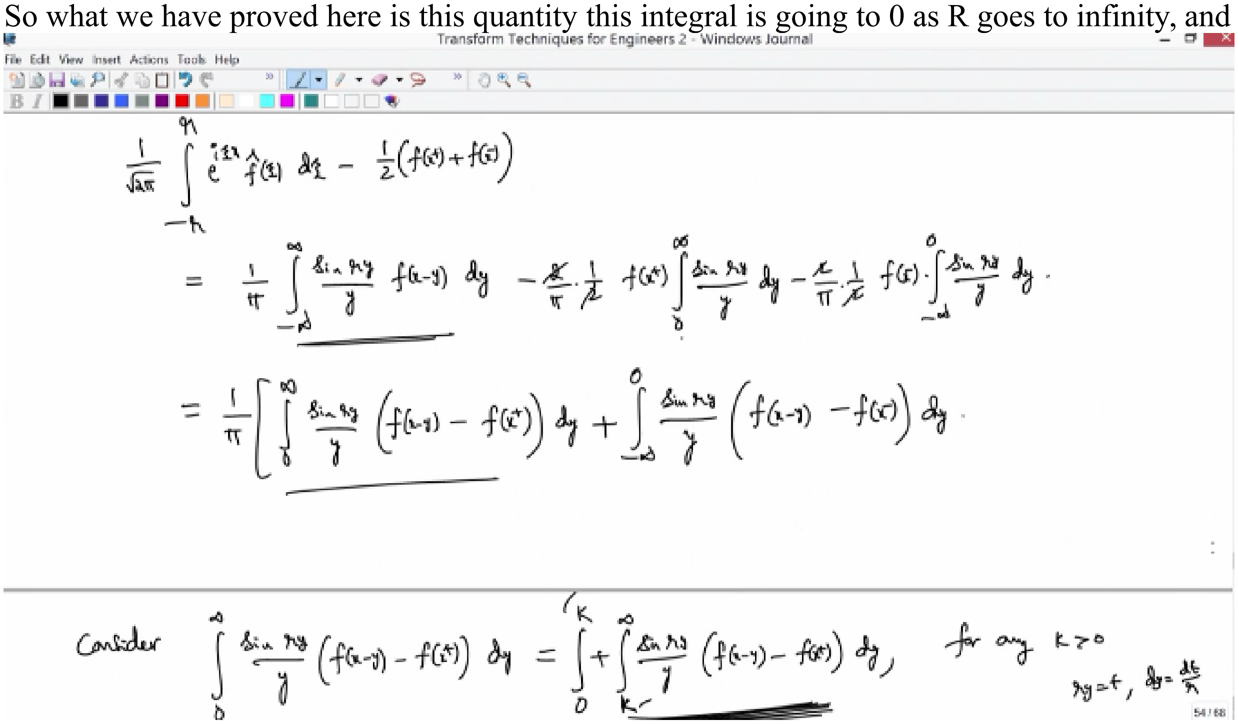


Lebesgue Lemma this can be made less than or into I/2 can be made less than epsilon for larger that means as R goes to infinity this integral 0 to K sine RY/Y F(x-y) – F(x+) DY goes to 0, that is the meaning, okay, because this I can make this 0, K to infinity for sufficiently large K first I made this is less than epsilon, okay, for some big K by choosing some big K I can make this less than epsilon, so I have that 1 epsilon I have.

Now for 0 to K integral, now this can be made less than epsilon so you have 2 epsilon, so that epsilon is arbitrary that is the meaning is, so for large R that is as R goes to infinity this overall 0 to K this goes to 0, okay, for large, so K is already we've chosen such a big value so that means as R goes to infinity K is already goes to infinity, okay, so in that way so we don't write like this so what we see is for large R this quantity can be made less than epsilon, that is what we want to put it, so that means for sufficiently big K this quantity finally can be made at less than epsilon, okay, so this means what I have is 0 to infinity sine RY/Y F(x-y) – F(x+) DY can be made less than 2 epsilon, okay for large R, for large R means for R is bigger than some capital R, okay, which is positive for R is bigger than some R, so this is exactly the meaning of limit R goes to 0 this quantity sine RY/Y F(x-y) – F(x+) DY = 0, so that is what is the meaning of this, R goes to infinity.



So what we have proved here is this quantity this integral is going to 0 as R goes to infinity, and



similarly the same way same technique you can show that this also goes to 0, as R goes to infinity so if I allow this R goes to infinity what you have is this total thing which will be, with R goes to infinity this is R goes to infinity, now what we have proved just now is this part is 0 and also you can prove the same way this part is also 0, okay, so that means limit of this is same as this quantity, because it's a fixed number, okay, so that means limit of this is equal to this one this is exactly what you want to show in your Fourier integral theorem, so this is exactly true.

Fourier integral theorem:

If $f(x)$ is an absolutely integrable function and piecewise smooth function in $(-\infty, \infty)$, then

$$\lim_{\lambda \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} \hat{f}(\xi)\, e^{i\xi x}\, d\xi = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(y)\, e^{-i\xi y}\, dy\, e^{i\xi x}\, d\xi$$

$$= \frac{1}{2}\left(f(x^+) + f(x^-)\right), \ \forall x \in (-\infty, \infty).$$

Proof: Observe that

$$\int_{-\lambda}^{\lambda} e^{i\xi(x-y)}\, d\xi = \left.\frac{e^{i\xi(x-y)}}{i(x-y)}\right|_{\xi=-\lambda}^{\xi=\lambda} = \frac{1}{i(x-y)}\left(e^{i\lambda(x-y)} - e^{-i\lambda(x-y)}\right)$$

$$= \frac{2\sin(\lambda(x-y))}{\ }$$

So what we have proved is the Fourier integral theorem in a rigorous way without having any, without any issue, everything is in legitimate here, we just made use of some of the ideas, some of the convolution product also we have used though we have not used this, I have not used this

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$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \lambda t}{t} \cdot f(x-t)\, dt$$

$$= \frac{1}{\pi}\left(f * g_\lambda\right)(x).$$

where $g_\lambda(x) = \dfrac{\sin \lambda x}{x} = \dfrac{\sin \lambda x}{\lambda x} \cdot \lambda = \dfrac{1}{1/\lambda} \cdot \dfrac{\sin \frac{x}{1/\lambda}}{\frac{x}{1/\lambda}}$.

$$= \frac{1}{\pi}\left(f * g_{1/\lambda}\right)(x),$$

where $g_{1/\lambda}(x) = \dfrac{1}{1/\lambda}\, g\!\left(\dfrac{x}{1/\lambda}\right)$, with $g(x) = \dfrac{\sin x}{x}$.

so we have I try to put it this I thought of using but I've not used, and anyway so this is doesn't matter, so from here it was just done in a more elementary way and we try to show that Fourier integral theorem, okay.

So now it's a legitimate to define from this you can define what is a Fourier transform and it's inverse transform the way we have defined, and then we have seen a lot of properties of the Fourier transform.

Now in the next video we will try to apply this Fourier transform and inverse transforms to solve some partial differential equations or boundary value problems for partial differential equations or differential equations, okay, we'll try to prove, we'll try to define those boundary value problems and then try to solve by this Fourier transform technique. We'll see the application of basically we see the applications of Fourier transform and in the next few videos. Thank you very much. [Music].

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