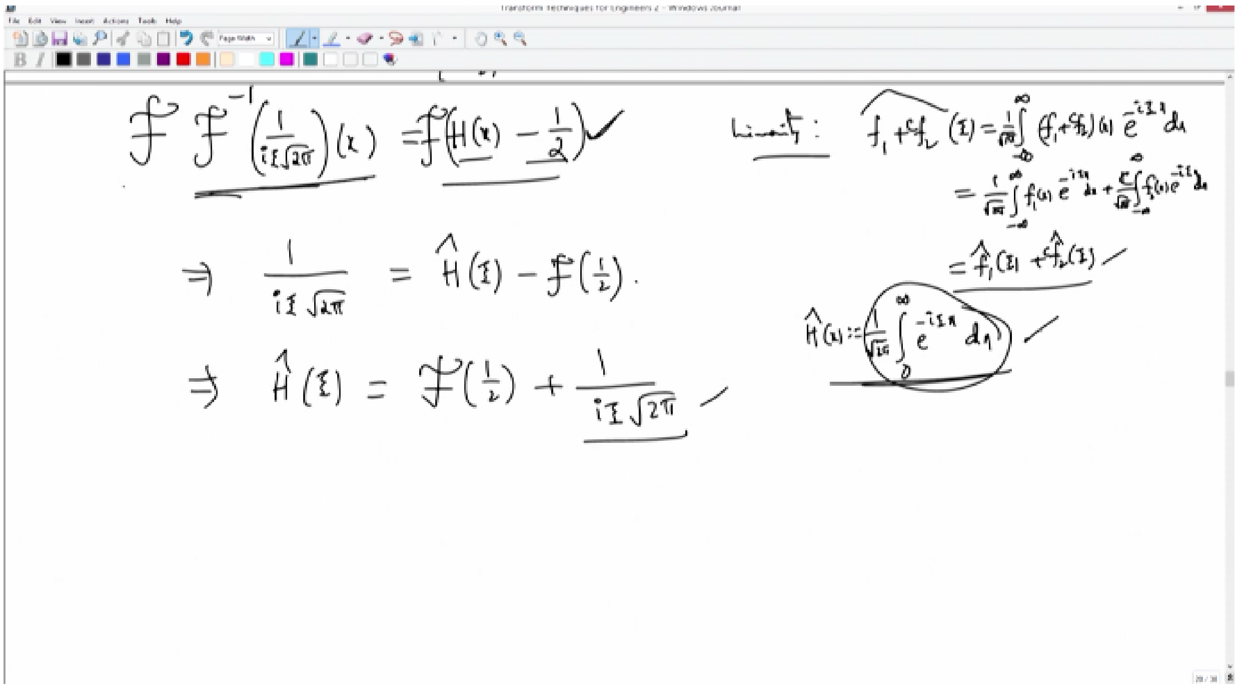
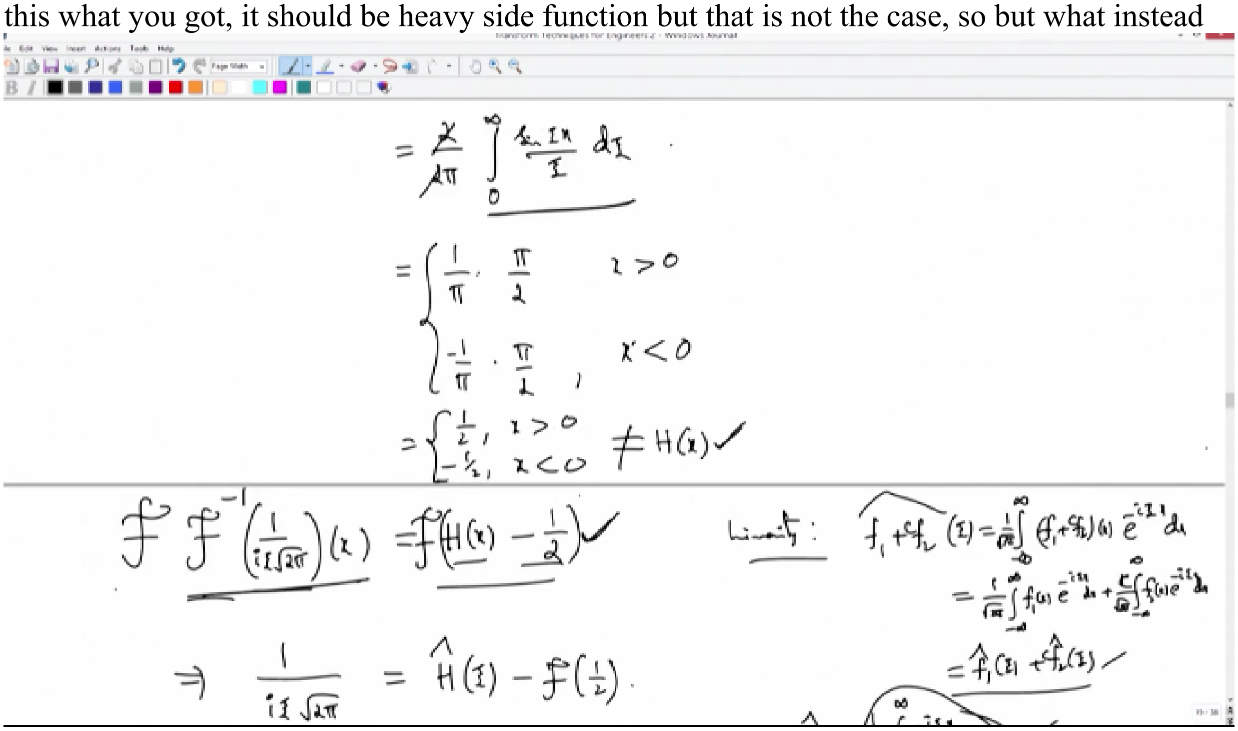
NPTEL
NPTEL ONLINE COURSE
Transform Techniques for Engineers
Use of Fourier transforms to evaluate some integrals
Dr. Srinivasa Rao Manam
Department of Mathematics
IIT Madras

Welcome back, in the last video we have seen what is the Fourier transform of heavy side function. Heavy side function is piecewise continuous function, and we have seen that it's Heavy side function is basically 1/I xi root 2 pi, 1 over I xi root 2 pi + some Fourier transformation of constant function 1/2, but as such if you directly calculate the heavy side function see that's integral, integral 0 to infinity 1/root 2 pi E power -I xi X DX, this is actually a definition of heavy side function.

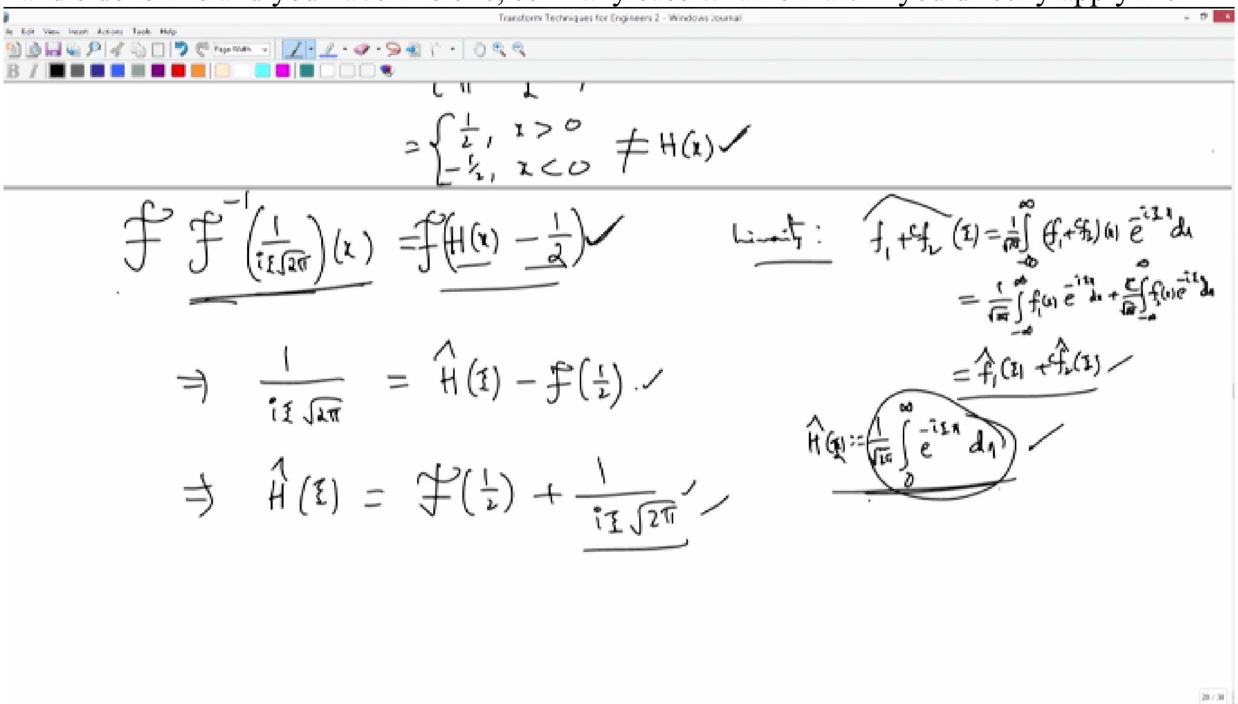


So this is actually, this integral is not in the usual sense, it's not defined, it's not defined so in that sense is actually you don't have this H cap(xi), it appears as a direct definition it is not a usual function, so that is expected clearly because this integral value is, integral is not converging integral, okay, so we have done in a different way so what we have done is we consider H(x) as a limit of some functions usual functions H alpha(x) and then you calculated this H cap of thing and in the process formally we have done, we brought this limit outside from this and which is not really legitimate, appears to be legitimate and if you have done you get this much but if you assume that this is actually the result you take the inverse transform of this what you got, it should be heavy side function but that is not the case, so but what instead



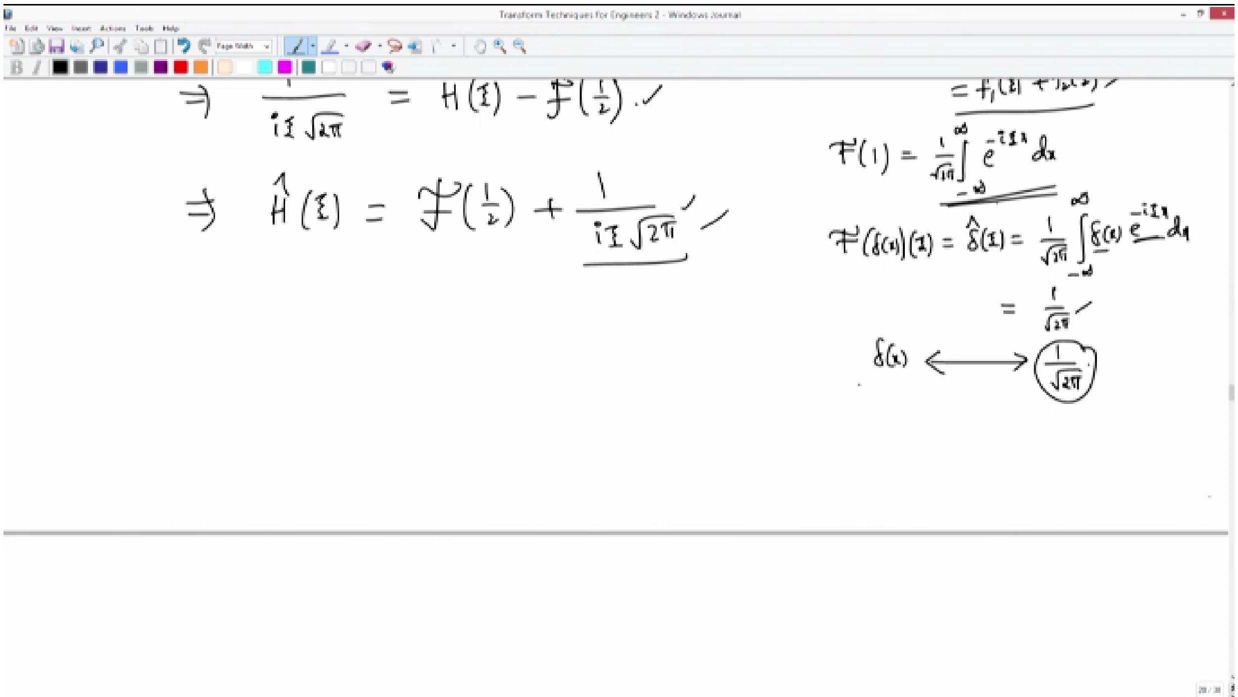
what you got is H(x) -1/2, so what you should get is the H(x) -1/2 whose Fourier transform is this one, 1/I xi root 2 pi.

So to get the Fourier transform of heavy side function so we can just take Fourier transform both sides again, because Fourier transform is basically can go and come back through Fourier and its inverse transform, so you can just apply both sides so get this one here and this right hand side is this and you have this one, so in any case what is that? If you directly apply the

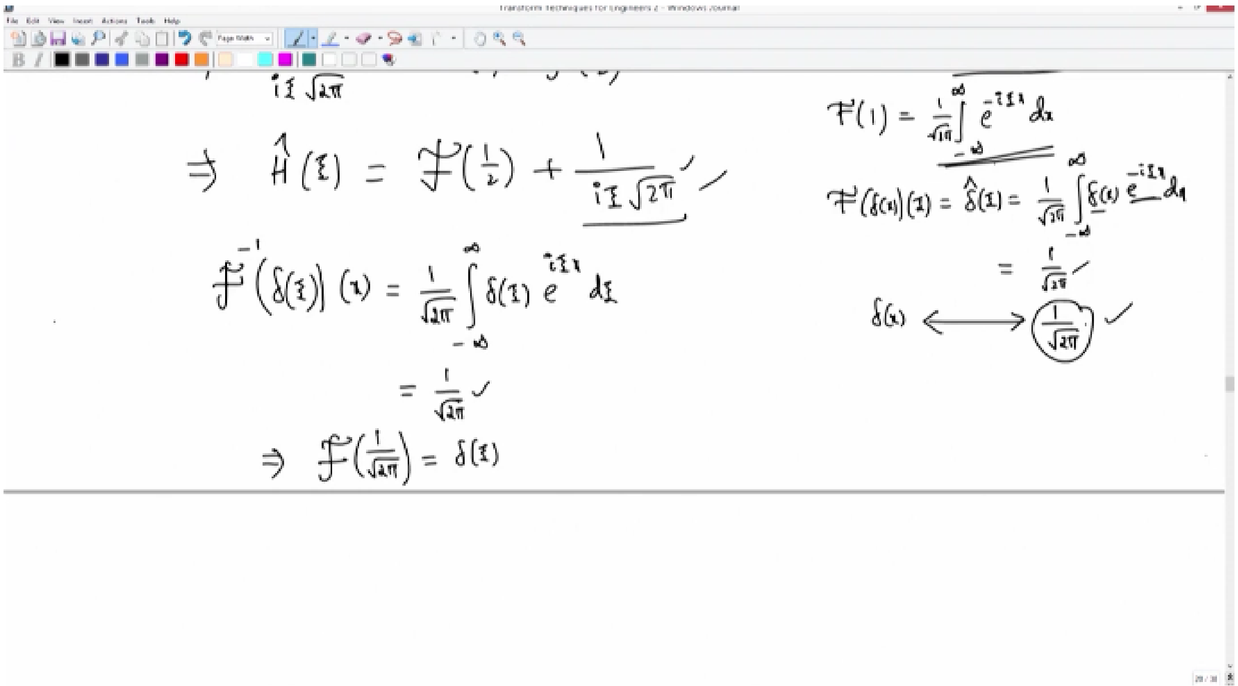


definition you are not seeing that this doesn't integral, doesn't converge that means you expect this Fourier transformation of heavy side function as is basically some generalized function, okay, so that is what actually we are going to get so.

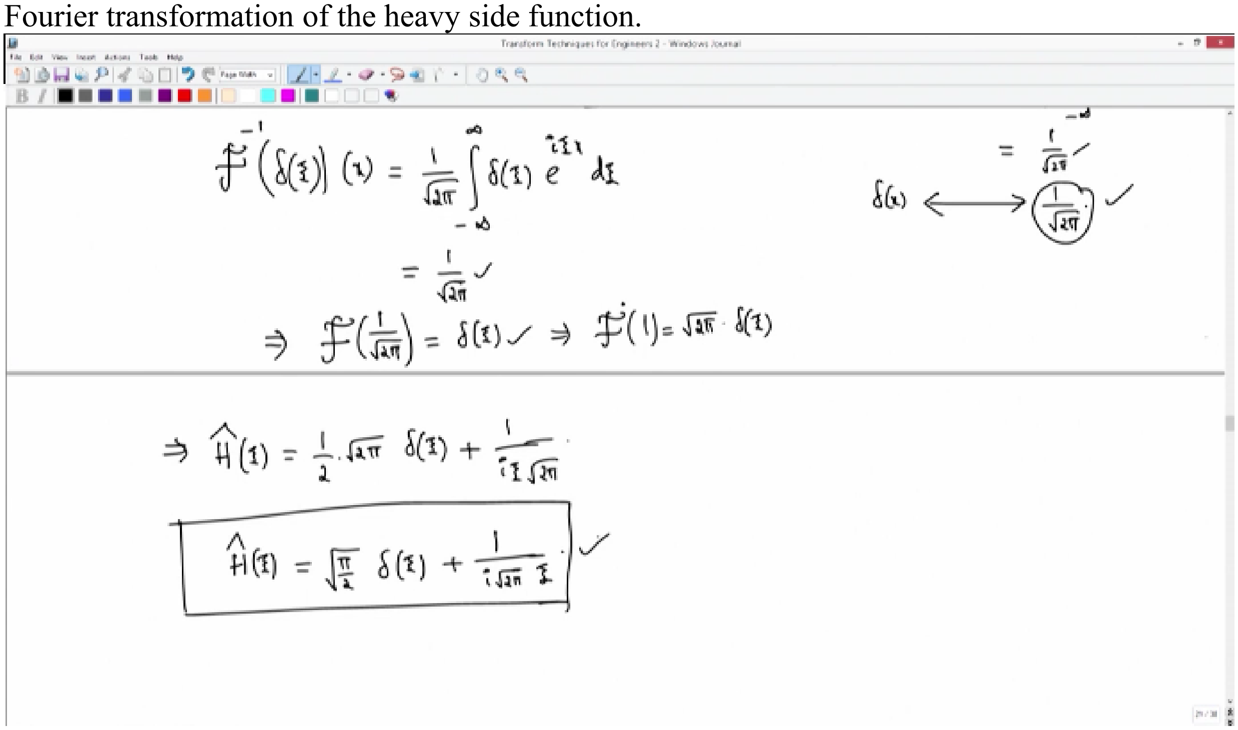
And now we have to see what is the Fourier transformation of, Fourier transformation of 1/2 or simply calculate 1 first, Fourier transformation of 1 is again 1/root 2 pi integral 0 to - infinity to infinity E power -I xi X DX, right, so again this does not converge, this is not a convergent integral, so you expect a Fourier transformation of 1 is some again, your some generalized function, okay, so I start with Fourier transformation of delta function of, so once you have the Fourier transformation of delta function is a function of xi that is actually del cap(xi) and we have seen that this is 1/root 2 pi is actually - infinity to infinity delta X E power –I xi X DX, so this is exactly, this value, the function value at X = 0 so that is 1, so you have 1/root 2 pi.



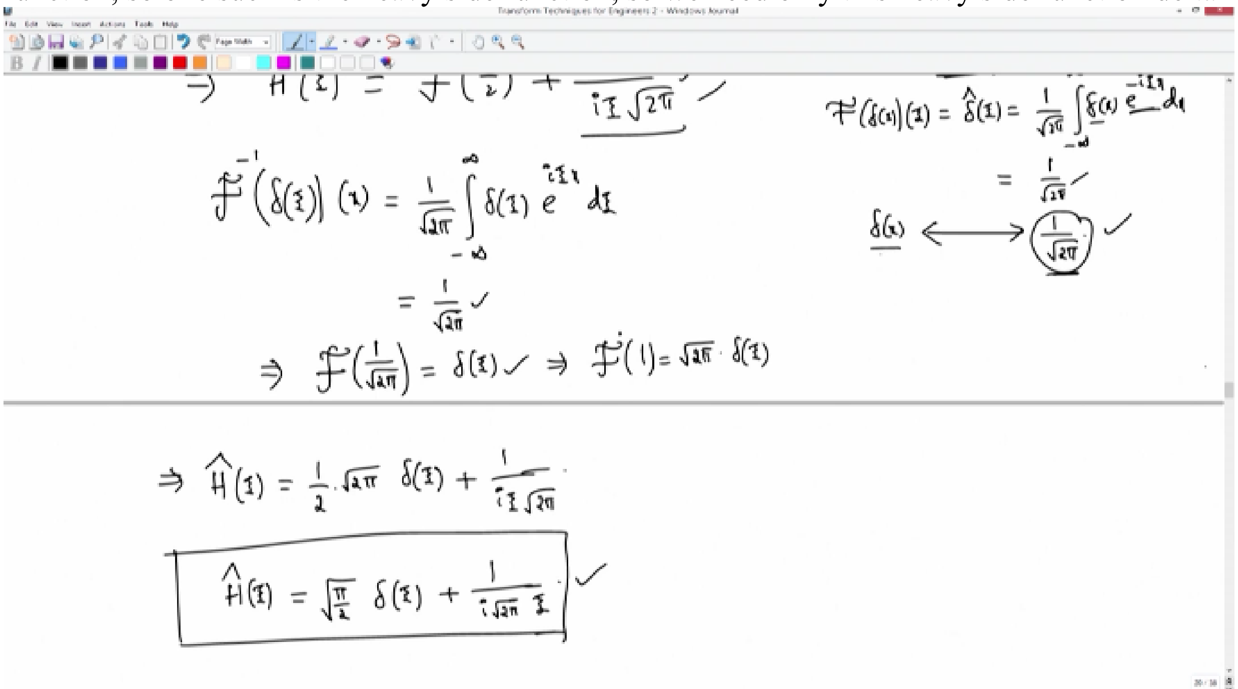
So that means if you, Fourier transformation of delta function goes to 1/root 2 pi, okay, and if you do the, if you take the inverse transform of this you get back your delta function. So inverse transformation is instead of this so you have 1/2 pi, so this is 1/root 2 pi, and 1/root 2 pi E power -I xi X DX is actually, inverse transformation is actually delta X okay, so this is what you have seen, so that means if you start with delta xi, delta xi as your some transformation, so delta xi you consider, you consider Fourier inverse transformation of this, then it will be a function of X which is by inverse transformation, by definition this is going to be - infinity to infinity delta xi, because xi is a real number, so we can do this delta xi E power I xi X D xi, so this is inverse transformation of delta xi, so this is also 1/root 2 pi.
So this tells me now if you apply both sides Fourier transformation what you get is Fourier transformation of 1/root 2 pi, it is actually a delta function okay, so this is a delta function of xi,



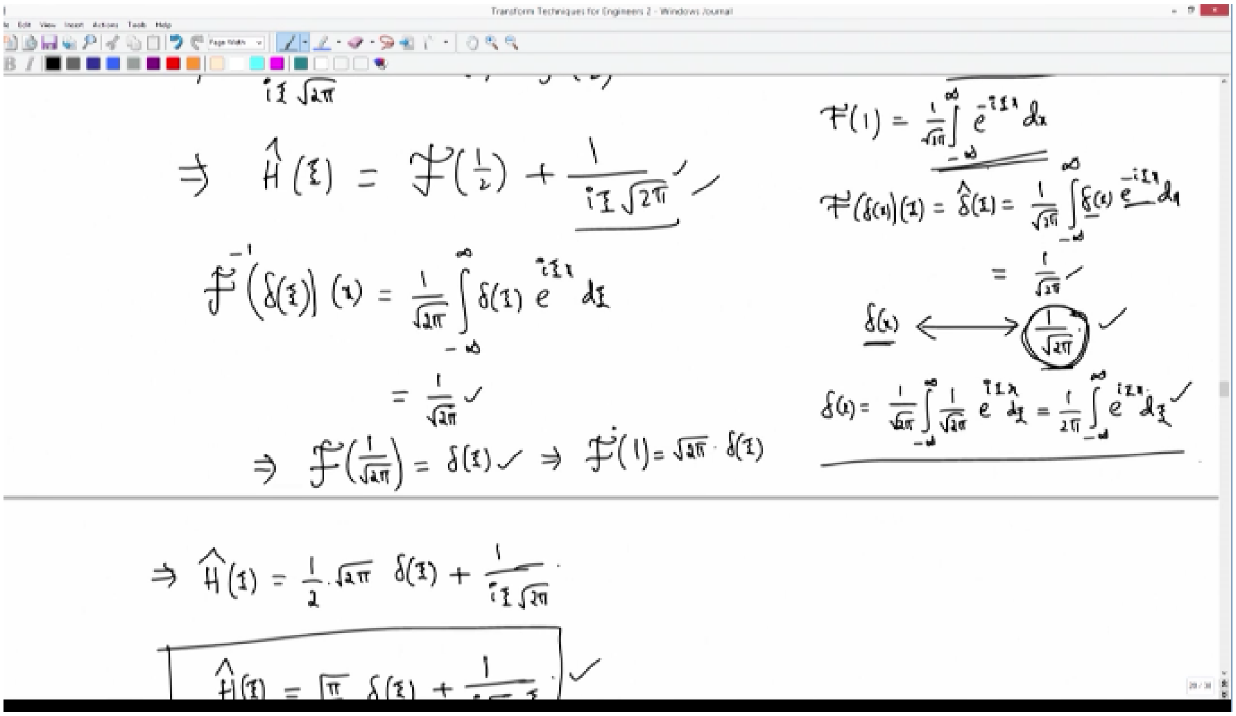
so this is actually what you have is Fourier transformation, yeah, fine, so inverse transformation of this delta function, delta cap of, it's not delta cap, so you simply take the delta function is a function of xi, so you have this one, so this implies 1 by, so Fourier transformation of 1 is equal to root 2 pi into delta(xi), so now this you go and substitute into your heavy side Fourier transformation of heavy side function which is function of xi, Fourier transformation 1/2 will be 1/2 times, Fourier transformation of 1 that is a root 2 pi delta function of xi + 1/I xi root 2 pi, so this is exactly what you have so, finally you end up the Fourier transformation of heavy side function as root pi/2 delta function of xi + 1/I root 2 pi into 1 over xi, so this is exactly the Fourier transformation of the heavy side function.



So you may come across this heavy side function or delta functions in applications, so that time when you need to apply the Fourier transformation so these are the transforms that are useful, so sometimes when you see that the Fourier transform of some function if we, that integral doesn't make sense or it is not converging it may basically it maybe some if you actually calculate it could be some delta function, some generalized function, okay, so such as delta function, so one such is the heavy side function, so we need only this heavy side function delta

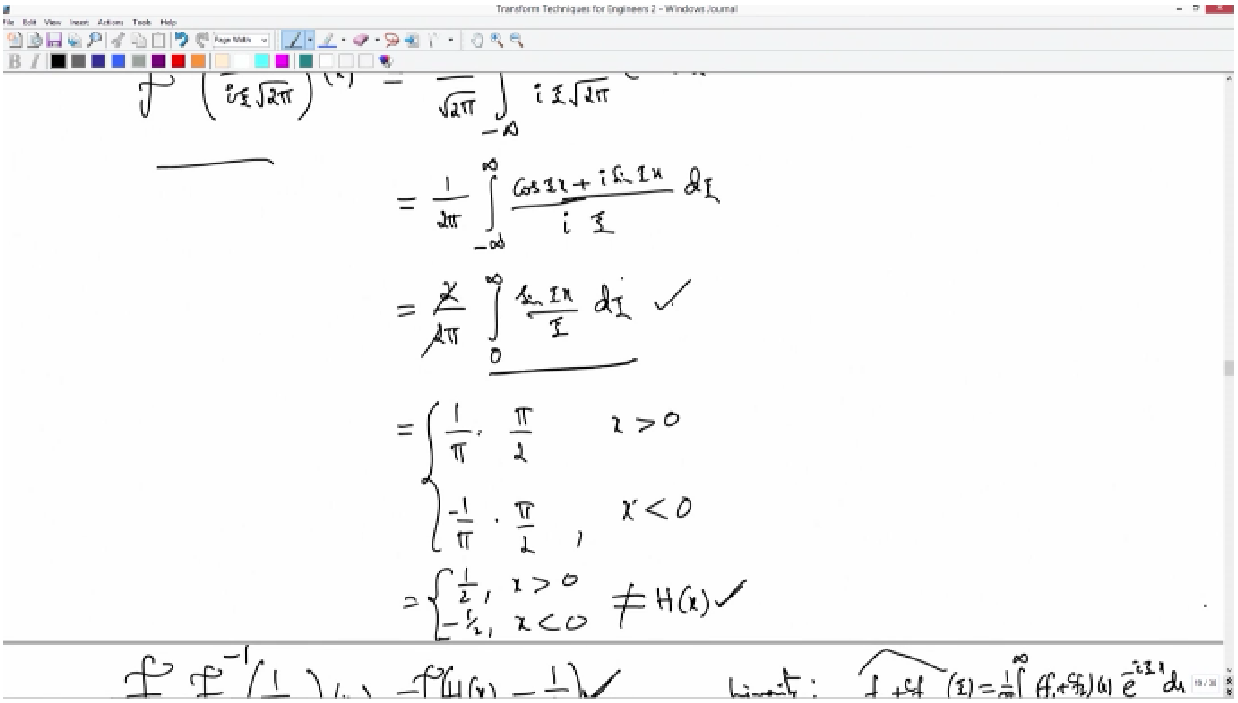


function whose Fourier transforms, so Fourier transformation of delta function goes to 1/root 2 pi inverse transformation of 1/root 2 pi will come back to delta function, so directly if you calculate the inverse transformation of 1/root 2 pi which is 1/root 2 pi - infinity to infinity, 1/root 2 pi this is a function and E power I xi X that is the inverse transformation of this which you think of delta function, right, so this is actually 1/2 pi - infinity to infinity E power I xi X D xi.

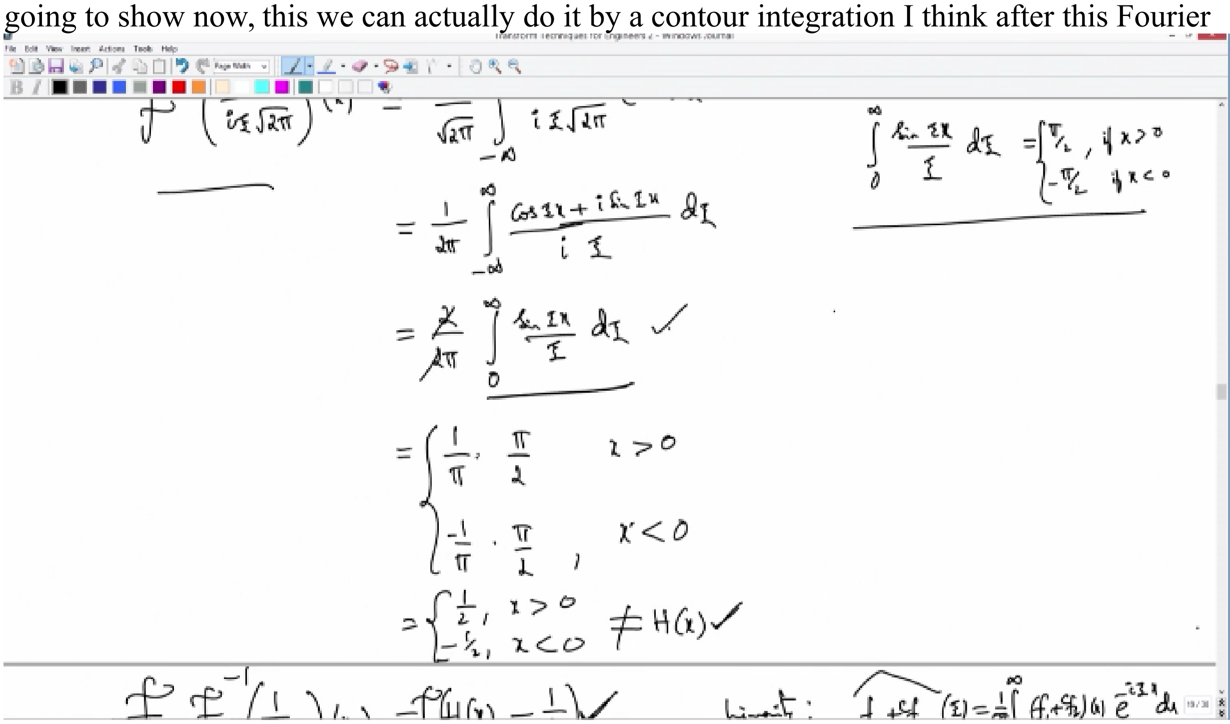


So this is another form for delta function, so another representation of delta function is so you have this oscillatory function E power I xi X, if you integrate from - infinity to infinity this is another representation of delta function, okay, so it actually tells you that if you fix X nonzero and what you have is E power is I xi X which is oscillatory and which is oscillatory like sine wave, so it gets canceled so finally 0, it's a kind of weekly, so that is the meaning of, because this equality is a weak thing, okay, so that's way so generalized functions when you, generalize function when you say generalized function is this, generalized function something is a generalized function that means the equality is a weak equality, okay, delta has some representation which has the limit of some though, okay.

We get another representation for this delta function, so with this we'll just do some more problems, and also earlier I'll try to do something, earlier we have derived while doing a problem earlier, in the earlier videos before I do this thing we have seen, yeah, so while doing

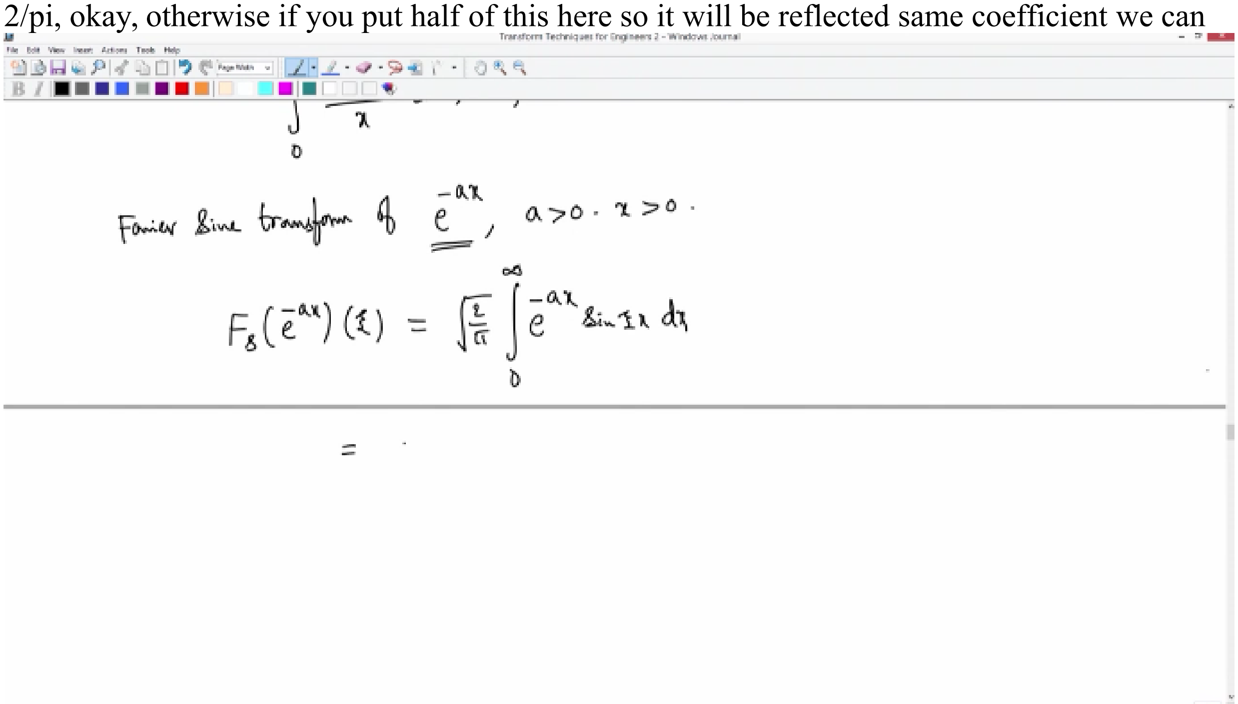


this Fourier transformation of 1/I xi root 2 pi we have used this integral, so this integral you can do many ways, you can evaluate this integral 0 to infinity sine xi X/X xi D xi this value is depending on what value of X? If X is positive this is actually pi/2 if X is positive, and if X is negative this is actually becoming –pi/2, that's what we have used here and this is what we are going to show now, this we can actually do it by a contour integration I think after this Fourier

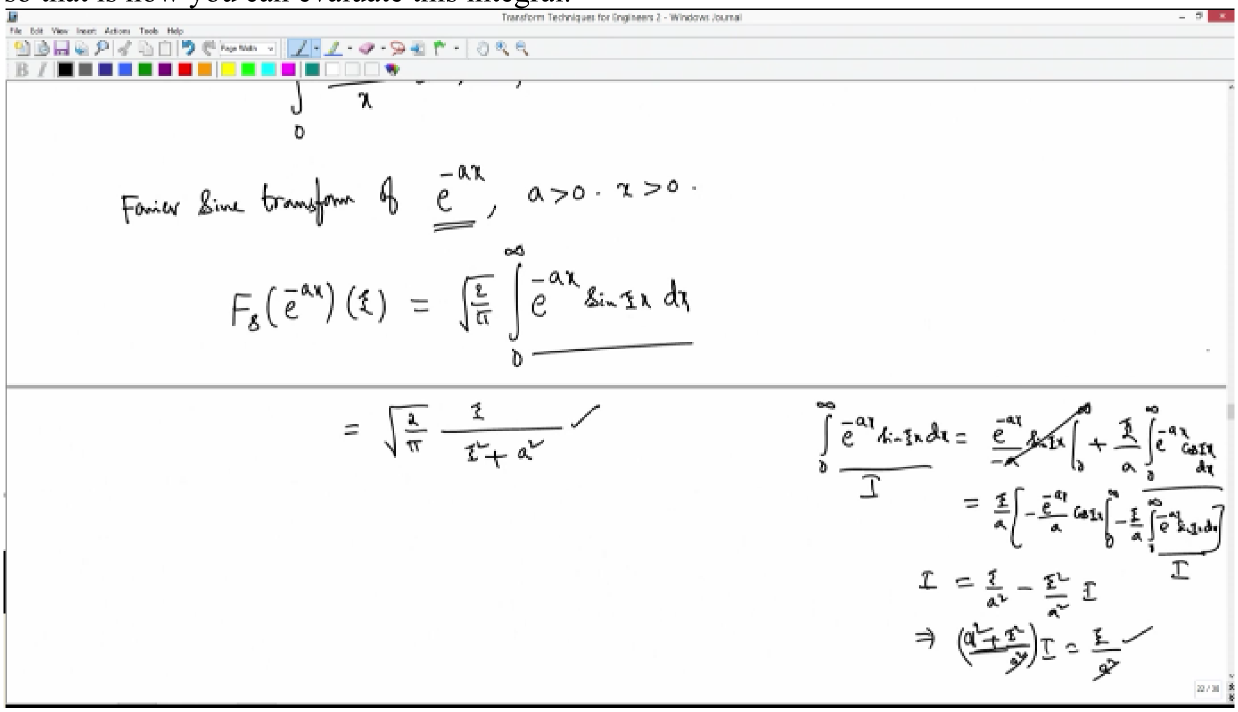


transformation before I do these applications of Fourier transformation, I'll try to give you some, I'll try to give the glimpses of complex variables how to do the integration in a complex plane, I’ll do the contour integration that is one way, one way you can do this, you can evaluate this integral by the contour integration technique, but now today we do only by Fourier

transformation because you know only Fourier transformation of some function may be comes something and when you fix those X values you may end up getting this integral, so through that way I will try to prove this result, we evaluate this integral sine xi X/X DX, okay for xi, xi belongs to, or okay, for different xi, xi is positive or negative, so we will try to show this one. What we do is, to do this, now not only we have a Fourier transform of a function full Fourier transform, if the function is defined only for positive values for example this function sine xi X/X this once you know, once you fix your xi, X is actually defined between 0 to infinity, so this is defined only one side 0 to infinity, so you can use either cosine transform or sine transform for this function.
So let's start with, so to get this integral in some Fourier transform of something, let us do Fourier transform, Fourier sine transform of E power - some AX let us say, A is positive, and X is positive, okay, so it's defined only for one side, so you choose such a thing so you have Fourier sine transform of E power –AX(xi) this is a function of xi, so this is equal to root pi/2 or rather 2/pi, right, so what is that we get this Fourier sine transform, cosine transform you have some coefficient even Fourier transform you have a coefficient which we split into 1/2 as 1/root pi the inverse transform is reflected as 1/root pi, so here we can get for Fourier sine transform root 2/pi, ok we split that coefficient as root 2/pi and the inverse transform it is root 2/pi, so without splitting also you can do, but the thing is if I define without this coefficient you have actually root 2/pi integral 0 to infinity, this function is E power -AX sine xi X, because it's a sine transform and you have DX, so this is your integral.
If without this one if I define in the inverse transform you get, you get the full coefficient that is 2/pi, okay, otherwise if you put half of this here so it will be reflected same coefficient we can

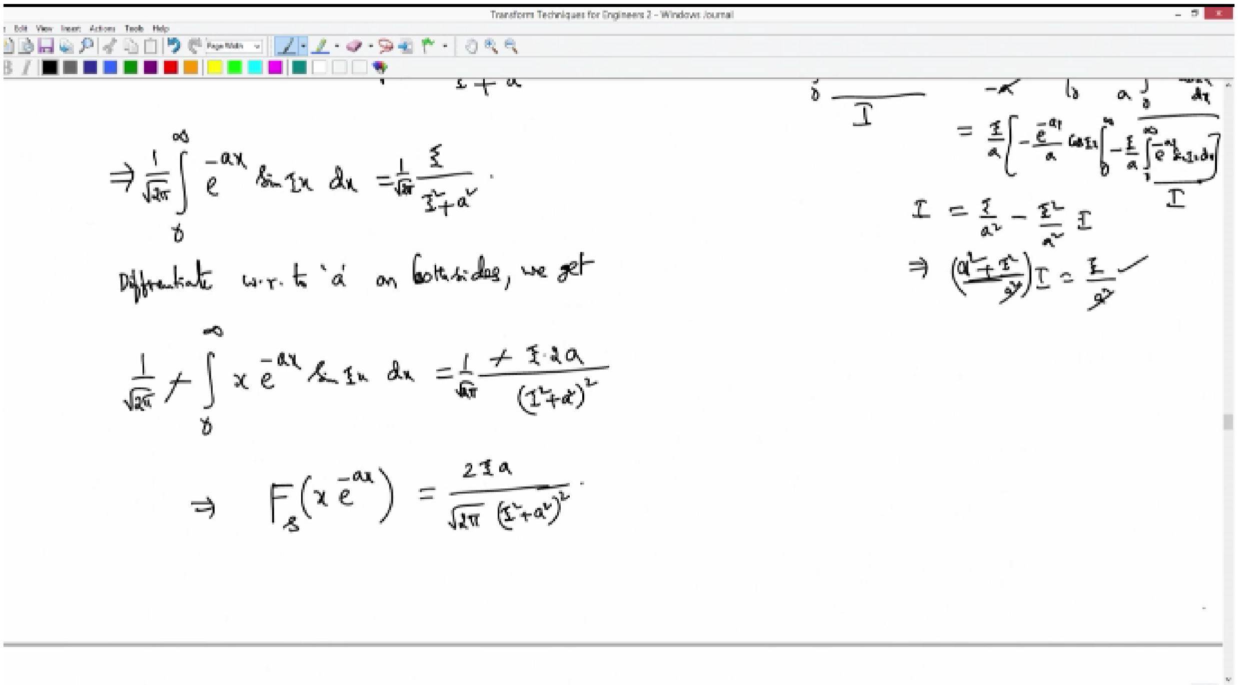


maintain in your inverse Fourier sine transform, so you can evaluate this integral and you see that it is 2/pi, so to avoid this what is the integral you get? You can see that this integral is, do this integration by parts as an exercises, this is elementary integral we can easily see that is going to be xi divided by xi square + A square, so how do I evaluate this? This is simple integral so E power -AX sine xi X DX, so you do one time integration, so you see that it is going to be E power –AX/- A sine xi X, you do it from 0 to infinity minus, minus plus so you have E power –AX/A and you have this integral 0 to infinity E power –AX, this you put it inside and then you have to differentiate this sine, so you have finally comes out as xi, so cos -AX now sine xi X DX, okay, this minus comes because your derivative of cosine is minus, so finally again here so you have xi/A square here, infinity part is 0, so at 0 contribution this is simply 1/A, so you have this –xi square/A square this is actually your original integral, so if you call this I, this is I, so you have I, so you can see that this is going to be 1 + xi square/A square into I = xi/A square, so A square A square goes, okay, so what you get is, I is xi divided by this, so that is how you can evaluate this integral.



So once you have this Fourier sine transform what I do is I differentiate this because A is positive, A is defined, A belongs to between 0 to infinity, so it is a differentiable function you can easily see that it is a differentiable function, A is never going to be 0, its 0 to infinity, so I can differentiate this quantity, this integral so I'll just from this what I get is 0 to infinity E power -AX sine xi X DX = xi divided by xi square + A square, both sides I differentiate with respect to A.

On both sides we get, what we get if you differentiate this is if you differentiate with respect to A, you ignore this, this is the integral with constants, if you use a Leibniz rule of differentiation so rule there is a constant that is nonzero, so that parts, two parts will be 0, so only the derivative of with respect to A is - integral 0 to infinity X E power – AX, sine xi X DX, which is equal to, and this one you can simply xi square + A square whole square, xi square + A square differentiate with respect to A that is 0, and you have a minus, and you have xi times into 2A, if you differentiate xi square + A square, so this minus minus goes, and what you end up is Fourier sine transform, this is actually and you also multiply so you can take and keep this as it is root 2 pi, if you want root 2 pi, and what you get up is 2 pi, 1/root 2 pi, what you end up is basically Fourier sine transform of X E power -AX = 2 xi A divided by root 2 pi xi square + A square whole square, actually I think we have to integrate to get the result, okay.



The same thing instead of differentiation what we should do is integration, so this is anyway this really, we really don't need is actually what we should have done the integration, so this equality you do the integration with respect to A, I will try how I do this, so let me write this, integrate the equality that means this one, this one with respect to A from, from A to infinity, then we get, what we get we can see, so what we get is 1/root 2 pi if you integrate this from E power -AX what you get if you integrate E power –AX/minus of that A divided by X, okay, this is the integration of with respect to A, you are integrating DA, integrating with the DA variable, so this one and you are doing it from A to infinity and sine xi X, this is as it is, this is equal to 1/root 2 pi and what you hear is you integrate from 0 to A to infinity xi divided by, xi comes out which is not a function of this, so 1 divided by xi square + A square DA, where is the meaning, okay.

So again so I can remove this 1/xi on both sides, so what you get is so if you see this this is contribution is 0 so you end up getting 0 to infinity sine xi X divided by X, right, so you are substituting A equal to A to infinity, A equal to A, so A you are substituting so in the place of A, you put A, one is A, so that is going to be E power -AX divided by X, okay, that's what you have, so basically what you have is only at A equal to A contribution that is minus minus plus, so you have at A equal to A divided by X into sine xi X DX, if you keep these constants these are actually Fourier sine transform of this function, this is equal to xi/square root of 2 pi and here this is from A to infinity, you try to write xi by, so xi square you take it out, so you have 1/xi, xi xi goes, here 1/xi and you have DA/1 + A/xi whole square, so what is this integral? Integral is, so you can take this 1/xi also inside, so D(a/xi) you can put it, 1/xi is a constant, so you can write like this, so this is, if I write A/xi as X, DX/1+ X square integral A to infinity, okay.

Let me use in a more elementary way, so A/xi, let A/xi = T, then that implies D(A/xi) = DT, okay, and what happens to your T? T is A/xi, when A = A so you have, what happens to your integral is, integral A/xi and this is going to be infinity divided by xi, xi is fixed constants, so

Integrate the above equality w.r.to 'a' from a to ∞; we get

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \left[-\frac{e^{-ax}}{x}\right]_{a=a}^{a=\infty} \sin \xi x\, dx = \frac{\xi}{\sqrt{2\pi}} \cdot \int_a^\infty \frac{1}{\xi^2+a^2}\, da .$$

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-ax}}{x}\sin \xi x\, dx = \frac{1}{\sqrt{2\pi}}\int_a^\infty \frac{d(a/\xi)}{1+(a/\xi)^2}.$$

$$a/\xi = t \Rightarrow d(a/\xi) = dt$$

$$= \frac{1}{\sqrt{2\pi}}\int_{a/\xi}^\infty$$

A/xi to infinity, and what you get is DT/1 + T square, so what is this one? This is 1/root 2 pi tan inverse of T from A/xi to infinity, so what is this one? This value is 1/root 2 pi, tan inverse infinity is pi/2, sine pi/2, so 1/cos pi/2 is 0, so that is this one, minus, minus tan inverse of A by, T is A/xi, right, so what I do here is this is, I allow as A goes to 0, I allow A goes to 0 both sides what you get is this Fourier sine transform of, or rather you see you cancel both sides if you want, okay, it's not required anyway, so 1/root 2 pi, 1/root 2 pi everywhere you can remove, so what you have, as A goes to infinity the left hand side becomes A to infinity, sine xi X/X DX which is equal to, okay, so you see that this is, when I substitute A/xi = T and I put

$$\frac{1}{\cancel{\sqrt{2\pi}}}\int_0^\infty \frac{e^{-ax}}{x}\sin \xi x\, dx = \frac{1}{\cancel{\sqrt{2\pi}}}\int_a^\infty \frac{d(a/\xi)}{1+(a/\xi)^2}.$$

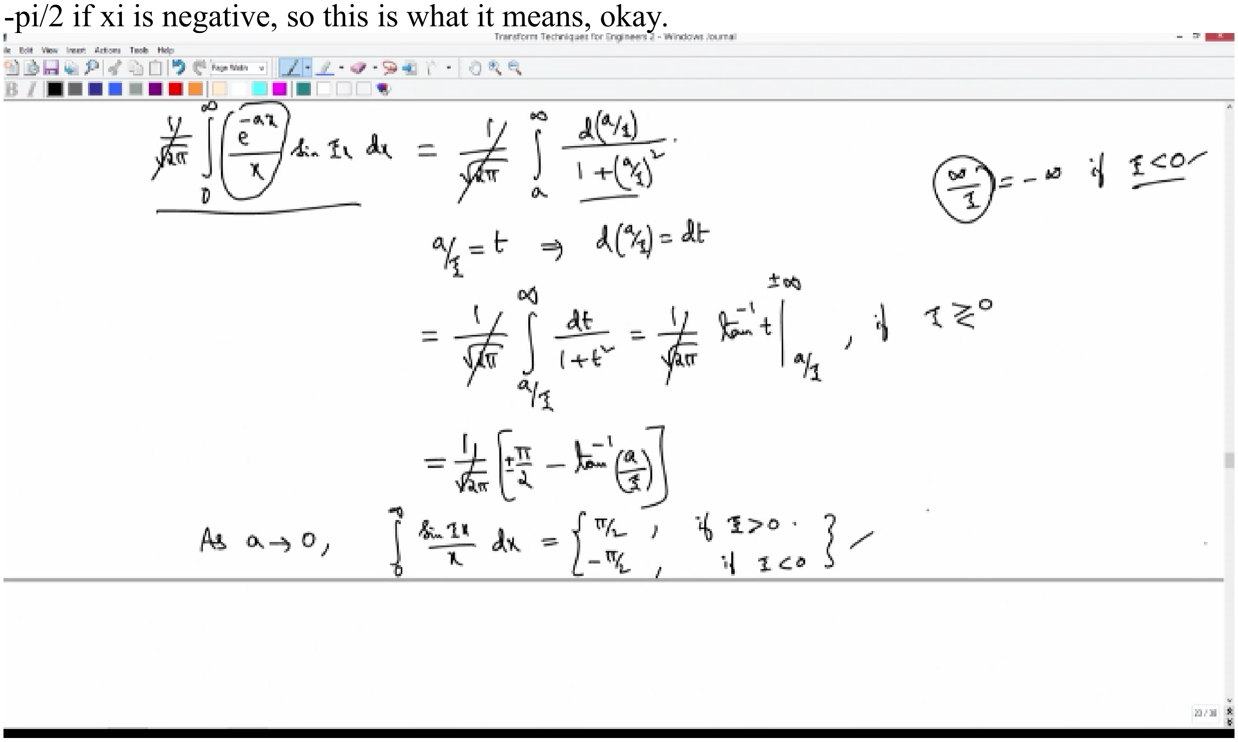
$$a/\xi = t \Rightarrow d(a/\xi) = dt$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}}\int_{a/\xi}^\infty \frac{dt}{1+t^2} = \frac{1}{\cancel{\sqrt{2\pi}}} \tan^{-1} t \Big|_{a/\xi}^{\infty}$$

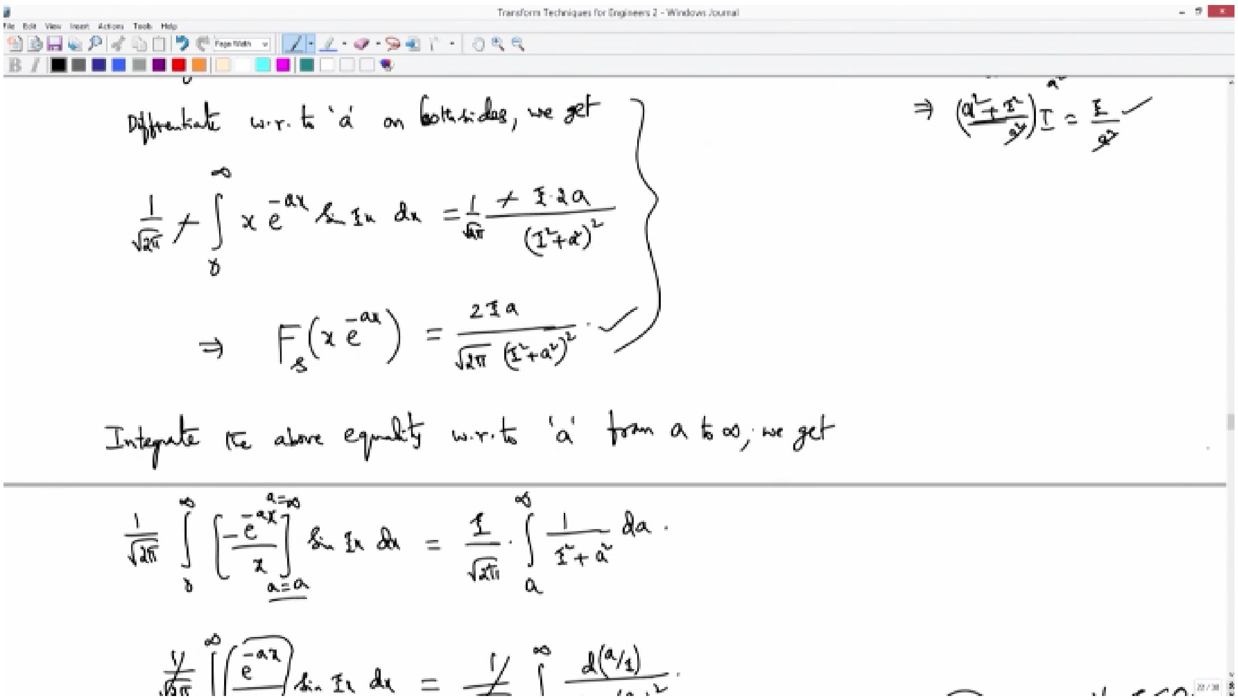
$$= \frac{1}{\cancel{\sqrt{2\pi}}}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a}{\xi}\right)\right]$$

As $a \to 0$, $\int_0^\infty \frac{\sin \xi x}{x}\, dx =$

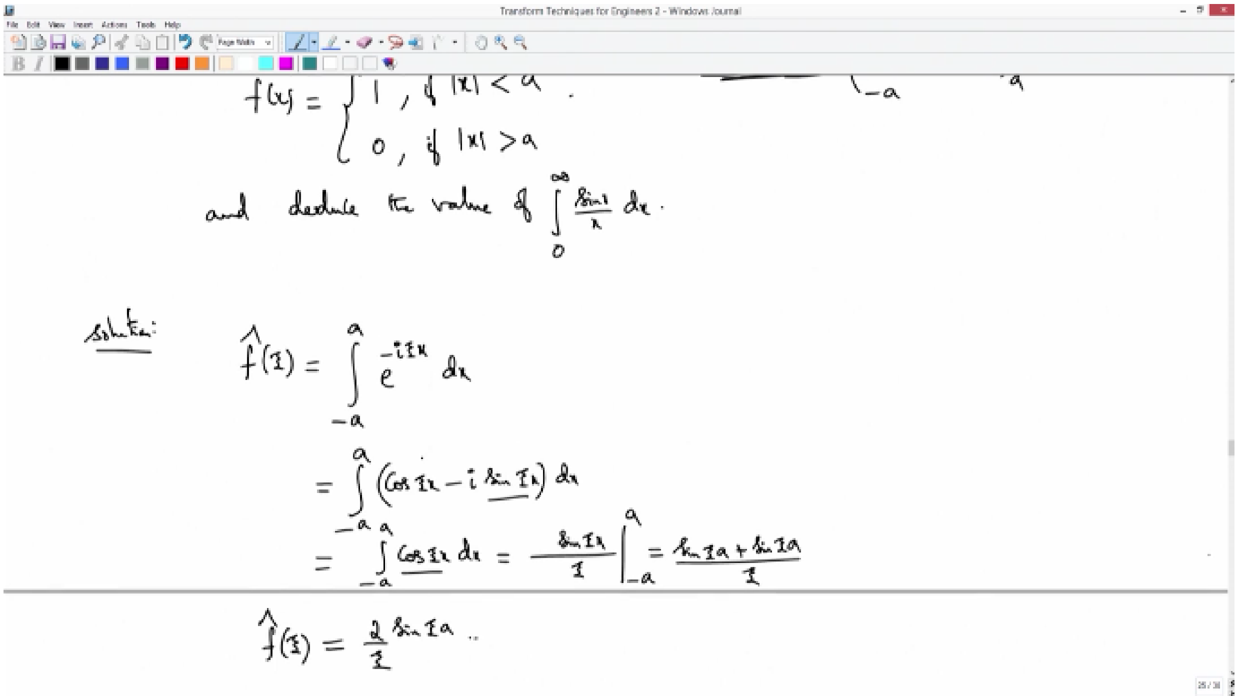
this, when A equal to A, A/xi, so this is a negative quantity if this is positive quantity if xi is positive, if xi is positive A is infinity this is going to be, T is going to be infinity that is fine, if xi is negative, okay, so this is where the difference is, xi is negative A/xi that is negative quantity, and here infinity/xi which is negative, infinity/-xi, xi is a negative quantity, this is actually minus infinity if xi is negative, okay, so that is where you have to see the difference, so if xi is positive what you get is pi/2 because and as A goes to 0 this is tan inverse 0, okay, so I have to write it here so this is going to be both + or – infinity, okay, so this is + or - infinity depending on if xi is less than, if it's positive this is an infinity, this is negative if I take xi as negative this is got to be – infinity, okay.
So when it is tan inverse of - infinity I should get + if it is xi is positive, at infinity it is + pi/2, at - infinity is actually –pi/2, so that's what is the case, so you see that clearly so if xi is this, so what you end up is if you allow A goes to 0 this tan inverse A/xi whether xi is positive or negative this is going to be tan inverse of 0 that is 0 otherwise it's simply pi/2 if xi is positive, -pi/2 if xi is negative, so this is what it means, okay.



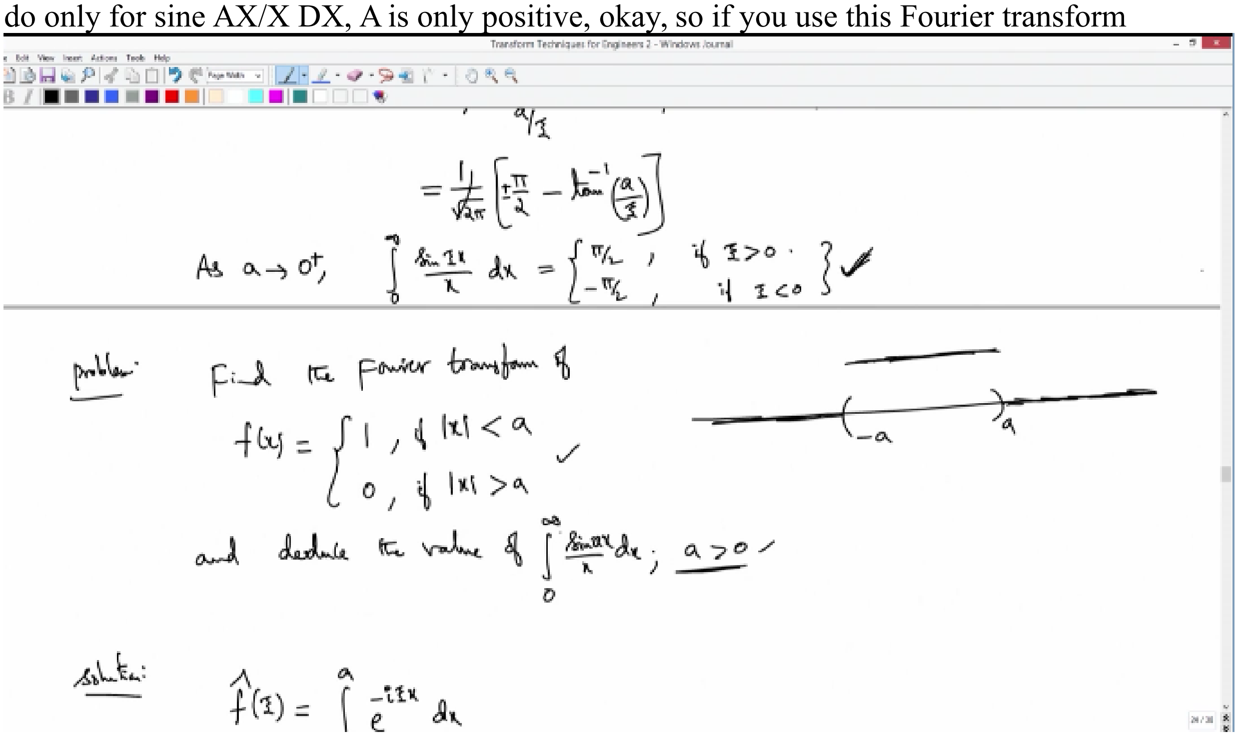
So this is the result you can actually, this evaluation of this integral you can also do by contour integration, and one way is just pick up E power –AX, X positive and A is also positive and when you allow A positive so you have to take positive side, because A is positive, okay. So in any case you could evaluate this integral which you have used earlier to find some Fourier transform of 1/I xi, inverse Fourier transform of 1/I xi you have used, 1/root 2 pi 1/I xi root 2 pi for that if you wanted to calculate the inverse Fourier transform that is where you have used this integral value so which we could show it by simple Fourier transformation of some simple function like E power -AX and then make use of this to evaluate this integral that is ultimately, okay.



So we'll do some more problems, let me do some more problems now to find, we will try to find Fourier transforms and then using that we can show something, so let us use, I'll start doing some Fourier transforms now, so I will define find the Fourier transform of the function F(x) which is defined as, if mod X is less than let us say some A and 0, if mod X is greater than A, such a function so if you can easily plot this is between minus, its open interval – A to A it is 1, otherwise it is simply 0, okay, such a function 0 and you have, this is 0, and you have 1 here, so such a function how do you find the Fourier transform? And then and reduce the value of integral 0 to infinity sine X/X DX you can do this problem, so solution, so what is the Fourier transform of F(x) that is F cap(xi), so because 0 outside A you have –A to A, function is 1, E power -I xi X DX, and this is actually integral -A to A cos xi X – I sine xi X DX, sine xi X in odd function so that is integral part is 0 this integral of that second term is 0, the only term remains is integral –A to A, cos xi X DX which is divided by xi – sine xi X, I think + right, if you differentiate this cos xi X into xi, so that xi xi goes, so what you have integrand, so this is from -A to A this you get sine xi A - of minus minus plus sine xi A divided by xi, so this is actually two times sine xi A/xi, this is your Fourier transform, okay.

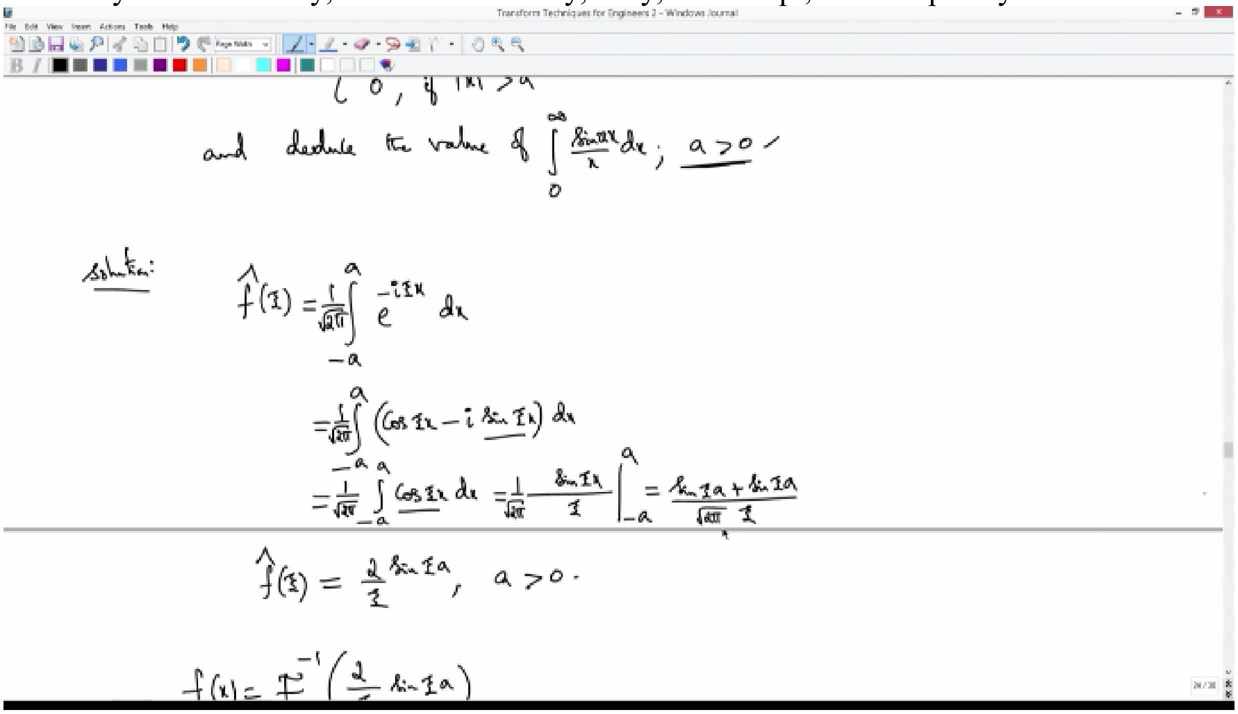


How do I get sine X/X or I think this is much easier, right, so you try to take the Fourier transform of this function maybe you are evaluating the same, so you can express this one, in fact you can get your sine xi X, right, sine AX/XA positive, A positive or negative doesn't matter, right, so we will see this one, so A is positive, so this is only restricted thing so we could do only for sine AX/X DX, A is only positive, okay, so if you use this Fourier transform

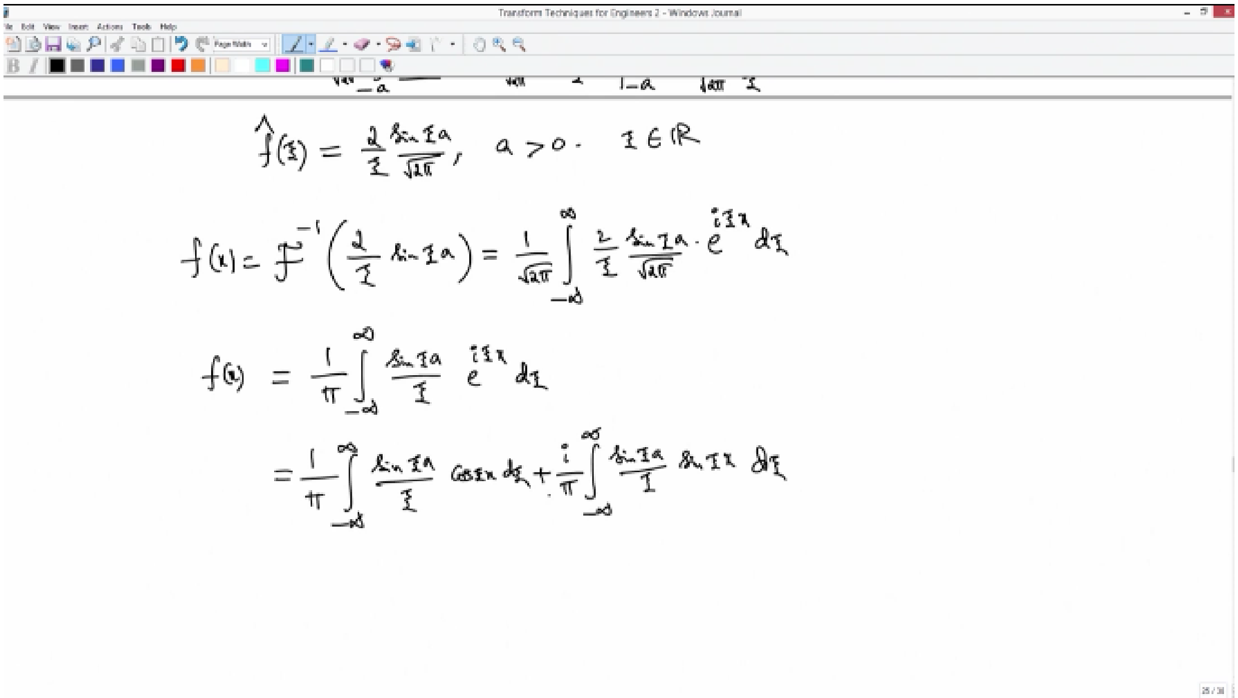


you could get only integral 0 to infinity sine xi X/X DX, xi is positive, only part of this result you can get it, you can also get it from this Fourier transform, so that's what it is here.
So if you take the inverse transform, this is the Fourier transform of this and A is positive anyway, and you have that you take the inverse transform you have F(x) is actually Fourier

inverse transform that is inverse Fourier transform of 2/xi sine xi A, I think I missed 1/root 2 pi, this is my conventionally, I define it this way, okay, 1/root 2 pi, 1/root 2 pi so you have root 2

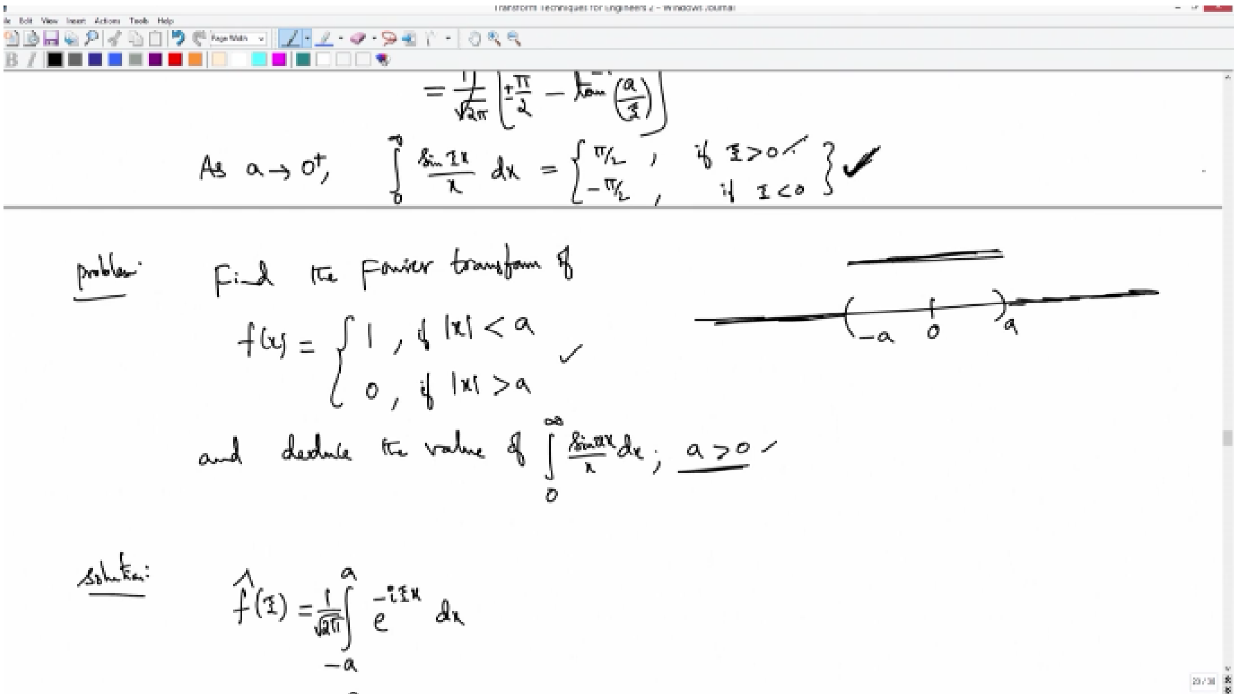


pi, so this was the Fourier transform and you take the inverse transform of this by definition, again you have 1/root 2 pi and you have from now xi is, xi belongs to full real line, so you have - infinity to infinity, 2 divided by xi, sine xi A, A is positive into E power I xi X D xi, so this is what exactly divided by root 2 pi, so this is 1 by, 2 2 goes you get 1/pi - infinity to infinity, what you get is sine xi A/xi E power I xi X D xi is my F(x), so you can expand this and evaluate - infinity to infinity sine xi A/xi, this is cos xi X DX + other part - infinity to infinity sine xi A/xi I times, so you have I comes out, sine xi X D xi, this is D xi, so you see that this is

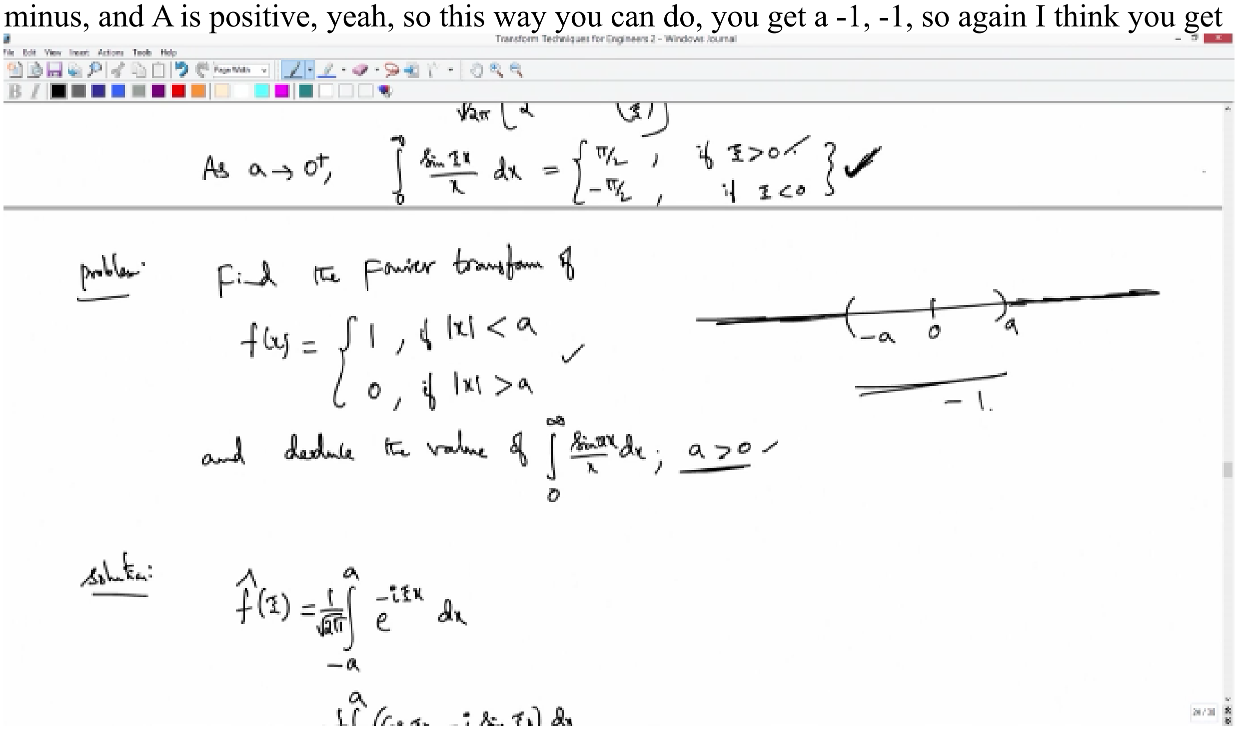


a real function, this is an odd function of xi, because the integral is from -infinity to infinity this whole function is you have 3 functions, product of 3 functions which are odd, so it is an odd function again, okay, so this is 0, this contribution is 0 so you end up getting 1/pi and this is even function, what you have is the even function here, so 2 comes out, 0 to infinity sine xi A/xi, cos xi X D xi is my F(x), yeah, this is my F(x).

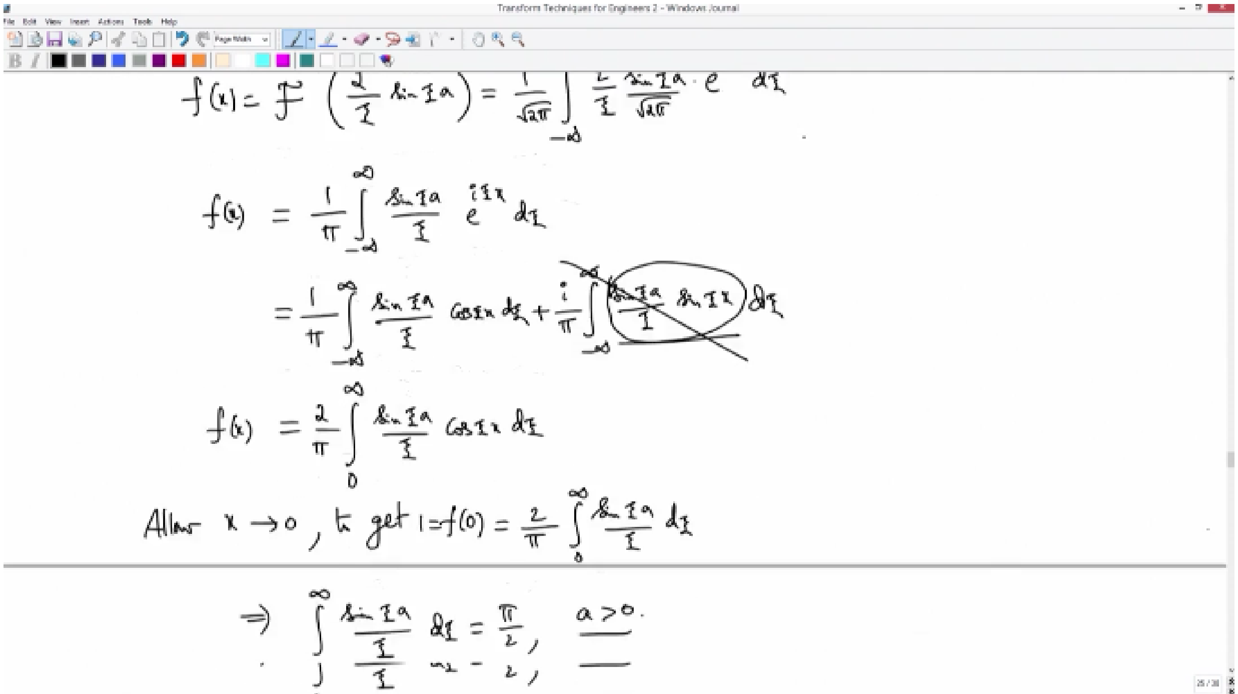
So what is my F(x)? So given what is your F(x)? One, if X is less than A, so if you allow X goes to 0, okay, allow X goes to 0 to get F(0) which is equal to 2/pi 0 to infinity sine xi A/xi, cos xi X as X goes to 1, so you have D xi, so you get F(0) is 1, F(0) is what? At 0 it is value is 1 so this is 0 its value is 1, so you have what you get is pi/2 so this integral sine xi A/xi D xi = pi/2, and A is positive that is what is given, okay, so there are many advantages just by using sine transform Fourier, directly if you apply the transform for Fourier transform for this function you can actually, you can also show this sine xi X/X, and part of it you could do, okay,



if you want A negative side can we also get the same result, other result that when A is less than 0, when A is less than 0 if you want, how you define you can define as -1 here, so if you take this to -1 for example if you take -1 what is the difference you get? -1 if you do you get this as minus, and A is positive, yeah, so this way you can do, you get a -1, -1, so again I think you get



the same, A is positive, no you don't, so you don't, so you get the same, even if you whatever you take you get the same say, part of this only one part of this result you will get, so by this same result you can get it from by many other, by taking Fourier transform of many functions,



okay, so far what we have seen is either by applying a Fourier transform on certain function or Fourier sine transform of an uncertain function we could evaluate certain integral, certain integrals that we want, we can evaluate certain integrals using just by Fourier transformations or that is either direct Fourier transform, full Fourier transform or sine or cosine Fourier transforms, okay.

So we will see some more examples how to evaluate certain integrals just by using these Fourier transforms, okay, later on so and then we'll move on to find the properties of the Fourier transforms and the main results that is a Fourier integral theorem which is done rigorously eventually, okay. So next few video we will concentrate on that, we'll see in the next video. Thank you so much.

Online Editing and Post Production

Karthik

Ravichandran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh Kumar

Web Studio Team

Anitha

Bharathi

Catherine

Clifford

Deepthi

Dhivya

Divya

Gayathri

Gokulsekar

Halid

Hemavathy

Jagadeeshwaran

Jayanthi

Kamala

Lakshmipriya

Libin

Madhu

Maria Neeta

Mohana

Muralikrishnan

Nivetha

Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Sorna

Subash

Suriyaprakash

Vinothini

Executive Producer

Kannan Krishnamurthy

NPTEL CO-ordinators

Prof. Andrew Thangaraj

Prof. Prathap Haridoss

IIT Madras Production

Funded by

Department of Higher Education

Ministry of Human Resources Development

Government of India

www.nptel.ac.in