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Transform Techniques for Engineers  
Fourier transform of a Heavyside function  
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# Transform Techniques for Engineers

## *Fourier transform of a Heavyside function*

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Welcome back, in this last video we have seen an intuitive proof for the Fourier integral theorem and based on that you take a signal of a non-periodic signal that's a function over full real line, and then if you apply the Fourier integral theorem you can see that the function is actually equal to some integral, so we have seen some examples and in the process you are actually calculating Fourier transform of the function inside the Fourier integral theorem, so when you're calculating the Fourier transform of such functions what you have is, you also have calculated what is the Fourier transform of a delta function.

While doing this Fourier transform of the delta function, you have some legitimate question that

$$\delta(x) = \frac{1}{\sqrt{2\pi}}$$

$$\delta(x) := \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases} = \lim_{\epsilon \rightarrow 0} f_{\epsilon}(x), \text{ where } f_{\epsilon}(x) = \begin{cases} \frac{1}{\epsilon}, & x \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{\delta}(i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) e^{-i1x} dx$$

$$\stackrel{!}{=} \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) e^{-i1x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} e^{-i1x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \left. \frac{e^{-i1x}}{-i1} \right|_{-\epsilon/2}^{\epsilon/2}$$

if you look at this, Fourier transform of grade, see this delta function, Fourier transform of this delta function which is  $1/\sqrt{2\pi}$  integral - infinity to infinity, and this delta function we replaced as this way, so delta function by definition is this which is a limit of some usual functions, normal piecewise continuous functions you can see here, they appear to be like, this looks like box kind of functions, so every time as you increase a, decrease epsilon it will be going, so it's finally it's going to be point mass function, as epsilon goes to 0 you have at 0 its Infinity, and at nonzero it is simply 0 so that is what is, that's the limit we have chosen so here that is the limit that's how you simply take this, this is not a function, this is clearly this is not a function which at 0 it is infinity, and at nonzero values it is 0, it's not a function but it is limit of usual functions as you can see, but once you write this limit of usual functions, for example

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\*  $\delta(x) = \frac{1}{\sqrt{2\pi}}$  ✓  $\lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) = \delta(x)$  ✓

$\delta(x) := \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases} = \lim_{\epsilon \rightarrow 0} f_{\epsilon}(x)$ , where  $f_{\epsilon}(x) = \begin{cases} \frac{1}{\epsilon}, & x \in [-\epsilon, \epsilon] \\ 0, & \text{otherwise} \end{cases}$

$\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) e^{-ix} dx$

$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) e^{-ix} dx$

$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\epsilon/2}^{\epsilon/2} e^{-ix} dx$

limit of  $f_{\epsilon}(x)$  as  $\epsilon$  goes to 0, so this means this is equal to, so you got delta function, right, so this limit usual way usually when you write this is converging to some  $F(x)$  this means a point-wise convergence, so for example if you fix  $X$  and this function, this sequence  $f_{\epsilon}$  for different  $\epsilon$  values of  $X$ , for different  $\epsilon$  values of  $f_{\epsilon}$  of fixed  $X$  that those sequence of, those values converges to that limit, limiting value will be  $F(x)$ . But here if you look at, if you take  $X = 0$  in the delta function case, when you put  $X = 0$  what you get is, this is infinity and what you end up here is  $f_{\epsilon}$  of 0 that  $1/\epsilon$  limit of  $1/\epsilon$ , so as  $\epsilon$  goes to 0 that is what happens at  $X = 0$ , so it's not converging that means this is not converging at  $X = 0$ , clearly it is not the point-wise convergence, these functions  $f_{\epsilon}(x)$  converges to delta  $X$  not point-wise, but any other place, every other place as  $X$  is not equal to 0, point-wise convergence is assured, okay, so after some  $\epsilon$  smaller and smaller this is actually becoming 0, so it is 0.

So when  $X$  is not equal to 0 point-wise convergence is there, but at  $X = 0$  it is not point-wise convergence, so this is not point, this limit is not, this converge, this means not  $f_{\epsilon}(x)$  is not converges to delta  $X$  point-wise, okay, so point-wise you fix  $X$  and then look at its convergence as a sequence of functions.

In mathematics is to say this is also called in which metric this converges, convergence once you write the convergence this means the distance that is a metric in which it is converging, this is real number, these are real numbers in the metric, modulus is the distance that is  $f_{\epsilon}(x) - \delta(x)$ , so  $\delta(x)$  is usual function that is how you define what is point-wise convergence, okay this goes to 0 as  $\epsilon$  goes to 0, that is the meaning, okay, so this is the meaning so point-wise convergences itself is not there so that means it's not even known uniform convergence, so if you know that is uniform convergence and you can take this limit inside and that is exactly these two are equal, in that case only these two are equal, but your convergence is not neither point-wise nor uniform convergence here, so what it means is actually this means some usual functions converging to some non-usual function, so this is not a generalized function, this generalized function, this is generalized delta function, delta function is an

example of a generalized function which is not a function but as a limit of usual functions, but not in the usual sense but as an average sense, so that means I'll try to multiply this  $F_\epsilon(x)$  usual functions, I multiply some function which is a smooth function that means infinitely many differentiable function, infinitely differentiable function outside some closed and bounded set it is 0, so that means it has a, there is also called in mathematics as a compact support.  $G(x)$  is infinitely differentiable function and  $G(x) \neq 0$ ,  $G(x)$  is the set of all which is nonzero,  $X$  such that nonzero is actually can put it in some closed interval  $A, B$  okay, so such a thing is where  $A, B$  are finite numbers, okay, that's what it is, that is the meaning of - infinity to infinity, so there are finite numbers, so if you can put this in some interval so this is called a compact support such a function you consider so  $G$  is infinitely differentiable function, and so  $G$  belongs to  $C^\infty$  so that is the meaning of infinitely differentiable function, if I write here  $C$  or  $C^0$ , if you write like this these are infinitely differentiable function whose wherever, the set of all  $X$  such that  $G$  is nonzero that's the domain at which  $G$  is nonzero that is all you can put it in a closed interval, so that why did you choose such a functions, you can choose any  $G$ , for all  $G$  when I multiply so this integral makes sense, so this integral it makes sense, so this is finite because  $G$  is differentiable and this  $F_\epsilon(x)$  is anyway it's integrable, so this product is also integral function, so if you take this averaging, so what happens? This means a limit of this that is the meaning, so limit of epsilon goes to 0 is actually equal to, again the average value of whatever happens here so this means so  $\delta(x) \int G(x) dx$ , and we know that this value is actually  $G(0)$ , so that is a meaning.

$$\delta(x) = \frac{1}{\sqrt{2\pi}}$$

$$\delta(x) := \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases} = \lim_{\epsilon \rightarrow 0} f_\epsilon(x), \text{ where } f_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & x \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \\ 0, & \text{otherwise} \end{cases}$$

$$\hat{\delta}(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\epsilon \rightarrow 0} f_\epsilon(x) e^{-izx} dx$$

$$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_\epsilon(x) e^{-izx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} e^{-izx} dx$$

generalized function.

$$\lim_{\epsilon \rightarrow 0} f_\epsilon(x) = \delta(x)$$

$$\Leftrightarrow \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_\epsilon(x) g(x) dx = \int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0)$$

So when you say this delta function is a generalized function which is a limit of usual functions what I mean is actually this, this is the definition of, okay, this is also definition of this limit is, limit of this if this is the case okay, so if you have the limit of you averages so this means this  $F_\epsilon(x)$  converges to delta  $X$  in an average sense, you are averaging that means your integration is an adding and, right, addition of all these things, so you are averaging it, so this is a kind of average, in average sense approx, and this kind of what do you say average value of it okay, so you consider the  $F$  value on an average between - infinity to infinity, so once you add it this is true for every  $G$ , for every  $G$  in  $C^\infty$  functions of real line okay,  $C$  or  $C^0$  infinity,

whatever,  $C_0$  means outside some closed and bounded interval this is 0,  $G$  going to be 0 okay, so you can say you can also say compact supports, and so  $C$  is a word for that so you use this one, so  $G$  is 0 outside for some closed interval once you fix  $G$ , so for every  $G$  if you take here if this is the case if this is true once you see that these are numbers, these are your usual numbers, okay, so you see that you know with delta function we have seen that property that this is actually  $G(0)$  it makes sense, these are numbers and these are numbers, this is the usual limit, this is the usual limit that modulus of this quantity minus this quantity can be made as small as possible as epsilon is close to 0, so that is the usual Euclidean metric, I mean that distance, Euclidean distance this converges to this, so this is the usual convergence but this is not, this is a weak convergence, weak limit, also called a weak limit, this is called weak limit and this is a usual limit, okay, usual limit in norm, okay, so this is what you see.

Handwritten notes on a whiteboard defining the delta function as a weak limit of a sequence of functions.

\*  $\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}}$  ✓

Def: (weak limit)  $\lim_{\epsilon \rightarrow 0} f_\epsilon(x) = \delta(x)$  if

$\delta(x) := \begin{cases} \infty, & x=0 \\ 0, & x \neq 0 \end{cases} = \lim_{\epsilon \rightarrow 0} f_\epsilon(x)$ , where  $f_\epsilon(x) = \begin{cases} \frac{1}{\epsilon}, & x \in [-\epsilon, \epsilon] \\ 0, & \text{otherwise} \end{cases}$

$\lim_{\epsilon \rightarrow 0} \int f_\epsilon(x) g(x) dx = \int \delta(x) g(x) dx = g(0)$  ✓

usual limit  $\forall g \in C_c^\infty(\mathbb{R})$  ✓

~~[ ]~~

$\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}} \int \lim_{\epsilon \rightarrow 0} f_\epsilon(x) e^{-ix} dx$

$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \int_{-\epsilon}^{\epsilon} f_\epsilon(x) e^{-ix} dx$

$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\epsilon}^{\epsilon} e^{-ix} dx$  ✓

So this is the meaning of this limit of usual functions converging to delta  $X$  in a weak, in the weak sense, okay, because this is not a function at all if it is a usual function then you can say, you can ask whether it is a point-wise convergence, certainly if it is a usual function so the limit of  $F_\epsilon(x)$  equal to some usual function  $F(x)$ , certainly it's at least it's a point-wise convergence, you can think of or you can also ask more questions like whether it is uniform converging or not, such a thing is not here so here it's only weak convergence when it is converging to some generalized function which is not the usual function you call it, it converges to that function on an average, that means what you are getting ending up as a limit is not a function, but if you multiply with some smooth nice functions and you integrate it, so integration makes sense with such a thing if it is, then you have such an integration is, these are numbers so that's a usual limit, so if such sees like, that is what I mean by this convergence, this limit of usual functions converges to a non-conventional function like generalized function or delta function here.

So delta function I have chosen such  $F_\epsilon$  like this, and whose limit is actually delta function, that's what you can easily see this, okay, so such a thing once you have this, this is exactly, this is actually the meaning of your, this is the meaning of this limit  $F_\epsilon$  goes to 0 is delta  $X$ , that is the meaning of exactly this one, right,  $F_\epsilon$  now I can take this, this

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\*  $\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}}$  ✓

Defn: (weak limit)  $\lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) = \delta(x)$  if  $\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) g(x) dx = \int_{-\infty}^{\infty} \delta(x) g(x) dx = g(0)$  ✓

where  $f_{\epsilon}(x) = \begin{cases} \frac{1}{\epsilon}, & x \in [-\frac{\epsilon}{2}, \frac{\epsilon}{2}] \\ 0, & \text{otherwise} \end{cases}$  ✓

generalized function.   
 usual limit  $\forall g \in C_c^{\infty}(\mathbb{R})$  ✓

~~[ ]~~

$\hat{\delta}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\epsilon \rightarrow 0} f_{\epsilon}(x) e^{-ix} dx$  ✓

$\stackrel{?}{=} \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f_{\epsilon}(x) e^{-ix} dx$  ✓

$= \frac{1}{\sqrt{2\pi}} \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{-\frac{\epsilon}{2}}^{\frac{\epsilon}{2}} e^{-ix} dx$  ✓

actually goes to delta function, right, so now it makes sense, so this is true, okay, this is a weak sense so that is the meaning of it, okay, so that's how you can justify this bringing this limit outside, okay.

And in the last video we also have seen, we have defined what is a heavy side function, so heavy side function is this, so which is 0 on the left side and it's 1 at X equal to, X positive side. And one can easily see the delta function which you know as a generalized function is a derivative of some this heavy side function, heavy side this function is well defined piecewise continuous function and it is not differentiable function at X = 0, so but if you can differentiate at that point it is actually, it's not defined, the derivative at X equal to not 0, but you can differentiate any generalized function and you can differentiate any N number of times, so that is, that if you study generalize functions you will understand that, so in that sense we look at

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

$$\delta(x) = \frac{d}{dx}(H(x))$$

$$\hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} (H(x)) = i\xi \hat{H}(\xi)$$

$$\hat{H}(\xi) = \frac{-i}{\xi \sqrt{2\pi}}$$

$$\hat{f}'(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\xi x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx = i\xi \hat{f}(\xi)$$

this delta function as a derivative of such a thing, we will see, we will try to prove how this delta function is, in this video we will try to prove this delta function is nothing but a derivative of this heavy side function.

So if you look at this  $H(x)$  0 if  $X$  is less than 0, and 1 if  $X$  is positive, okay, and what is that you want, so you want right hand side  $D/DX$  of  $H(x)$ , so what is this? By definition this is the limit of  $H(x + \Delta x) - H(x)$  divided by  $\Delta x$ , as  $\Delta x$  goes to 0. So what I'd choose, if I choose  $\Delta x$  goes to  $0^+$  okay, if  $\Delta x$  is this, what happens to this limit? This limit is  $\Delta x$  goes to 0, so if it is positive this value is 1 okay, even if  $X$  is not equal to 0 or rather if  $X$  is equal to 0, if  $X$  is greater than or equal to 0 suppose you define it like this, it doesn't matter, okay, so let's not define anything value at  $X = 0$ , so if  $X$  is positive what happens?  $X$  is positive,  $H(x)$  itself is 1,  $-H(x + \Delta x)$  is which is also 1, so which is still positive divided by  $\Delta x$ , okay, so this is the meaning, so if this is the case and you see that this is 0, value is 0 for  $X$  positive,  $X$  positive derivative is 0, so what happens at  $X$  negative side, limit  $\Delta x$  goes to 0 so the moment you say  $X$  positive either  $X$  positive or 0 or side, 0 side yeah so let us take only  $0^+$  side, now we will see what happens when you take  $X$ , what happens to this limit? Limit  $\Delta x$  goes to  $0^-$ , okay, both plus or minus, okay, so no, no, when  $X$  is positive as  $\Delta x$  goes to 0 either positive side or negative side of  $X$ , okay,  $\Delta x$  you reduce, even if  $\Delta x$  is very small  $X - \Delta x$  is still positive such way you can choose your  $\Delta x$  sufficiently small so that  $X - \Delta x$  is still positive.

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\frac{d}{dx}(H(x)) = \lim_{\Delta x \rightarrow 0} \frac{H(x+\Delta x) - H(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1 - 1}{\Delta x} = 0, \quad x > 0 \checkmark$$

$$= \lim_{\Delta x \rightarrow 0^-}$$

So in that way, in both the cases delta X is going to 0 this is always H(x + delta x) is 1, -H(x) when X is positive, okay, if X is positive this is always true and you go delta X is a positive side or negative side.

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$

$$\frac{d}{dx}(H(x)) = \lim_{\Delta x \rightarrow 0} \frac{H(x+\Delta x) - H(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{1 - 1}{\Delta x} = 0, \quad \text{if } x > 0$$

$$\frac{d}{dx}(H(x)) = \lim_{\Delta x \rightarrow 0} \frac{H(x+\Delta x) - H(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0}$$

if  $x < 0$

Now you look at a derivative of H(x) as a limit delta X goes to 0, H(x+ delta x) -H(x)/delta X, now if you choose if X is negative, what happens? Then limit delta X goes to 0, what happens? H(x-), X is negative I can add or subtract if delta X is, depending on delta X is the negative or positive X + delta X or X - delta X because X is negative, what is still negative? So you have 0 - X is negative 0 divided by delta X, this is also 0, okay, in both the cases you see that D/DX of H(x) is actually 0 if X is not equal to 0, that is what you have seen this is exactly almost

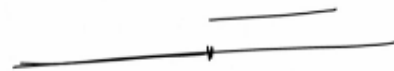


matching for delta when X is not equal to 0, what happens here? So if you choose now D/DX of H(x), at X = 0, if you want to calculate this you take limit delta X goes to 0, H(delta x) - H(x)/delta X, so H(0) so this is 0, this is by delta X, so what is this one? This is H(delta x), delta X is positive or negative, so you have a 0 - H(0), H(0) let us here itself we will let us choose as 0, or let's start with this side, at 0 you take it as 1, if you take like that this is going to be -1 divided by delta X, limit delta X goes to 0, so this is actually infinity, okay.

So what exactly, how did we define H(x) earlier? We define negative side is 0, so let's choose a negative side is 0, and positive side only it's just 1, like a step, okay here up to 0 it is 0 and from more than one it is 1, so something like that if you choose, if you choose like this H(delta x) is 1, because delta X goes to 0+ let us take 1 - H(0) is 0 in that case this is delta. If I choose here limit delta X goes to 0-, anyway once this is not defined, this limit, this derivative has a limit which is actually infinity, so but if I choose let me see what happens if I choose limit delta X goes to 0-, 0- means and you have H of, it's delta X will be negative, H of negative is 0, -H(0) is 0 divided by delta X which is 0, so it's only one limit is 0 other limit is infinity, so basically that means it is not defined, okay, when you say something is infinity even your delta function like this when you say delta function X equal to infinity means it's not defined, that is the meaning of writing infinity, so here both the limits are not defined here, both sides two different



$$\delta(x) = \frac{d(H(x))}{dx} = \begin{cases} 0, & \text{if } x \neq 0 \end{cases}$$



$$\begin{aligned} \left. \frac{d(H(x))}{dx} \right|_{x=0} &= \lim_{\Delta x \rightarrow 0} \frac{H(\Delta x) - H(0)}{\Delta x} \\ &= \left. \begin{aligned} \lim_{\Delta x \rightarrow 0^+} \frac{1 - 0}{\Delta x} &= \infty \\ \lim_{\Delta x \rightarrow 0^-} \frac{0 - 0}{\Delta x} &= 0 \end{aligned} \right\} \end{aligned}$$

limits this may be 0, this one side limit is 0 other side what you have is, other side limit is simply infinity so it's not defined, so that way you what you have is you have a infinity if X = 0, it's not defined, so that is the meaning of a delta function, so this implies derivative of delta function is nothing but simply H dash(x).

What does it mean? Again if this is in the weak sense, okay, so this means so when I write something which is, it's a usual function which is piecewise continuous function which you are writing a derivative which is C, it's not differentiable function but you are differentiating it, so some non-differentiable function you are differentiating that is not possible but you have to say, you have to look at it as when you are equating with the delta function which is a generalized function, so when you say some two generalized functions are equal again it's like averaging

sense, so this means in a usual way this is  $H'(x)$  into some  $G(x)$  on an average, so if wherever it is defined - infinity to infinity  $\Delta x$  is same as  $\delta x G(x)$   $\Delta x$  this is true for every  $G$  is infinitely differentiable function on that full real line whose, which is nonzero, nonzero values of  $x$  for which  $G$  is nonzero, set of all  $x$  and the values of  $x$  for which  $G$  is nonzero you can always put it in a closed interval, this is called a compact, compact set closed and bounded, you can put closed and bounded set you can put it, so that's what the meaning, so once you choose any for every  $G$  in this, if this is true then you say that this is actually true, so this is like a weak equality, that is earlier you have a weak limit, this is a weak equality, it's not a usual equal sign this is the meaning, so something if you are equating with the generalized function that is basically you're equating in a weak way, okay.

The image shows a handwritten derivation in a software window titled "Transform Techniques for Engineers 2 - Windows Journal". The derivation is as follows:

$$\frac{u(x)}{\Delta x} \Big|_{x=0} = \lim_{\Delta x \rightarrow 0} \frac{1-0}{\Delta x}$$

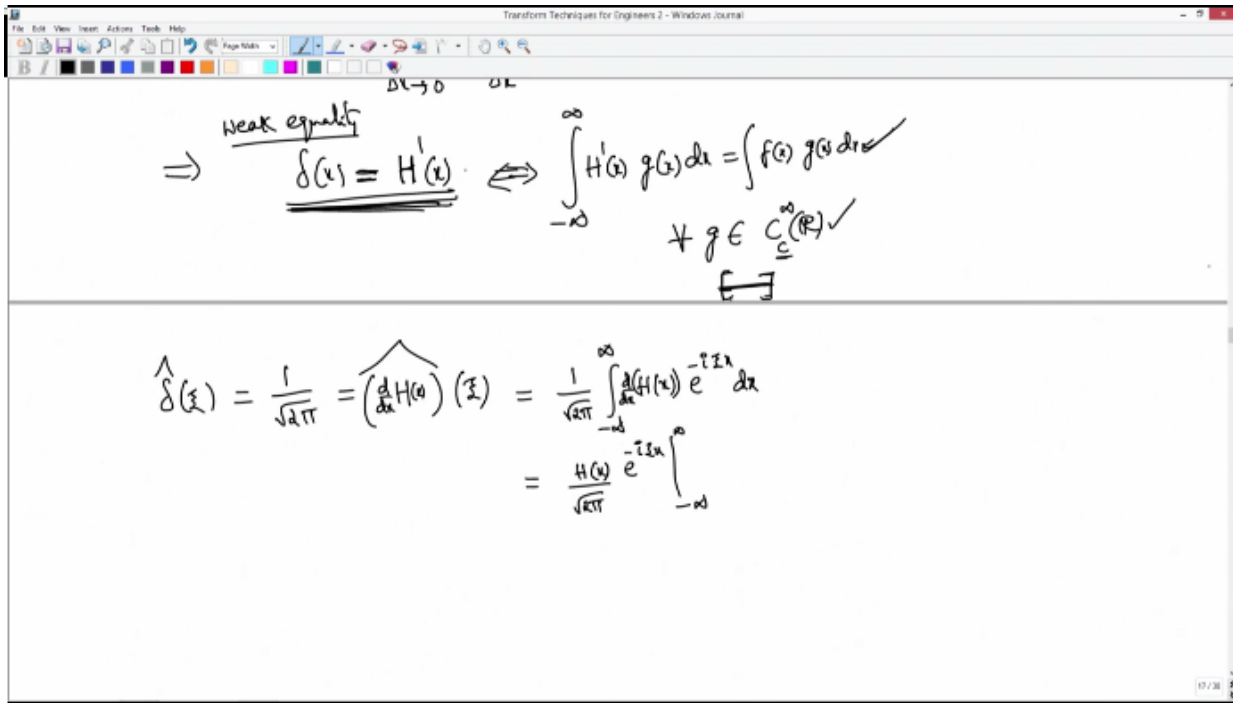
$$= \left. \begin{aligned} &= \lim_{\Delta x \rightarrow 0^+} \frac{1-0}{\Delta x} = \infty \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{0-0}{\Delta x} = 0 \end{aligned} \right\}$$

Then, it states:

$$\Rightarrow \text{Weak equality} \quad \underline{\underline{\delta(x) = H'(x)}} \Leftrightarrow \int_{-\infty}^{\infty} H'(x) g(x) dx = \int_{-\infty}^{\infty} f(x) g(x) dx$$

with the condition  $\forall g \in C_c^\infty(\mathbb{R})$ .

A weak means this is averaging sense, you take this average so that is true that is meaning of weak equality, so when you say this delta function is the derivative of the heavy side function that is in a weak way, weak sense you are talking about, so we have already seen what is the Fourier transform is delta function, so you have the Fourier transform of derivative, derivative of  $H$ , so let us do what is a, so you have already seen the Fourier transform of delta function is, so what is the value you get as a Fourier series of delta function is  $1/\sqrt{2\pi}$ , right, no, what is that?  $1/\sqrt{2\pi}$ , delta function is  $1/\sqrt{2\pi}$  that is once you write this cap as a Fourier transform it's a  $\xi$  variable,  $1/\sqrt{2\pi}$ , so this is the meaning of  $H' D/DX$  of  $H(x)$  for which you take, if you take a cap Fourier transform this, but if you choose this what is this one?  $1/\sqrt{2\pi} \int_{-\infty}^{\infty} H(x) e^{-i\xi x} dx$ ,  $H'$  dash here, okay, so  $D/DX$  of this, so this after if you do the integration by parts here  $H(x)$  into  $e^{-i\xi x}$  do it from  $-\infty$  to  $\infty$  by  $\sqrt{2\pi}$ , so we will have a problem here, right, so if you do this limits,



limits  $H(x)$  as  $X$  goes to infinity it is actually 1, at infinity it is 1 right is a constant function, so this is 1 and  $E$  power  $-I \xi x$ , and this is minus that is 0, so what you have is, so this is again, this is not the way to do this, this is not the way, even if you want to do see if you use the Fourier transform of  $DF/DX$ , Fourier transform of this, this is the derivative and what you get, you got you have seen is if  $F$  is at infinity, if  $F$  at  $+$  or  $-$  infinity is 0, what you have seen is this is going to be  $I \xi x F \text{ cap}(x)$ , okay.

So derivative, Fourier transform of the derivative you have seen that is this, if this is the case, but here for the function  $H(x)$  and this is not true, only  $H$  at  $-$  infinity it is 0, but  $+$  infinity it is 1, so this way, this is not the way to proceed to get this Fourier transform of, this is a delta function, that is okay, but if I want Fourier transform of  $H \text{ cap}(x)$ , how do I go about? How do I proceed to find this  $H \text{ cap}(x)$ ? Okay, so to do this, so this idea that I know is a derivative to use the derivative because I know the derivative is exactly delta function, so that I cannot do because this condition is not satisfied, because if you do the integration by parts at  $+$  infinity it still remains, so this is actually, this what you are getting here is  $1/\text{root } 2 \text{ pi}$  or integral  $-$  infinity to infinity  $H(x) E$  power  $-I \xi x X DX$ , so this is exactly your, this is 0 so  $H(1) E$  power  $-I \xi x X/\text{root } 2 \text{ pi}$ ,  $\xi$  is infinity, so there is no meaning for this, minus of course you have to differentiate so that will give you  $I \xi x$ , differentiation of exponential function, so that's what it is, so you have plus  $I \xi x/\text{root } 2 \text{ pi}$  this is your  $H \text{ cap}(x)$ , so there is no meaning, this limit does not exist as  $X$  goes to infinity, so such a thing, so that's why it doesn't work.

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$$\hat{\delta}(\xi) = \frac{1}{\sqrt{2\pi}} = \widehat{\left(\frac{d}{dx} H(x)\right)}(\xi)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d}{dx} H(x) e^{i\xi x} dx$$

$$= \frac{H(x)}{\sqrt{2\pi}} e^{-i\xi x} \Big|_{-\infty}^{\infty} + \frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) e^{-i\xi x} dx$$

$$= \frac{e^{-i\xi \infty}}{\sqrt{2\pi}} + \frac{i\xi}{\sqrt{2\pi}} \hat{H}(\xi)$$

$$\frac{d\hat{f}}{d\xi}(\xi) = i\xi \hat{f}(\xi)$$

$\int_{-\infty}^{\infty} f(x) dx = 0$

If I want  $\hat{H}(\xi)$  ?

So, how to get this? If I get this, if this makes sense then I can get this  $\hat{H}$  in terms of  $1/\sqrt{2\pi}$  - of this divided by this that will give you, okay. So how do I go about this? What I do is, I choose my  $H(x)$  as a limit of some usual functions, so generalized function you can view, see if you take usual functions  $F$  and  $G$ ,  $F$  is equal to  $G$ , means for every  $X$  this is true if this is the case this is a strong way of telling, so the strong equality, so this is also once it is strongly equal implies it is also weakly equal that means I can simply multiply  $F$  into some  $H(x)$  which is equal to  $G(x)$  into some  $H(x)$ , whose integral value, okay, this is always true for every  $H$  in  $\mathbb{C}$  infinity( $\mathbb{R}$ ), right, this is always true so strong implies this, that implies weak  $F(x) = G(x)$  weak, okay that's the meaning of weak inequality, weak equality, so in that sense if  $H(x)$  is a heavy side function I can write this as a limit of some  $H_\alpha(x)$ , and then as I take  $0$  to  $0^+$ , what is this? Where  $H_\alpha(x)$  is, what I do is I simply multiply  $1$ , where ever it is  $1$ ,  $0$  if  $T$  is less than or equal to  $0$ , and then you have a  $T$  positive, sorry  $X$  is positive side, what I do is I multiplied with  $E$  power  $-\alpha X$ , instead of  $1$  I chose this one  $-\alpha X$ ,  $\alpha$  is positive anyway, because it's going from  $0$  to,  $\alpha$  going to  $0^+$  plus side, so if I choose that like this these are  $H_\alpha(x)$  are all piecewise continuous functions, and if this is actually a strong, this I don't say strong because these are usual functions, but yeah these are strong, this is strong implies this is also weak, okay, if you look at this here very strongly equal, so  $H(x)$  is well-defined function but it is piecewise continuous function.

Here if I choose at  $X = 0$  both are same, at  $X$  is a positive as  $\alpha$  goes to  $0^+$  is going to be  $1$ , so that way both are strong equality what you have, so I write like this and then I apply, I try to find what is my  $\hat{H}(\xi)$  here using this, then what I get is  $1/\sqrt{2\pi}$  integral  $-\infty$  to  $\infty$  what you get is  $H(x)$  into  $E$  power  $-i\xi X$   $DX$ , this is the definition of Fourier transform, and this is  $1/\sqrt{2\pi}$ , and now I can put this  $H(x)$  as this limit of this, limit  $\alpha$  goes to  $0^+$ ,  $H_\alpha(x) E$  power  $-i\xi X$   $DX$ , now because of this, this is actually your  $H(x)$  and this is exactly equal to your weak limit and strong limit, right, so if I have this limit of this is

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If I want  $\hat{H}(\xi)$  ?

$$= \frac{H(\omega) e^{-i\xi\omega}}{\sqrt{2\pi}} + \frac{i\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(\omega) e^{-i\xi\omega} d\omega$$

$$= \frac{e^{-i\xi\omega}}{\sqrt{2\pi}} + \frac{i\xi}{\sqrt{2\pi}} \hat{H}(\xi)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(\omega) \cdot h(\omega) d\omega = \int_{-\infty}^{\infty} g(\omega) h(\omega) d\omega, \quad \# h \in C_c^\infty(\mathbb{R})$$

$$\Rightarrow f(\omega) = g(\omega) \text{ (weak)}$$

$$\underline{H(x)} \stackrel{\text{strong}}{=} \lim_{\alpha \rightarrow 0^+} H_\alpha(x), \text{ where } H_\alpha(x) = \begin{cases} 0, & x \leq 0 \\ e^{-\alpha x}, & x > 0 \end{cases}$$

$$\hat{H}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) e^{-i\xi x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\alpha \rightarrow 0^+} H_\alpha(x) e^{-i\xi x} dx$$

same as, so this is also weak sense they are same this limit is, limit of this is equal to H(x), because this is strong equality, this is also weakly this is true that means a limit of this is, this means 1/root 2 pi limit of alpha goes to 0+ and you have - infinity to infinity H alpha(x) E power - I xi X DX, so your CC infinity or so only thing is here I have chosen CC infinity

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$$\hat{H}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} H(x) e^{-i\xi x} dx.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \lim_{\alpha \rightarrow 0^+} H_\alpha(x) e^{-i\xi x} dx$$

$$= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \rightarrow 0^+} \int_{-\infty}^{\infty} H_\alpha(x) e^{-i\xi x} dx.$$

$$\Rightarrow f(\omega) = g(\omega) \text{ (weak)} \quad \# h \in C_c^\infty(\mathbb{R})$$

because when I say in the weak sense when you multiple, when your multiplication function this nice function so that this integral makes sense, because here I assume that the integral makes sense, if at all the integral makes sense with this Fourier transform E power - I xi X, that's why we have chosen so, you don't ask why this is not, this function is not CC infinity

function, but this is 0 everywhere, wherever  $E^{-ix}$  is nonzero on the full real line which is not closed unbounded.

As long as this whole integral makes sense that is what is a point, so here this makes sense  $H_\alpha$  of  $E^{-ix}$ , here this integral makes sense, okay, and once you have this limit, limit  $\alpha$  goes to 0 that is  $H(x)$  if at all it makes sense if that means this limit, if this limit is usual way if it converges to something and then that means it converges to that, okay, because it is strong implies weak, so weak limit so you can write this step, from this step to this step, so this is equal to  $1/\sqrt{2\pi}$  and this limit  $\alpha$  goes to  $0^+$ , what happens here? This I can write now as  $0$  to infinity  $E^{-\alpha X}$  into  $E^{ix}$   $DX$ , so what is this one?  $1/\sqrt{2\pi}$  limit  $\alpha$  goes to  $0^+$ , and this integral is a usual integral that is  $\int_0^\infty e^{-\alpha X} e^{ix} dX$ , so  $E^{-\alpha X} e^{ix}$  is  $E^{-(\alpha - i)X}$ , so  $E^{-\alpha X} e^{ix}$  into  $X$  divided by  $-(\alpha - i)$ , this is from  $0$  to infinity. At infinity because of  $\alpha$  is positive,  $\alpha$  is a real number  $E^{-\alpha X}$  that makes it  $0$ , and what you get is  $1/\sqrt{2\pi}$  only  $0$  contribution will be there,  $\alpha$  goes to  $0^+$  and what you had  $0$  this is minus minus plus we get  $1$  divided by  $\alpha - i$ , so this is exactly equal to  $1$  divided by  $i - \alpha$   $\sqrt{2\pi}$ , so this is my, this is exactly what I got as  $H(x)$ , right.

You still, I think, you look at this, this is strong equality and if you want to still justify this question, so this limit of this  $H_\alpha$  and this is a limit of  $H$  of this multiplied with this  $G(x)$  and we don't know whether this is finite or not, right, see if this  $E^{-ix}$ , if it is a differentiable function which is in nonzero all the, if it is in  $C^\infty$  in this notation if this is the case and we know that this is finite, but here we cannot say whether this integral is finite or not, okay, in that sense though we have done formally still this, this is still questionable because  $H(x)$  is this and we substituted this now this step, this step is still questionable because we don't know whether this is finite or not, because the multiplication function this integral is finite but this integral you don't know whether it is finite or not, so in any case formally if you do this if you do formally this what you get is this one, so this is still questionable, right.

So if this is true, so what happens to your inverse Fourier transform? Inverse Fourier transform let me write this inverse of  $F$  of, script  $F$  of Fourier transform is script,  $F$  inverse of  $1$  divided by  $i - \alpha$   $\sqrt{2\pi}$ , so this you should get it as a function of  $X$ , what is this one? If you calculate you should get back your  $H(x)$  okay, if this is true I should get back here  $H(x)$  that is the question, okay.

So let's choose, let's start with doing this, so this is equal to by definition  $1/\sqrt{2\pi}$  integral  $-\infty$  to  $\infty$   $1/i - \alpha$   $E^{ix} dxi$ , that is the meaning, okay, so what is this one? So first one is, this you can rewrite  $1/2\pi$  and this is  $-\infty$  to  $\infty$   $\cos xi X$ ,  $\cos xi X + i \sin xi X$  divided by  $i - \alpha$ ,  $dxi$ , and you see  $\cos xi X / i - \alpha$  is a odd function from  $-\infty$  to  $\infty$ , so that integral will be  $0$  so what you get is  $1/2\pi$ , I I cancel here, that you get  $-\infty$  to  $\infty$  this is an odd function so you have  $0$  to  $\infty$ , this is  $\sin xi X / i - \alpha$  is even function, so that you have two times of that,  $0$  to  $\infty$   $\sin xi X / i - \alpha$ ,  $dxi$ , so this is exactly  $1/\pi$ , so  $1/\pi$  into the value of this when  $X$  is positive, okay, when  $X$  is positive so this if you calculate this we will try to prove this one, if this value is, value of this is  $X$ , this value is  $1/\pi$  this is here, this value is if  $X$  is positive this value is  $\pi/2$ , and this value is, value of this integral is  $-\pi/2$  if  $X$  is negative.

$$\begin{aligned}
 \mathcal{F}^{-1}\left(\frac{1}{i\xi\sqrt{\pi}}\right)(x) &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{i\xi\sqrt{\pi}} e^{i\xi x} d\xi \\
 &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\cos \xi x + i \sin \xi x}{i \xi} d\xi \\
 &= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\sin \xi x}{\xi} d\xi \\
 &= \begin{cases} \frac{1}{\pi} \cdot \frac{\pi}{2} & x > 0 \\ -\frac{1}{\pi} \cdot \frac{\pi}{2} & x < 0 \end{cases}
 \end{aligned}$$

If  $X = 0$  that is anyway 0, so we don't define it at all, okay there. So what you see is, there's nothing but  $1/2$  if  $X$  is positive,  $-1/2$  if  $X$  is negative, okay, but this is not equal to your  $H(x)$ , so that means there is something wrong here, so it's not really this value, okay, so this is wrong, this is actually wrong, okay, it's not really wrong but there is something missing here, so that is what we see, so but what is this one? This is exactly equal to, this is not  $H(x)$  but this is equal to  $H(x) - 1/2$  if you do like this,  $H(x) - 1/2$  if you do this is  $0 - 1/2$ , when  $X$  is less than 0 this is this, when  $X$  is positive it's going to be  $1 - 1/2$  that is  $1/2$ ,  $1 - 1/2$  that is  $1/2$  anyway, so this is actually  $H(x) - 1/2$ , okay.

Fourier inverse transform of this function it is actually equal to this one, so that will give you, so through this instead of directly calculating when you do this direct calculation to find this  $H$  cap even though you have this strong inequality you introduced there is still a question,

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$$\begin{aligned}
 & \lim_{\alpha \rightarrow 0^+} \int_{-\infty}^{\infty} H_{\alpha}(x) e^{-iI x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \rightarrow 0^+} \int_0^{\infty} e^{-\alpha x} e^{-iI x} dx \\
 &= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \rightarrow 0^+} \int_0^{\infty} \frac{e^{-(\alpha+iI)x}}{-(\alpha+iI)} dx \\
 \hat{H}(I) &= \frac{1}{\sqrt{2\pi}} \lim_{\alpha \rightarrow 0^+} \frac{1}{\alpha+iI} = \frac{1}{iI\sqrt{2\pi}} \quad \times \\
 \mathcal{F}^{-1}\left(\frac{1}{iI}\right)(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{iI} e^{iI x} dI
 \end{aligned}$$

bringing this limit inside okay, so bringing this limit outside this is still questionable and even the formally if you do it, so that's why somewhere you went wrong, somewhere it is not true so it's not really true, okay, this is not true, but if this is true but it is almost, but this helps you to find the Fourier inversion of this one this function, when you calculate what you see is not  $H(x)$

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$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\cos Ix + iI \sin Ix}{iI} dI \\
 &= \frac{\times}{2\pi} \int_0^{\infty} \frac{\sin Ix}{I} dI \\
 &= \begin{cases} \frac{1}{\pi} \cdot \frac{\pi}{2} & x > 0 \\ -\frac{1}{\pi} \cdot \frac{\pi}{2} & x < 0 \end{cases} \\
 &= \begin{cases} \frac{1}{2}, & x > 0 \\ -\frac{1}{2}, & x < 0 \end{cases} \neq H(x) \\
 &= H(x) - \frac{1}{2} \quad \checkmark
 \end{aligned}$$

but this is  $H(x) - 1/2$ , so if I bring this what, what you have is Fourier inverse transform of  $1/I$  is  $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{iI} e^{iI x} dI$  a Fourier inversion, Fourier inversion this is what you have, so you bring this Fourier inversion to this side, if I multiply Fourier both sides Fourier transform of this and this becomes  $1/I$ , this is function of  $X$ , right, function of  $X$ , so this will become Fourier inversion of



X, Fourier transform of X is finally, see this Fourier transform of this function of xi is overall function of X, when you apply Fourier transform function of X it is a function of xi, that is  $1/\sqrt{2\pi}$  xi, which is equal to here  $H \text{ cap}(xi)$ .

Fourier transfer of two functions because it's linear, Fourier transform is  $F \text{ cap}(xi)$  is actually rather one of the property of the Fourier transform is  $F1 + F2 \text{ cap}(xi)$  is actually can split it into two parts, integral  $1/\sqrt{2\pi} \int_{-\infty}^{\infty} F1 + F2(x) \text{ E power } -I \text{ xi X DX}$ , so this is actually, you split this into two parts  $F1(x) \text{ E power } -I \text{ xi X DX} + 1/\sqrt{2\pi} \int_{-\infty}^{\infty} F2(x) \text{ E power } -I \text{ xi X DX}$ , what are this? This is exactly  $F1 \text{ cap}(xi) + F2 \text{ cap}(xi)$  so it's a linear property, so this is a linear property, linearity of the Fourier transform, so if you use this here Fourier transform of  $H1$  is  $F1$ ,  $-1/2$  is  $F2$  so if you do that you will see that if you multiply even constant here, and you have a constant here, so that is going to be  $C$  times of this, so  $C$  is  $-1$ , so you have a minus Fourier transform of I don't put a cap on  $1/2$  function, constant function, I write like this  $1/2$ , Fourier transform of this, so that will give me  $H \text{ cap}(xi)$  the Fourier transform of the heavy side function is Fourier transform of  $1/2$  constant function +  $1$  divided by  $I \text{ xi root } 2 \text{ pi}$ , which is not exactly this, but you have to add with Fourier transform of  $1/2$  so this is wrong, somewhere we missed this Fourier transform of  $1/2$ .

So if we know Fourier transform of  $1/2$  that is  $1/\sqrt{2\pi} \int_{-\infty}^{\infty} 1/2 \text{ E power } -I \text{ xi X DX}$ , that is what is the meaning. So how to find this or rather so, this is also meaningless, this is not you cannot integrate this function, so instead so to find this the question is what is this Fourier transform of a constant function, so we have already seen that Fourier transform of a delta function is  $1/\sqrt{2\pi}$ , okay, so that means Fourier transform of  $\sqrt{2\pi}$ , root 2, or rather root  $\pi/2$  is equal to, okay, so how do I do this? Fourier transform of simply bring it here, so you want it  $1/2$  right, so you want  $1/2$ , Fourier transform of  $1/2$  is, so you have this Fourier transform of this heavy side function is you have this  $1/I \text{ xi root } 2 \text{ pi} +$  something which is a Fourier transform of a constant function, we will try to find this Fourier transform, once we know this Fourier transform of a constant function, and we will find eventually, then we know exactly what is a Fourier transform of the Heaviside function.

So we'll try to find this Fourier transform of a constant function in the next video, once we have that, we have the Fourier transform of heavy side function, so we will see all this in the next video along with properties of the Fourier transform. Thank you very much. [Music]

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