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Transform Techniques for Engineers

Fourier integral theorem-an informal proof

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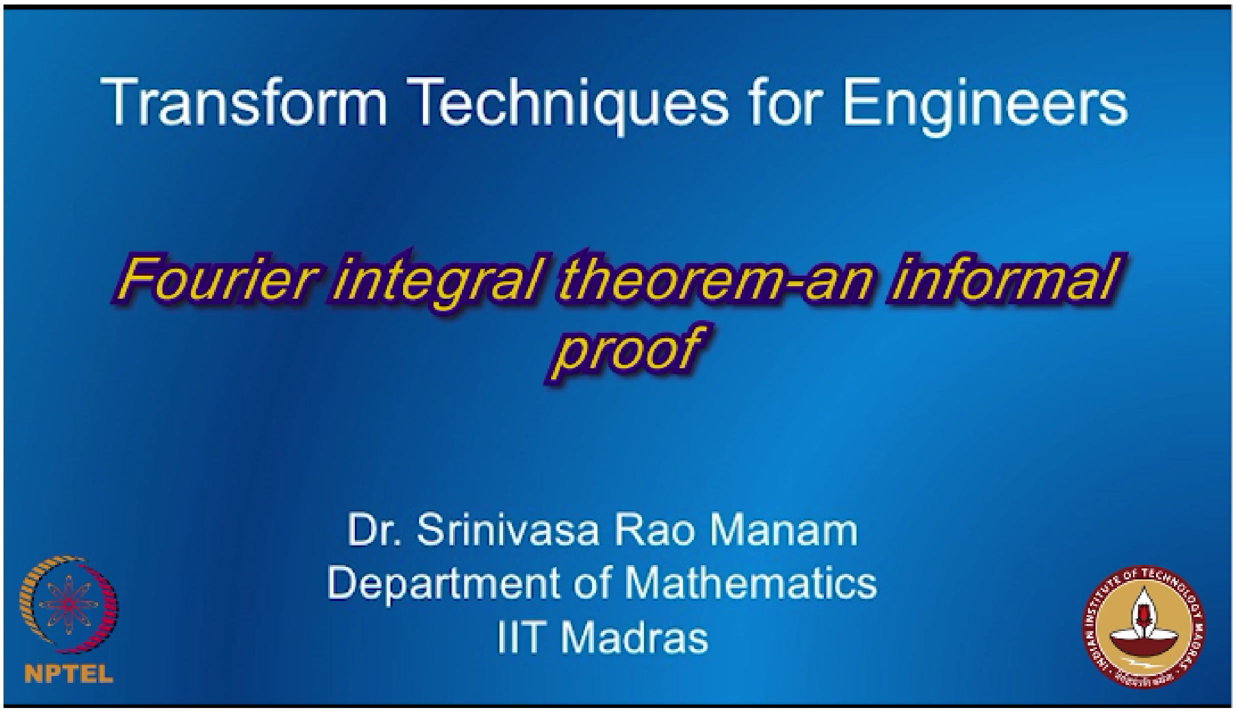
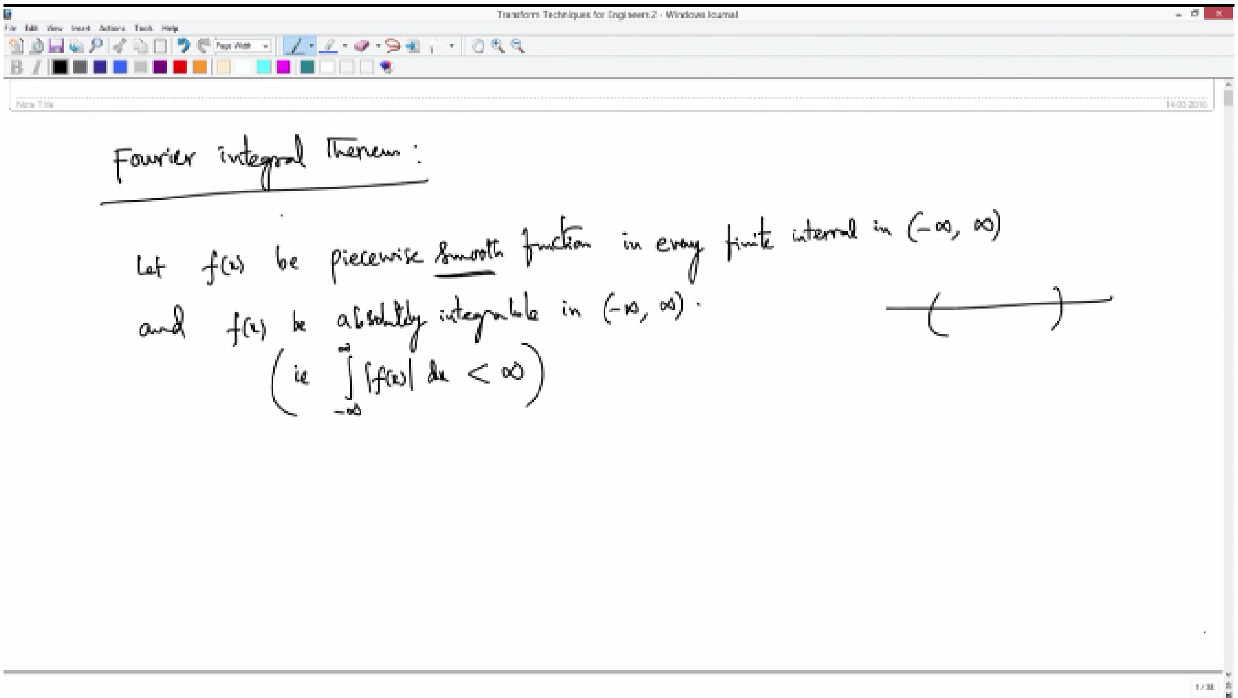


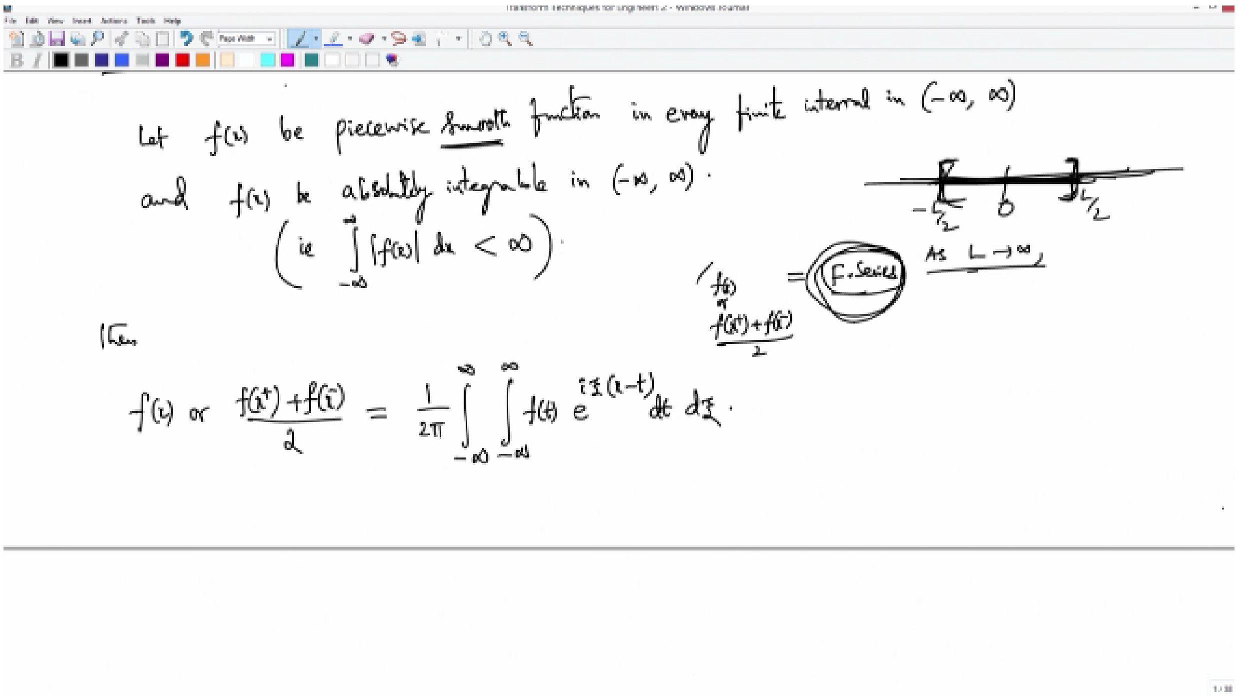
Welcome back, this video we will start Fourier transform of a non-periodic function that means a function or a signal that's defined over a full real line. So to do this we have to link with Fourier series that we have already aligned, you just build on from that to define, now we'll try to derive this Fourier transform.

So I'll just write what is a Fourier so there's a something called Fourier integral theorem, I'll just derive Fourier integral theorem, Fourier integral theorem this tells you that if you have a signal, so let me give what is that we consider, let F(x) be a piecewise, piecewise smooth function, piecewise differentiable function, and we know that it is Fourier series exist and it's converging point wise, and if it is piecewise smooth that means F double derivative is piecewise continuous function, then we have seen that it is a Fourier series is uniformly convergent, so that's why we assume this is a piecewise smooth function, that means at least 2 derivatives are piecewise, F double dash is piecewise continuous function.

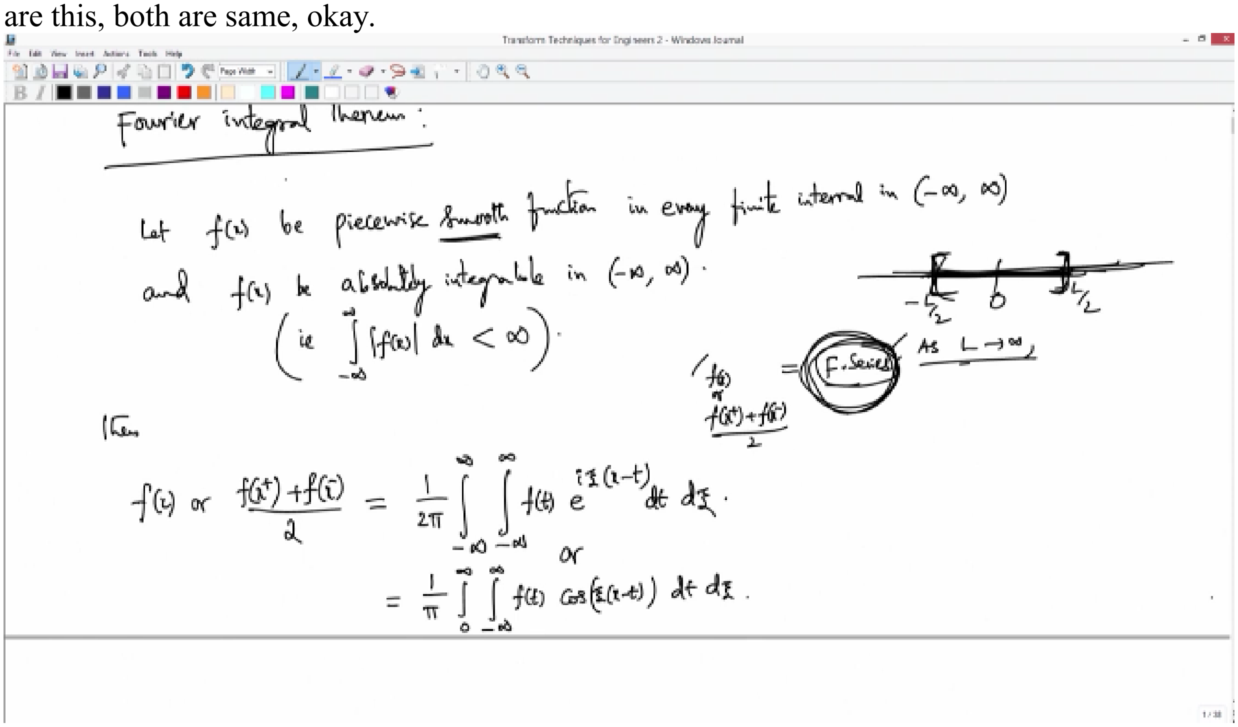
What we do is, this is a non-periodic function so how I assume is, in every finite interval this is a piecewise smooth function, okay, every finite interval in - infinity to infinity, that means full real line to have, you should take any finite interval so on which is the piecewise smooth function and it is absolutely integrable function, and F(x) is absolutely integrable in full real line, that is, that means that is what I just show you, so this - infinity to infinity mode F(x) DX is finite, that is the meaning of this, okay, so let me not write this, that is this.



So on every piece of a, finite piece of interval, this is piecewise continuous that means you have only finitely many jump discontinuities, and if at all there is a discontinuity it is a jump discontinuity, other than those places, remaining places it is actually twice differentiable function minimum, so that's what is the meaning of smooth here.
What we need is the Fourier series has to converge uniformly so for that what we mean by smoothness is true derivatives that is piecewise continuous function, and over a full real line - infinity to infinity it is integrable absolutely, such a function, such a smooth signal we have considered and we know that if we take the finite interval from –L/2 to L/2 if you take any finite interval like this then we know that is Fourier series is exist because it's a piecewise smooth function on every finite interval which is this, if I choose my finite interval as –L/2 ro L/2 you have a Fourier series, so that Fourier series is actually converges to as L goes to infinity, so what happens? You have a Fourier series which is equal to F(x) or either F(x) or if it is discontinuous point you simply take the average of it, F(x+) + (Fx-) divided by 2, so we have seen that the Fourier series, Fourier series over a finite, finite interval means you can extend as a periodic function, so if it is your finite interval you have a Fourier series that is actually equal to F(x), if F is continuous at X, if it is not continuous what we have is an average value of it.
As L goes to infinity this Fourier series becomes, I'll just write here and the left-hand side as it is, so what we get is so if such a function if I choose then what I get is either F(x) or F(x+) + F(x-) divided by 2, so this is equal to, so the Fourier series as L goes to infinity it becomes 1/2 pi, and you have a - infinity to infinity, this is an integral - infinity to infinity, F(t) E power I xi X-T DT D xi, so this is what it becomes this Fourier series actually becomes a double integral here, as L goes to infinity.

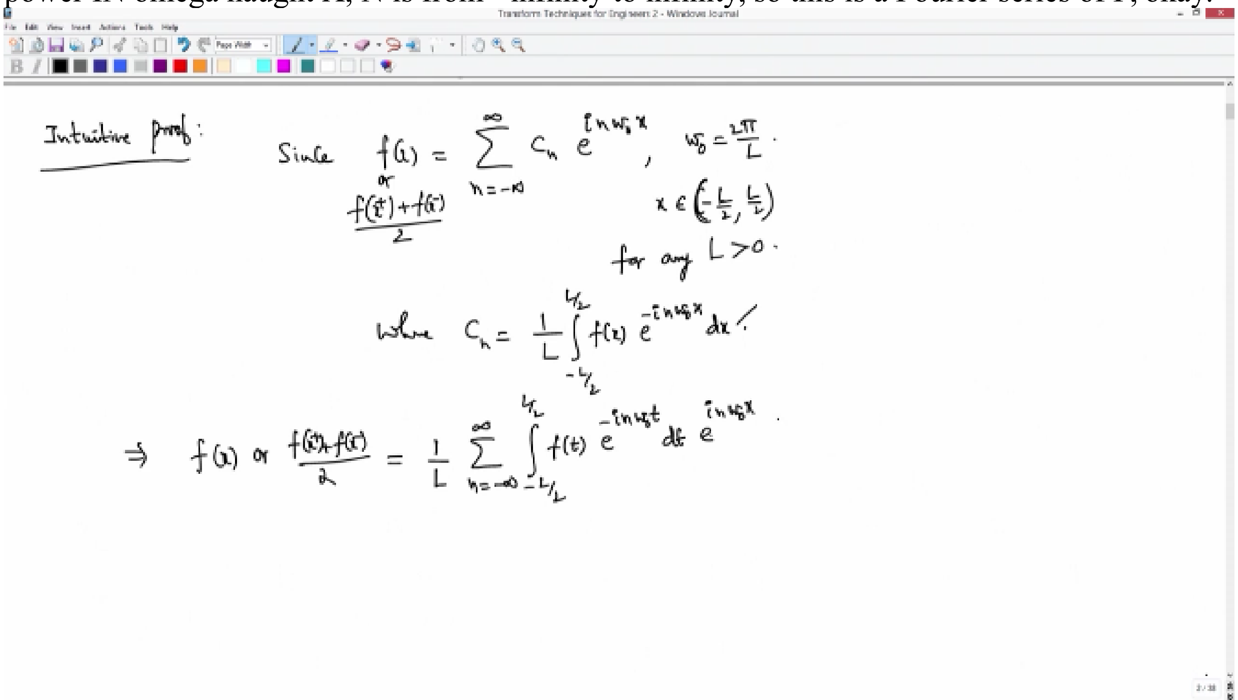


So if you allow this interval eventually to infinity, because every finite interval if you choose L is any big number is still this Fourier series exists and this is this, so such a Fourier series what happens as L goes to infinity it becomes this double integral, so that's what we are going to show either this or you can just do, you can also show that this is actually 1/pi - infinity to infinity, actually 0 to infinity, - infinity to infinity F(t) cos xi times X-T DT D xi, so that these are this, both are same, okay.



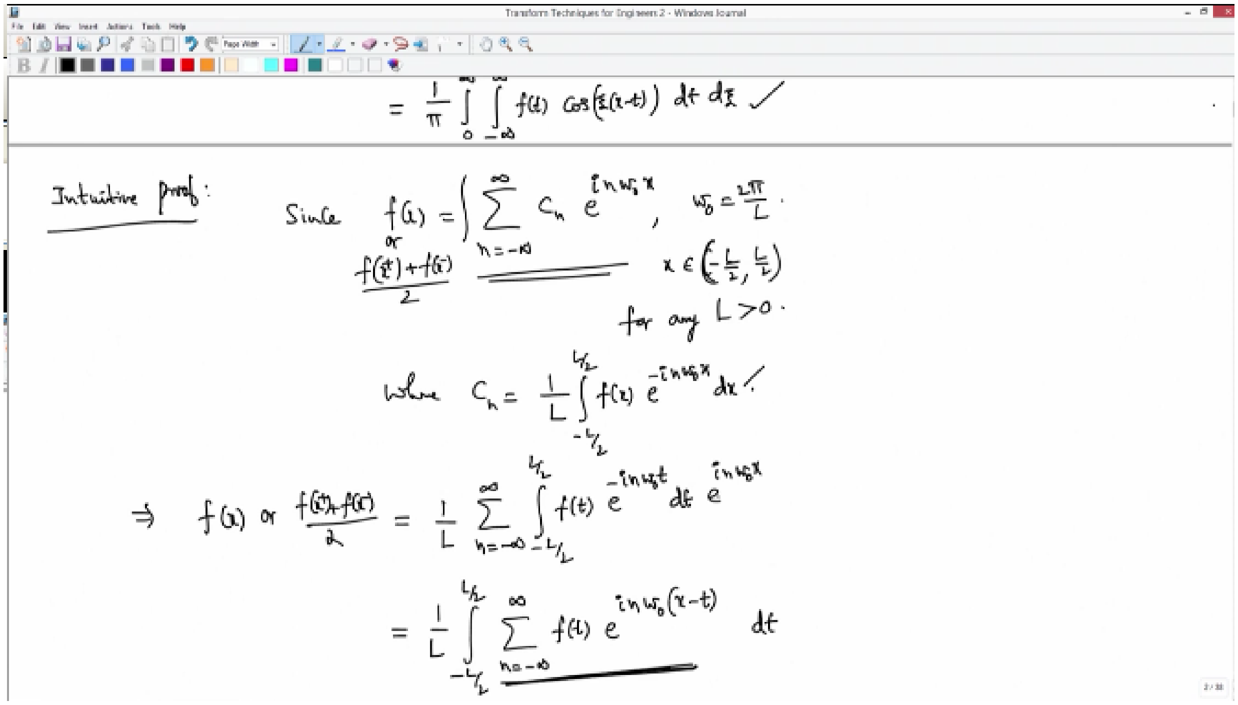
This is what is a Fourier integral theorem, so in this video we'll just show that if such a function, such a signal, non-periodic signal you get this, this is always true we will try to give you intuitive proof somewhere it's not so rigorous, so will try to give you intuitive proof of this theorem, and also from once you know if you accept this I can simply define what is a Fourier transform, so transform is involving from this integral theorem you can define its Fourier transform and its inverse Fourier transform just from this relation, okay, so before I do this, so I’ll try to give you the intuitive proof of this Fourier integral theorem from the Fourier series over a finite interval.

So intuitive proof I'll tell you why it's called intuitive? So since you have the Fourier series over a finite interval we have F(x), this is of course we have a Fourier series, Fourier coefficients E power IN omega naught X, N is from - infinity to infinity, so this is a Fourier series of F, okay.



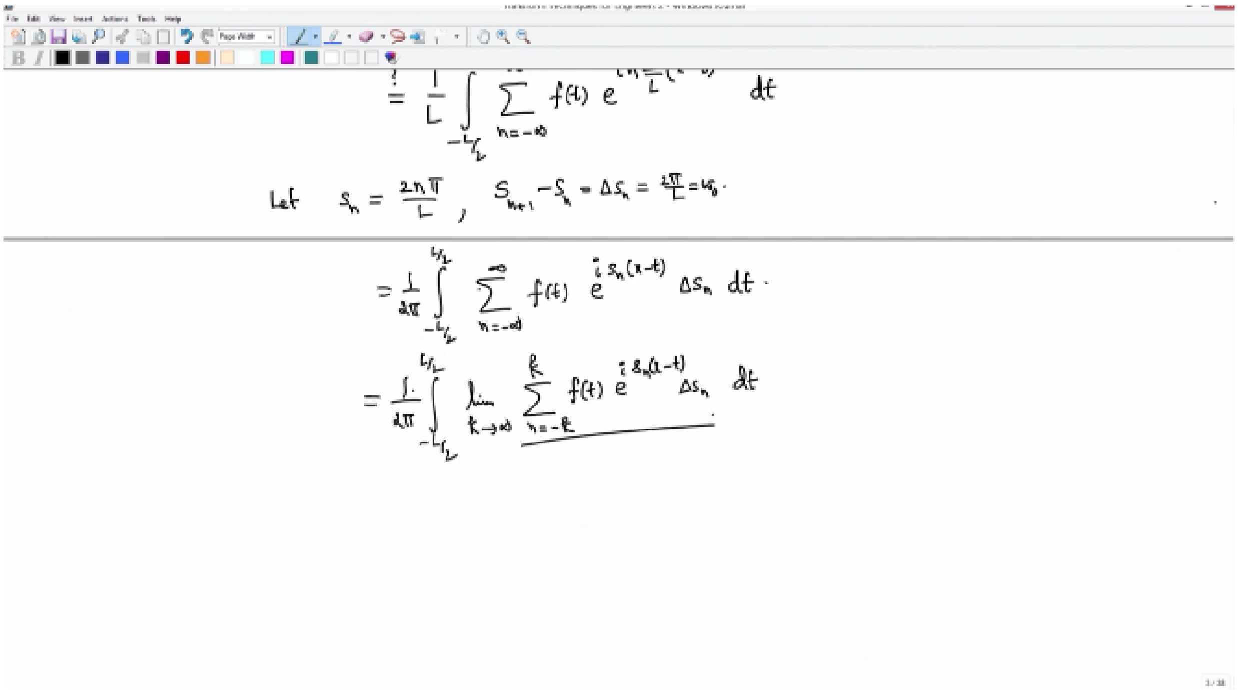
And what is omega naught? Omega naught is 2 pi/L so let me write this, so this is valid in X belongs to –L/2 to L/2, this is true for any L positive, and then what you get is where CN is actually 1/L –L/2 to L/2 this F(x) E power - IN omega naught X DX, so this is by definition of, this is what we have already seen in the earlier videos, if you take, if you have any signal on finite interval you can extend it as periodically to up to outside, outside this interval, so you have a periodic signal which is already assumed to be smooth so that you have a Fourier series that converges to F, and you have a Fourier coefficients defined by this way.

So what I do is now I start with the left hand side, so what happens to this F(x)? F(x), so either this or if it is discontinuous point which is an average of it, both of that you can choose so this or F(x+) + F(x-) divided by 2 this is equal to, so I choose 1 by, I substitute the CN into this Fourier series so that you get 1/L, N is from - infinity to infinity, CN is –L/2 to L/2, F of, let me use this dummy variable, so F(t) E power -IN omega naught T DT into E power IN omega naught X DX, sorry there's no DX, that's a series, I substitute only CN’s here, so this is equal to 1/L, N is from, so because the Fourier series is and since this is absolutely integrable function, and so assume that you can, I’ll just justify why this –L/2 if I take this integral outside, what you get is N is from - infinity to infinity, F(t) E power –IN omega naught T-X, -T so you have X-T, right, so I’ll just remove this somehow, X-T into DT, so this series is actually, if this series is uniformly convergent then you know that this, you can allow this integral inside this, is it true?



We know that the Fourier series is uniformly convergent, that means this is convergent uniformly, you can allow integration if you pull integration here you can take it inside, but as such here you don't know about this series, okay, so this series we will just see whether this series is uniformly convergent - infinity to infinity, and if you look at this part F(t) into E power IN omega naught(x-t) this modulus is always less than or equal to mode of F(t), and this is sigma with N, N is from - infinity to infinity, we have a sigma from - infinity to infinity, we don't know this is infinity, right, F(t) is a nothing to do with N, so you have a sigma 1, that is infinity, so you cannot really say that this is uniformly convergent just by M test so that is where we, that's why we somehow call this is not rigorous, proof is not rigorous, assume that is possible to bring this integral outside, so we're not justifying exactly so all the steps here, so we will write, that's why we are calling intuitive proof, we'll take this, assume that you can bring this integral just outside this infinite sum, okay.

So you can now, if you could do this so this is still questionable that is why it is a, it's not rigorous so this is equal to 1/L –L/2 to L/2, so let me do something here, so before I do this so let me define what is SN? SN is, it depends on N so that's why I'm calling it SN 2, and W naught is 2 pi/L so you call this 2N pi /L, so if you take this what happens to SN+1 –SN which is delta SN, which is equal to simply 2 pi/L all right that is exactly what W naught. Delta SN is this, okay, so if I use this what happens to this? This will be 1/L –L/2 to L/2, and now you have this sum becomes N is from - infinity to infinity, F(t) E power IN, so 2N PI by, so that is your SN into X-T into delta SN, so what is delta SN? Delta SN is 2 pi/L, 1/L is already there so 2 pi/ L means 1/2 pi if you multiply you have 2 pi/L, so I divide with 2 pi so I multiplied 2 pi, already 1/L is there so that I bring it inside, so 2 pi/L so this is exactly is my delta SN into DT. So this is 1/2 pi –L/2 to L/2 and here if I try to write as K goes to infinity I try to write this as -K to +K, F(t) E power I SN(x-t) delta SN DT, so this sum, this sum as delta SN goes to 0, when



is delta SN goes to 0? That means as L goes to infinity so you can write as L goes to infinity delta SN goes to 0, okay, so therefore this becomes 1/2 pi and this is -L/2 to L/2, so I'll just write it as a limit, L goes to infinity –L/2 to L/2 and you have this limit K goes to infinity and this becomes as delta SN goes to 0 so you have this limit you can write this as N -K to K, F(t) E power I SN X-T delta SN DT, so this is what we are looking at it, only this part with this limit. As limit L goes to infinity that is delta SN goes to 0 so what happens to this part, and together so that is where we don't have the rigorousness, so is it really legitimate, it's not mathematically so, you simply intuitively seeing that this is also true here, so if you allow this is the integral sum so what you have is –L/2 to L/2 so that is as L goes to infinity is going to be – infinity to infinity, and then and here this is the limit which is actually from -K to K as delta SN goes to 0 so F(t) E power I SN is the xi variable, some new variable xi (x-t) into D xi, so this is what it becomes.

Let $s_n = \frac{\quad}{L}$,

$$= \frac{1}{2\pi}\int_{-L/2}^{L/2} \sum_{n=-\infty}^{\infty} f(t)\, e^{i s_n (x-t)}\, \Delta s_n \, dt .$$

$$= \frac{1}{2\pi}\int_{-L/2}^{L/2} \lim_{k\to\infty} \underline{\sum_{n=-k}^{k} f(t)\, e^{i s_n (x-t)}\, \Delta s_n} \, dt$$

As $L \to \infty$, $\underline{\Delta s_n \to 0}$.

$$= \frac{1}{2\pi} \underline{\lim_{L\to\infty}} \int_{-L/2}^{L/2} \lim_{k\to\infty} \underline{\sum_{n=-k}^{k} f(t)\, e^{i s_n (x-t)}\, \Delta s_n} \, dt$$

$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, d\xi \, dt$$

Now we have this limit K goes to infinity that makes it this is from - infinity to infinity so into DT, so this is exactly what happens to your Fourier series as L goes to infinity intuitively, okay. If I choose D xi as, put xi = - xi, so what happens? What you see is 2 pi - infinity to infinity this is for DT, I write like this, this DT for this outer integral, and inside if I put D xi that is minus, and if this limit becomes infinity to - infinity and I have a D xi – D xi, so this minus this limits you can interchange again now, so you have a minus infinity to infinity, F(t) E power - I xi (x-t), so this is also true, so both are same, okay, so if you just use the change of variables, and what you see is this, both are same.

$$= \frac{1}{2\pi}\int_{-L/2}^{L/2} \lim_{k\to\infty} \underline{\sum_{n=-k}^{k} f(t)\, e^{i s_n (x-t)}\, \Delta s_n} \, dt$$

As $L \to \infty$, $\underline{\Delta s_n \to 0}$.

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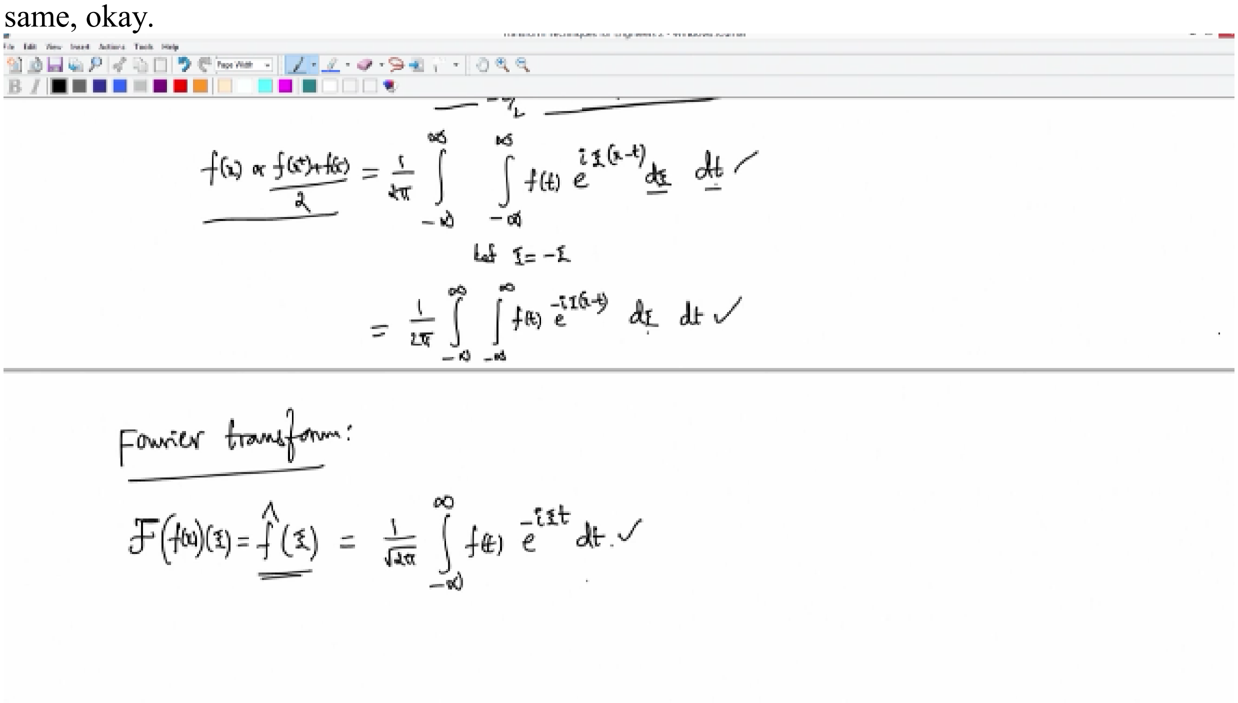
$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, d\xi \, dt$$

Let $\xi = -\xi$

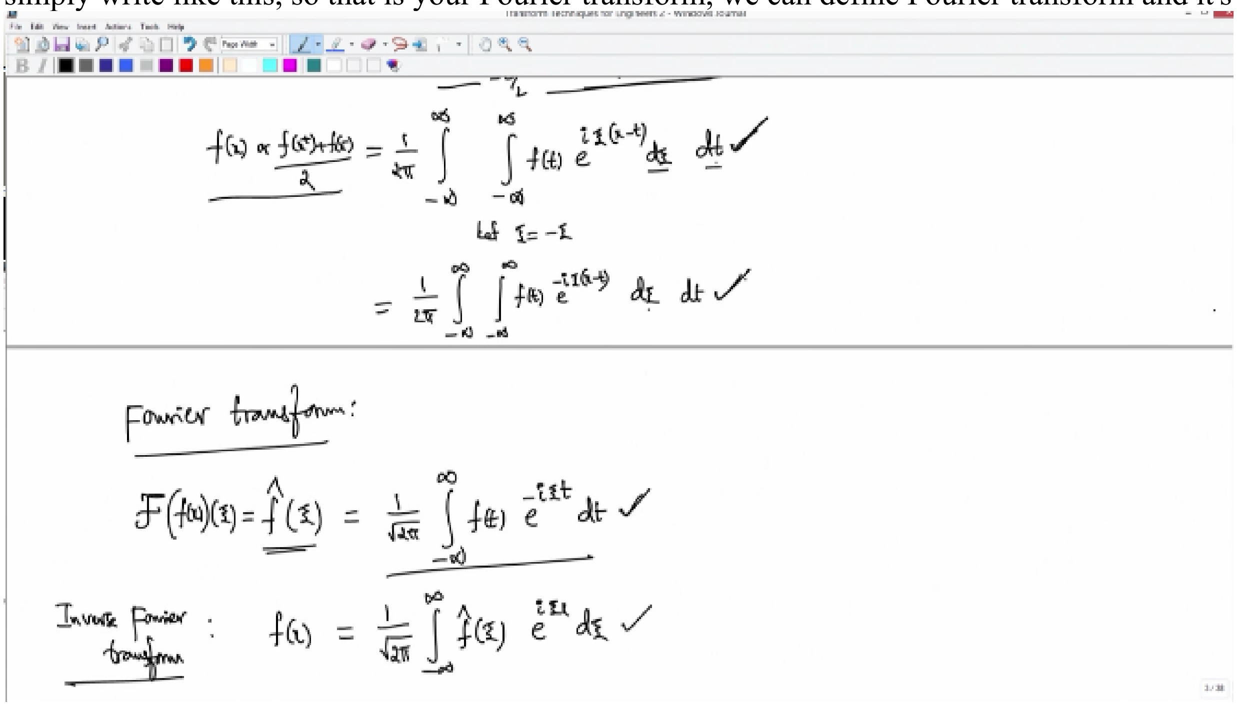
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t)\, e^{-i\xi(x-t)}\, d\xi \, dt$$

Now this is exactly what happened, as L goes to infinity somehow, as intuitively or not so rigorously we have seen that this function has become a double integral here, so iterative integral, because this xi part and T part these are the two parts, so you have F(t) into E power I xi T, I xi X into E power I -I xi T, so this is what you have, so T part functions of T and xi not really separated, but there is a kind of a double integral here iterate integral, you do with DT first and you do D xi, okay, so these are the iterative integrals you get which is, they are finite if this integral as such it exists as a double integral you have all the actual double integral, that is double integral from - infinity means it's over R2, so it's like DT D xi, it's together, okay, and whatever is the integrand, so all are same if it’s the integral is - infinity to infinity, these are iterative integral with D xi DT separately or this should be same, all are same if it is integrable over a full real line, so this is the DT separately, these two iterate integrals and this double integral all are same if it is actually integrable, and they all exist they all should be same. So anyway, so what you get is this Fourier series becomes this iterated integrals, double integrals. Now from this we can define what is Fourier transform? So let me define, what is Fourier transform? Fourier transform, how do I define? So given a function F I can define from this what is my Fourier transform, so my Fourier transform now is you can define many ways or the way you have defined earlier F cap(xi) this is actually script F function f(x) which is going to be variable xi, so all are same, so I prefer this notation so F cap(xi) is for a function F(x) if you take the transform, this becomes a function of sai okay, and which is a Fourier transform, and you define it as, so I split this 1/2 pi into both 2 parts 1/root 2 pi, so there is another row 1/root 2 pi is left there, so 1/root 2 pi and I choose this - infinity to infinity, F(t) E power -I xi T DT, so if I use this, so you also assume that this is a iterate, one iterative integral, other iterative integral is you can also rewrite it’s, you can rewrite them as iterate, other iterate integral you can do the order of integration you can interchange, so if you do that you have DT D xi all are same, same, okay.



So here so DT D xi are D xi DT all are same, both are same, so if you interchange here you get DT D xi, so that DT with this inner integral that is my Fourier transform, okay.

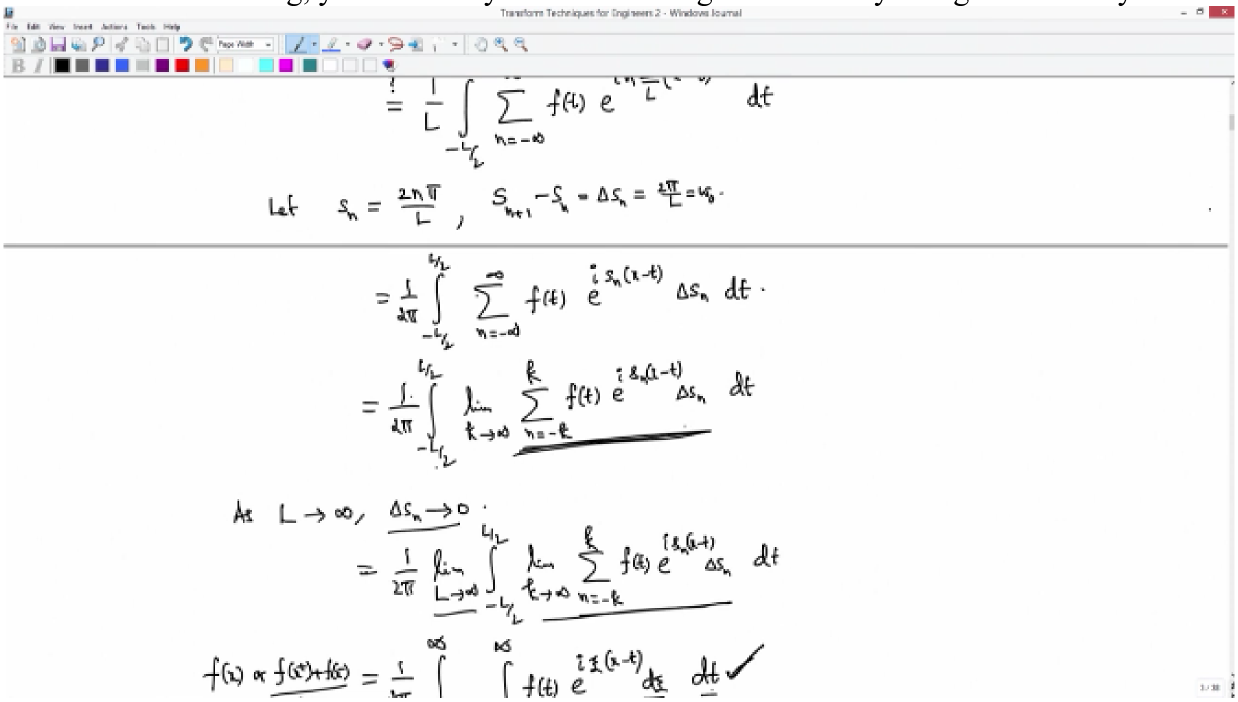
And what is left is once you have, once you say this is what is your Fourier transform, this integral, this integral double integral itself what you get is F(x) which is equal to 1 by, what you have already, this is what we define as F cap, so you already you have root 2 pi left there and this outer integral - infinity to infinity, and inside what is left? You have E power I xi X and you have D xi, okay, so that is what you have and what is remaining whatever you already defined this as like this so you have F cap(xi), so this is your inverse transform, and it's inverse transform, inverse Fourier transform, so from this Fourier integral theorem we can actually simply write like this, so that is your Fourier transform, we can define Fourier transform and it's



inverse transform like this, if you use this part some people may define either, so if you use this part for example you can also rewrite, you can define 1/root 2 pi - infinity to infinity F(t) E power - I xi, so it’s going to be + I xi T, so I xi T DT if you define it, if you use this part, this double integral you can define this as your Fourier integral and then in that case what happens to your inversion which is Fourier integral, this double integral itself 1/2 pi into - infinity to infinity.

So what you have written is F cap(xi) what is left is now T power – I xi X, I xi X D xi, of course here also you have to interchange the orders of integration so that you can define this part and this part, okay, so when you have so that's why most of the textbooks you may see whether this or this as your definition of Fourier transform, and it's inverse transform is if you choose E power - I xi T in the inversion you have E power + I xi X, okay I xi T, so because it’s function of X we were writing I xi X.

So anyway if you have exponential negative if you choose here you have exponential +I you have to choose in inversion, so if it is plus here you are getting negative, so this is how you define your Fourier transform, and it's inverse transform, okay, so this is the definition of Fourier transform so you can put it like this, this is from so inverse Fourier transform is coming from Fourier integral theorem, Fourier integral theorem, integral theorem which we have not proved only intuitively we have seen it's proof, okay, because of its many restrictions many we impose many things, we assume many things we can, we assumed we can bring this integral outside this infinite sum here, you somehow you allowed as L goes to infinity so you have not never seen such a thing, you have only seen here as L goes to infinity - M goes to 0 only this



part is actually that integral, integral - K to K, and then -K to K and at the same time simultaneously this integral also becoming - infinity to infinity, so such a thing intuitively it's actually true, so we will eventually see when we give the rigorous proof this is actually true, so that's why it's called intuitive proof, so through which we have defined Fourier transform and it's inverse transform.

And this again, this part this either of this integrals you can rewrite as, so if you split one integral, if you split xi part, xi integral if you split so let me take this double integral, so if you take this double integral 1/2 pi, remark 1/2 pi - infinity to infinity, this is DT integral. Now what you have is inside, inside is this one - infinity to infinity F(t) E power - I xi (x-t) or + xi (x-t) DT D xi, so DT D xi also you can put it so this what I do is I try to integrate from, so this D xi part I try to integrate from 0 to infinity, so inside as it is F(t) E power I xi (x-t) DT D xi + again I write 1/2 pi 0 to - infinity 0, - infinity to infinity F(t) E power I xi (x-t) DT D xi, so what I do here is you try to, you change this variable D xi as – D xi, so you have it's going to be infinity to sorry, this is - infinity to 0, I split it into - infinity to 0 here, and 0 to infinity is here, so this - infinity so D xi, if you replace I = - xi so you have a minus you will get a minus here because of D xi is – D xi and this limit becomes infinity to 0 and because of this minus you can rewrite this as 0 to infinity.

And then what happens here? This is going to be minus, because of xi you have to replace with –I, so this is what happens with plus, so now you try to add them together and what you see is 0 to infinity - infinity to infinity these are all common F(t) E power I xi, this is plus, this is minus right here, F(t) and what you have is E power I xi (x-t) + E power - I xi (x-t) DT D xi, so this is exactly two times, those 2 2 cancel so what you have is pi goes, so 2 - infinity to infinity F(t), this is cos xi (x-t) DT D xi, this is my F(x) or F(x+) + F(x-) divided by 2, so this is what it

Remark: $\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, dt\, d\xi = \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\xi(x-t)}\, dt\, d\xi \; +$

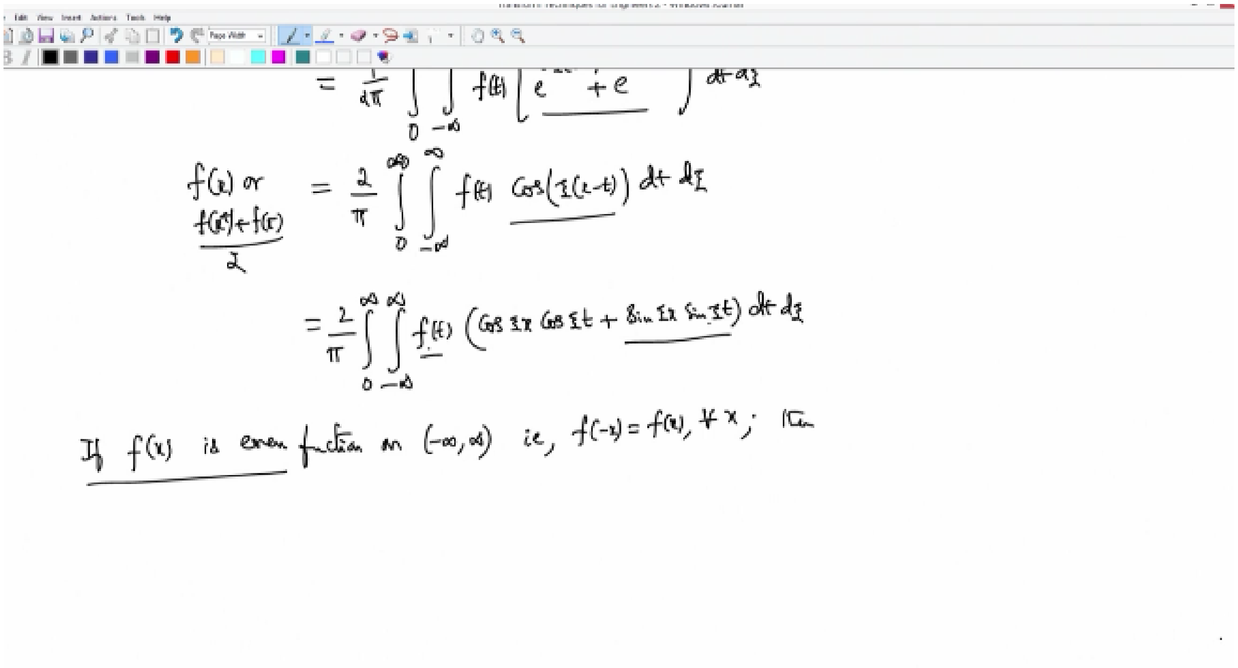
$$\frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{-i\xi(x-t)}\, dt\, d\xi$$

$$= \frac{1}{2\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\left[\underline{e^{i\xi(x-t)} + e^{-i\xi(x-t)}}\right] dt\, d\xi$$

$$f(x) \text{ or } \frac{f(x^+)+f(x^-)}{2} = \frac{2}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\, \cos(\xi(x-t))\, dt\, d\xi$$

becomes, both are same this is also Fourier integral theorem, intuitively we have shown that this is true now. So in this argument to show from here to here, from this to with writing in terms of cosine there is no intuition, this is a rigorous, both are same.

Once you have this, now you can define what is called a Fourier transform, Fourier cosine transform, so what is this one? You try to expand this, so you have 2/pi, 0 to infinity, - infinity to infinity F(t), now cos, cos thing you expand so you get cos xi X, cos xi T + sine xi X sine xi T DT D xi, this part if you see both are same so these are all same, okay, but if F(x) is even function, F(x) is even function on - infinity to infinity, that means F(-x) = F(x) for every X, if such is the case then what happens to this? If F is even this is even, and what happens to this part, this part - infinity to infinity that is DT integrals, that is DT integral is, if F is even because



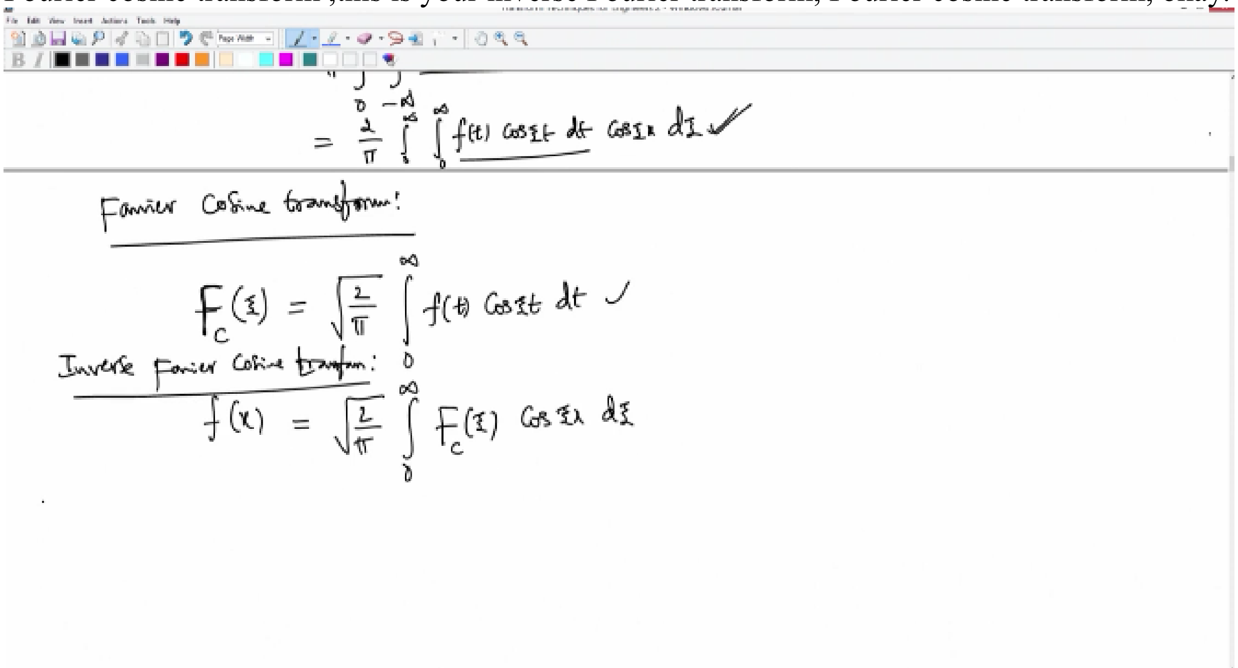
of sine function this is odd function, this is odd, so this part this whole thing F is even sine xi T, sine xi T DT that is F(t) into sine xi T that is odd function, so - infinity to infinity integral will be 0, so DT integral will be 0. So what you get is F(x) or the average part of it is, 2/pi integral 0 to infinity, and what you left with is - infinity to infinity F(t) cos xi X cos xi T DT D xi, this is what you have, if F is even function, because if F is even function this T integral with the second term is 0, so you have only, first term is left, you are left with only first term, because -infinity to infinity F(t), if you look at the second term sine xi T DT, of course you have, what you have is sine xi X, 0 to infinity part D xi, so this integral because this integrand F(t) sine xi T, F(t) into sine xi T is odd function now, because it is odd function - infinity to infinity this is 0, so that's way that makes it completely 0.

So left with only first term, so that is cos xi X cos xi T, so from this now I define what is called Fourier cosine transform, so you can define, so if you have a function that is defined only on the positive real line that is 0 to infinity, or if F is a function that is symmetric with X = 0, that is symmetric function at about X = 0, that means it defined and it is an even function then you can define a Fourier cosine transform on positive side that is from this Fourier integral theorem what you get is a Fourier cosine transform, so I'll define FC here, I prefer this notation, FC means a Fourier cosine instead of cap, what I have is capital F cosine(xi), so at the end you have xi, so here also again I split 2/pi as a with root, 1 is for the inverse transform, 1 is for the transform, so you have, so what happens to this double integral? So when you write 2 pi so you have, see here 1/2 pi when you write 2 cos xi thing, so 2 2 goes, so you have only 1/pi here, so you have only 1/pi, now you have only 1/pi here.

Now this you can also rewrite 1/pi 0 to infinity, and because now this function F is even cos xi T, both are even function so you have, so you can write this as 2 times integral 0 to infinity, F(t) cos xi T DT, cos xi X comes out which is nothing to do with DX D xi.

So now from this you can define what is your Fourier cosine transform, that is here root 2/pi 1, root 2 pi I have taken this is 0 to infinity, F(t) if your function is F(t) given signal cos xi T DT, so this is your Fourier cosine transform.

Now this integral, Fourier integral theorem will give you inverse transform that is F(x) which is one more root 2/pi is left, and this whole thing you put it inside so that is 0 to infinity and this part you have written with root 2/pi so you have, that is FC(xi) cos xi X D xi, so this is your Fourier cosine transform ,this is your inverse Fourier transform, Fourier cosine transform, okay.



So when you have, so you can from the Fourier integral theorem you can rewrite when F is even function and you can rewrite the Fourier integral theorem and then finally from which you can define your Fourier cosine transform and Fourier inverse cosine transform, so if F is odd function, or F is defined only from 0 to infinity you can extend it as an odd function to full -infinity to infinity, and in that case we can again do the same way and we can see that, we can define through which you can rewrite Fourier cosine integral theorem, and from which you can define Fourier sine transform and it's inversion. So we will see that in our next video. Thank you so much.

[Music]

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