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TRANSFORM TECHNIQUES FOR ENGINEERS

Introduction to Fourier Series

Dr. Srinivasa Rao Manam

Department of Mathematics

IIT Madras

So in this lecture we'll start with Fourier series. So before we start the Fourier series if you are an electrical engineering student in the signal and systems theory you have you know what is the signal. Signal is basically a real valued or complex valued function. We'll just define what is a signal in a systems theory. If you have some system like this kind of imagine, black box so you give the input and what comes out is output. So this black box you call it a system and input is function. o you give for all times. So let us say input I of T, so it's an input O of the or let us say E of T. This is electric signal you can think so that is you give this input and you get the output as the E of T. So for each time you give I you get E if it is a real value. So this is going this is coming through going through a system so that can be governed or described by differential equation or some way some other thing okay. So it's linear or non-linear that's – this is so basically for each I of T so for an input you have an output so that is a real valued function kind of a signal is I'll just define what is a signal. Signal so let us say f of t is a real valued real or complex valued function of time. So if it's a real-valued is a simple so for this complex so it takes so any complex number if I just recall Z is a complex number which is you can represent with R2 so you can put it in the plane so as X plus iY, X,Y belongs to real. So this and in a different form that is Euler form this is the length of this complex number amplitude of the complex number into E power I this is the theta is the angle okay.

So this is Euler representation of the complex number. So you have f of t which is – you call it so you have a different kinds of signals. So is a real if it is a real valued or a complex valued function we call it signal. Okay. So we start with the Fourier series of what is a Fourier series. So basically we deal with this real valued or complex valid function. We want to represent in terms of fundamental signal. So what are the fundamental signals so for example if you know what is the signal like this if from this system some simple fundamental signals are a sine T, sine T sine let us say Omega T, cos Omega T, Omega is the frequency okay. These are the fundamental signals fundamental signals. Okay. So you want to represent general signals so or general function f of T in terms of these fundamental signals okay.

So they're different times types of signals you have. So if T is the time continuous time variable so you call continuous signal. If T is discrete time so it is called a discrete signal okay. So this system so let's say L of L, L is the operator so which is a system L acting on I of T the input if it's a linear so L is linear. So linear system when do you say it's a linear system L is linear if it's a usual way so you we can make – you can take a linear combination of the inputs two inputs C F1 plus F2 if this is same as if you if you give this as the input what comes out is the C comes out as a constant this is the output for F1 input plus output for my input F2. So if this is the case then you call this a linear system.

So what do you give input is actually your signal so that is the real or complex valued function which is maybe continuous or discrete depending on whatever it is. Okay. So depending on whether it is a discrete signal or complex continuous signal so if it satisfies this you say that it's a linear, the system is -- the described system is a linear if it satisfies this.

As I said in the introduction so in natural sciences and engineering sciences most of the phenomena are repetitive. So for example as you can see vibration of a string in mechanical system or in electro-magnetic signals emitted by transmitters in satellites they are all repeated. So this is basically a repetitive process in the repetitive phenomena in natural and engineering sciences. So mathematically we can put this as so we call this – so the system so that phenomena the input or the outputs the signals okay we can represent as a periodic function. So that gives because they are repetitive in nature natural so many phenomena are repetitive so they're basically vibration of a string all this okay, pendulum of a clock in a mechanical system. In electrical voltage and current in electromagnetic systems okay. Voltage and currents in electrical networks these are all repetitive. So you basically in the nature what you have is the these signals are basically time periodic okay. So that means so after so between zero to T or time T the signal is a repetitive after 2T so what is it just repeats with its two time period T. So zero to T you have a signal so we have a different thing and it repeats from T to 2T and so on. So such signals we can represent as periodic functions. Mathematically they are periodic functions. So a function so definition for this periodic function is a function is periodic if F of a function f of x, f of T is periodic, if f of T plus you give this period the same as f of T for every T, such a functions are periodic periodic functions. You know many of these periodic functions for example Sine T, cost T, all these are period functions, sine T sine, cos, cos T, sine Omega T, cos Omega T these are as I said is fundamental signals they're also fundamental signals there are fundamental functions sine T, cos T these are all periodic functions.

So if I have f of T which is periodic, a periodic function f of T. So let us draw this how it will be if it is a repetitive how does it look. So if it is zero, if it is a zero to t let us say it's like this, okay. What happens at T to 2 T? So again it's a repetitive again. Same way. So like this minus T to zero also so repetitive like this and so on. So such a repetitive process is what you see in the nature they are actually periodic function.

So can we represent such functions? As I said in the introduction we will do this Fourier series which is actually a kind of transform, Fourier transforms takes so Fourier transform on periodic functions is actually a Fourier series, Fourier coefficients. So which is a Fourier transform and it's inverse transform is basically what you get is Fourier series. Okay. So we will see what it is before we define what it is a Fourier coefficients we will define for a time signal f of T will just define what is Fourier coefficients. So you start with the periodic function. Let f of X be a periodic function with period. So instead of calling it as a function of time I'm just trying – now onwards I will just write f of x as in a mathematical symbols will use a function of f of x which is a kind of you can imagine as a signal with x as a time variable. So f(x)be a periodic function with period L let us say then now what is its frequency if it is a repetitive. So if it is repetitive like this it has a periodic function then the frequency of the signal of f of x is so, W naught so we call this fundamental frequency which is 2 Pi by L. So you can think of sine function like this which is 0 to 2 Pi is a period so on par with this a sine function so fundamental signals those sine and cosine functions if you imagine. So 2 Pi by that their time period is 2 Pi so whose frequency is 1. So fundamental frequency is 1 so the sine 2t is also periodic function but whose time period is 2, 2 Pi by whose time period is Pi so this is 2 so you have like this it will go on so frequency is 1 and 2 depending on what you have here.

So we will define this. Then the frequency is defined like this. This is the fundamental frequency. Why you call this fundamental will come to know little later. So Fourier coefficients I will define what is a Fourier coefficient. Fourier coefficient defined so some integrals so I'm just giving so far each n so I'm just defining for each n which is from 0, 1, 2, 3, onwards for each n you define and with so I'll define it as an integral 2 by L minus L by 2 2 L by 2 f(x) so there's a time signal what is the input you have given. So this one so the signal you take and you multiply with these cosines and sines. So cos periodic functions those are fundamental periodic functions, cos n Omega naught. So Omega naught is this of x dx this is a function of x with this you multiply and you take it from minus L by 2 to 2 L by 2 because of its period is L. Okay. Imagine how you – so it's defined from minus L by 2 to L by 2. This function is defined like – defined over L by 2 and L by 2 to plus L by 2 plus L again it's repeated whatever is the function here this repetitive because it's a periodic function. So an is this and bn if I define like this 2 by L now this is f(x) now you multiplied with sine, sine n Omega naught x dx. So what are these? These are kind of frequencies. So this is n need to, W naught is a fundamental frequency, nw naught is a second fundamental second frequency so and so on. Okay. This is the first frequency. This is a second frequency n equal to 2 we have n equal to zero. It's just a constant okay. So this is our fundamental frequency and you have n Omega naught n equal to 2 and 2 onwards there are other frequencies.

So these are if they exist. These are Fourier coefficients when they exist. What do you mean by when they exist? So this integral should make sense? So f is such a signal such a periodic function for which you multiply with cosine and sine it should be integrable. So you should be able to integral, this integral value it should be finite. So all these an and bn should be finite for n equal to 0, 1, 2, 3, onwards. So what is actually happening? So once you define what is – these are the Fourier coefficients. So this is actually a transform. So what you are going is our time signal to some discrete values n is from 0, 1, 2, 3, onwards. So you have a time signal. So that you break it into a frequency so discrete frequencies. So here these are ans and bns these are the discrete frequencies. Okay. So n is basically n is 0, 1, 2, 3, onwards okay. So n into W naught these are the frequencies for discrete frequencies because n is running from 0, 1, 2, 3, and so on. So Omega naught starting with n equal to 1 is a frequency fundamental frequency, 2, 3 and so on will get different other frequencies. So how many? They are all discrete because n is running from 1 to infinity or 0 to infinity.

So you have a periodic function which is a time signal and you break this into discrete frequencies okay. Frequencies those are n W naught. So if I write like this so there is a 1-1 correspondence. So what is this 1-1 correspondence? So you have a periodic function, f(x) or the time signal and this side what I get you are given – you have given f(x) a time signal what you get is Fourier coefficients. You have Fourier coefficients here. So these are ans and bns. N is from from 0, 1, 2 ,3, and so on. So you have - if you give this input as f(x) okay. So what you get is a Fourier coefficients these are these things. So now, so these are – this is your Fourier transform on a periodic function. So these are our Fourier transforms. Also called Fourier transforms of f of x a periodic function f of x. So these an, bn are basically they define Fourier transform of a periodic function. So once you get a Fourier transform they are in that discrete space, the discrete numbers okay.

Now the question is so basically you go from a time signal to a discrete frequency signals right. You break it into different frequencies. So what happens so you should be able to retrieve back whenever you want after some process you want to retrieve it back. Is it possible to do that? Yes that is answer is yes so you will be able to come back from -- if you are given a Fourier coefficients an, bn numbers. A0, a1, and so on b0, b1 and so on if you are given. If you know all this all these Fourier coefficients, some numbers these are simply numbers. You should be able to reconstruct your f of x. So that is a Fourier series. So your transformation is from the periodic function you define Fourier coefficients as your Fourier transform and inverse transform is if you are given so these frequencies discrete frequencies if you know with these discrete frequencies you should be able to reconstruct. So if you are given this with these discrete frequencies you will be able to reconstruct your signal f of x. So that is Fourier series.

So this way is a Fourier transform. So other way. So this is inverse Fourier transform or here because of the periodic function in this case inverse Fourier transform is nothing but Fourier series. So what is this? How does this look?

So I can just reconstruct f(x) as in terms of ans, bns, and the fundamental functions, our fundamental signals like cosines and sines. So a0 divided by 2 plus Sigma n is from 1 to infinity, we have an cos n Omega naught X plus bn sine n1 Omega naught so this is a Fourier transform. This is a Fourier series. So if you know an naught an and bn so if you are given here you can get your function f(x) in terms of these just by multiplying sine and cosine functions. Okay. Sine and the fundamental signals like fundamental functions sine n Omega naught x and cos n Omega naught x you multiply and take this series. This sum is actually converges point-wise to f(x) under some conditions on a function f(x).

so what are those conditions? So you have to see right now we don't know exactly. So if you are period not all periodic functions you can get like this but under some sufficient conditions under certain conditions on f under certain conditions on a periodic function it's always possible. This Fourier series you can always – given any function f(x) periodic function f(x) you can always define a Fourier transform just Fourier transform is Fourier coefficients you can always define only thing you have to make sense is these integrals should be finite okay.

So as long as these integrals are finite for all such functions f(x) for all such signals, general signals you can simply get their Fourier coefficients. So one transform, transforming these two discrete space and then you should be able to come back then only it make sense. So that is the inverse Fourier transform. That only once you go to the – once you transform something you should be able to come back then only it is useful. So to get a Fourier series that means once you define these Fourier coefficients using them you can derive this Fourier series but that should be giving you your function. when is – the question is when is the Fourier series actually is originally is your signal f of x original function f of x original periodic function f of x. So under some conditions -- under some sufficient conditions on f(x) this star, the star is true for each x for each fixed value of x, fix x this is a series of numbers that converges to some number that is exactly equal to f of x once you fix x. Okay. So what are those conditions? I'll just give you briefly so if f is piece wise continuous if you know what is continuous function I hope you all understand what do you mean by know what is I hope you all know what is a continuous function. Continuous function is if you know the [Indiscernible] [0:23:28] knowing continuous function piecewise continuous function. I define piecewise continuous function. It is a continuous function as a pieces okay. If I give you for example like this between let us say a to be b function if it is like this, this is piecewise continuous function. So what happens you can think here so piecewise continuous function is it has a finite number of places. You have a discontinuity. A continuous function. So a function with finite number of discontinuities. So those discontinuities are also so you know the discontinuance are of many types. So one is 1 by x is discontinuous at x equal to 0. At x equal to 0 this goes to infinity. Okay. So that is also discontinuous. So these are but they are defined at that zero. At zero it should be finite so these are jump discontinuities. They are jump discontinuity. So that means so wherever it is a discontinuous at those values both the left hand side and right hand side the function is a finite value. Function is defined. Limit of those functions. Here this side limit f of x is this value and this side limit is a value here. So they are finite values okay.

So you have a finite number of discontinuities a function is finite is called is a piecewise continuous function. Roughly this is the definition okay. So we will define now formally later when as when require. So this is one condition. So piecewise continuous this is a one sufficient conditions but this proof is required. If you have a signal f of x that is piecewise continuous function like this then the Fourier coefficients you can always define because this is the content, pieces of finite number of pieces which are continuous functions you know those are integral values. These integrals are finite for such function. They say always finite so Fourier coefficients are well defined. Using this Fourier coefficients you make the sum that is always converges point-wise to f of x. Point-wise means you fix your x that value is actually equal to the signal value at x equal at such fixed value of x okay. So if it is a piecewise continuous function this is always true but the proof is a little difficult but what we prove later on is if a piecewise a smooth function. So that means continuously a differentiable function. This means f of x. so if you have f is a piecewise continuously differentiable function f dash of x is a piecewise continuous function, as you see f dash of x is defined at every point between A to B and the A and B. F dash is defined everywhere, okay, except at those points. So those limits left-hand sides of – left's left-hand limit of f dash. f dash at b means the definition of the derivative what is the derivative of f dash of x at b at x actually you see f of x plus h minus f of x this is the definition of derivative. So h as h goes to 0. So this is the if 0 plus or minus both are same okay, zero means zero both plus or minus 0. If you take plus that is x right side x plus h so you basically you're looking at the derivative at this point, for example you are looking at x is value here this is x plus h right so you have this side. This is the derivative. So derivative from the right side. At this point so derivative this side. So that the limit this side and if you take h goes to 0 minus that is you finally get x minus positive h you are going – you're taking the limit here. So you're looking at the tangent at this point okay, the limiting value here for this function here.

So f dash of x which is defined where f is differentiable at all those points. First of all it should be defined it is differentiable at every point okay. So f is differentiable and if you look at the derivative of derivative f(x) f dash derivative of f that is f dash of x that is piecewise continuous function okay. Then f is piecewise differentiable function. Okay. So continuously you should use piecewise differentiable function is if f dash is the piecewise continuous function then we say that f is f of x is piecewise differentiable function. So if this is the case if your signal f of x is piecewise differentiable function then you can easily show that this Fourier series is actually converging to the signal f of x at every point x okay.

So I'll prove this maybe in the next video so what – later videos will try to prove this result. Now I will just take some function f of x, periodic function f of x. So how do we look at the periodic functions? We will just would take some periodic function. We will try to calculate this Fourier series and we can actually see that we will just find the Fourier series and if you – for the time being if you assume this result under these conditions if piecewise differentiable function f is piecewise differentiable function you can always assume that you can believe that F is this, this series whatever you calculate this one that actually converges point-wise to f of x. This we will actually prove it in the later videos. Right now we will take some f of x we will try to calculate the Fourier coefficients ans and bns and then I will make this series. That series is actually converging to f of x if you believe these sufficient conditions okay. So let us start with f of x. So example, let's take the say a signal like this f of x equal to let's say mode x with period. It's a periodic function so what do you mean by periodic function mod x? If you want if you take the periodic function which is mod x, it should give the function with period so that means if it is it defined x belongs to let us say 0 to 1 if you say like this what does it mean? So how does this look? Function looks like this this is 0, 1 so you have mod x that is 0 to 1 or let us say minus 1 to 1. So minus 1 to 1. So you have minus 1 so this is 0 and this is 1. So at x equal to 0 to 1. This is 0 to 1 like this and here this is the negative. So if x is negative this is going to be minus minus x. So minus x minutes you should write like this is what is your function.

So minus minus 1 to 1 is this one. So what is the period? Period is 2 for this. So 1 to again 1 to 3 so you repeat the same. This is 2, you repeat like this. So this is how you can extend always like this, 3, 4, 5, so because it's a period 2 it will go on like this. So you can define so – 1, - 2, - 3, - 3, - 4, - 5, and so on. So this is how it looks mod x. Mod x function with if it is defined at -1 to 1 I can always extend it to full real line like this. So that is a periodic function. So whatever this function which I am drawing here this function is a periodic function because it's a repetitive and this period is minus 1 to 1. So that is 1 minus 1, 1 minus of minus 1 that is 2 period is 2. So such a function first you calculate what is its an. So an. So what is period? Period is 2, so L is 2. So L is 2 so what is the value of an so just look at what is an so an is 2 by L that is 2 by 2 integral minus L by 2 that is minus 1 to 1 f of x which is mod x cos n. So what is our W naught fundamental frequency that is 2 Pi by L that is 2. So this is Pi. 2 Pi by L so n Pi x, dx. So these are your Fourier coefficients. ans n is from 0,1, 2, 3, onwards and you can also have bn which is also 2 by L, L is 2 and this is L by 2 minus L by 2/ 2 L by 2 that is minus 1 to 1. L is 2 by 2 so that is 1 minus 1 and you have minus x instead of cos now you multiply with sine n W naught is Pi x, dx. Again n is from 0, 1, 2, 3, onwards. N equal to 0 this anyway this b0 is always 0 because of sine. Sine 0 is 0 and you put n equal to 0 so is running only from 1 to 1 onwards okay.

So you can calculate now again an. What is an? So this is minus 1 to 1. So you can write 0 to 1 mod x is X, X into cos n Pi x, dx plus minus 1 to 0 if it is a negative mod x is minus X, cos n Pi x, dx. This is what it is. So if you calculate this one so you can easily integrate and find this value. When you put x equal to 0 what is the value of so n equal to 0 so you get a naught, a naught value that is integral 0 to 1 x dx, minus minus 1 to 0 x, dx. So this is going to be x square by 2 0 to 1 minus x square by 2 minus 1 to 0. So this is going to be 1 1 by 2 minus minus plus 1 by 2. So it's going to be 1. Okay. So a naught value is 1 and if n is not equal to 0, n equal to naught, that means n equal to 1, 2, 3 onwards. So you can calculate what is an. An is so you can do the integration by parts here. So that is going to be sine n Pi X by n Pi into X 0 to 1 minus integral 0 to 1. What is this one? So this is going to be sine n Pi x by n Pi into derivative of x, there is one dx. So this is 1 and minus other integral, minus other integral you can do the same. So that is going to be sine n Pi x by n Pi into X 0 minus 1 to 0 here, that goes minus miunus plus into - 1 to 0 sine n Pi X by n Pi into derivative of x is 1. So we have dx. This is what you have here. So sine n Pi X equal to 1 sine n Pi that is 0 n equal to 0, x equal to 0 also this is 0. So it's going to be n is any way nonzero so this is going to be 0. This is 0 and what you have here is minus 1 by n Pi so cos n Pi x by n Pi 0 to 1 this one, this is also 0 because of sine this is also 0. So you get here here also you get 1 by n Pi again. So here cos n Pi x by n Pi 2 - 1 to 0. So this is so 1 by n square Pi square cos n Pi so what is the value of this cos n Pi is what is the cos n Pi minus 1 power n okay cos n Pi value is minus 1 power n and minus and this is 1, cos 0 is 1. So this is what you have minus 1 by again n Pi square here 1 minus again cos n Pi. So that is going to be minus 1 power n. So this is going to give you minus 1 by n square Pi square so you have 2 by 2 times -1 power n is your an. So n is from 1, 2, 3, onwards. So this is your an. So similarly you can calculate your bn. Okay.

So you can calculate your bn. Bn if you do the same so you can do as an exercise and bn is what you just calculated take it as an exercise and do it. Simple integration.

Then because f of x is a periodic function and it is mod X so mod X you can clearly see that mod X is a continuous function between minus 1 to 1 with period 2 which is repetitive as you can see. It's not even piecewise. It is simply a continuous function between minus 1 to 1 and which is periodic. It's repetitive with every two x values. So for every subsequent interval of length 2 you see repetitively. So that is the piecewise continuous function. So you can calculate ans and bns as you see f of x which is mod X which you can see that a naught a naught is a naught by 2 so a naught is 1 so you have 1 by 2 plus a sigma n is from 1 to infinity, an is is 2 into -1 power n divided by n square Pi square times cos n Pi x plus one more is bns once you calculate this bns you know exactly so bn times sine n Pi x, sine n Pi x. So this is what it is. Okay. This is your Fourier series. You can represent x belongs to minus 1 to 1 this is always true. So you fix your x value here this series is the series of numbers that converges to value mod X at that point x, fix your x as x naught mod X not equal to this half plus this series which should replace x by x naught that series of numbers so that converges that value is actually equal to mod X. so this is your Fourier series.

So we are still we don't know exactly. We just believe that this Fourier series is actually converging to this f(x) we will have to prove it to actually believe finally to accept that this Fourier series is actually converging to f of x. We will do it as a theorem in later videos that this Fourier series always converges pointwise to this function under some sufficient conditions on f of x and the signal is piecewise continuous function which means differentiable function. Piecewise differentiable function if you have a signal you can get a Fourier series. That inverse transform is always exist and that converges to the signal itself. Okay. This is what will prove maybe in the later videos. Maybe I'll just try to get you what is bn in the next video and we will see exactly what is the representation for this mod X so that you can also conclude some mathematical results out of this Fourier series. Okay. That we will see in the next video.

Thank you very much.

Online Editing and Post Production

Karthik

Ravichandran

Mohanarangan

Sribalaji

Komathi

Vignesh

Mahesh Kumar

Web Studio Team

Anitha

Bharathi

Catherine

Clifford

Deepthi

Dhivya

Gayathri

Gokulsekar

Halid

Hemavathy

Jagadeeshwaran

Jayanthi

Kamala

Lakmipriya

Libin

Madhu

Maria Neeta

Mohana

Mohana Sundari

Muralikrishnan

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Parkavi

Poonkuzhale

Poornika

Premkumar

Ragavi

Raja

Renuka

Saravanan

Sathya

Shirley

Sorna

Subhash

Suriyaprakash

Vinothini

Executive Producer

Kannan Krishnamurty

NPTEL Co-ordinators

Prof. Andrew Thangaraj

Prof. Prathap Haridoss

IIT Madras Production

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