Graph Theory
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Lecture - 05
Part 1
Minimum Spanning Tree

Welcome to the 5th lecture on Graph Theory. Today we learn a special type of graphs called Trees. A tree is a connected acyclic graph and a collection of trees is called a forest. So, we learn few results related to trees and then we move to Minimum Spanning Tree given weighted graph how to find Minimum Spanning Tree of the given graph using Kruskal algorithm. So, let me formally define what is a tree?

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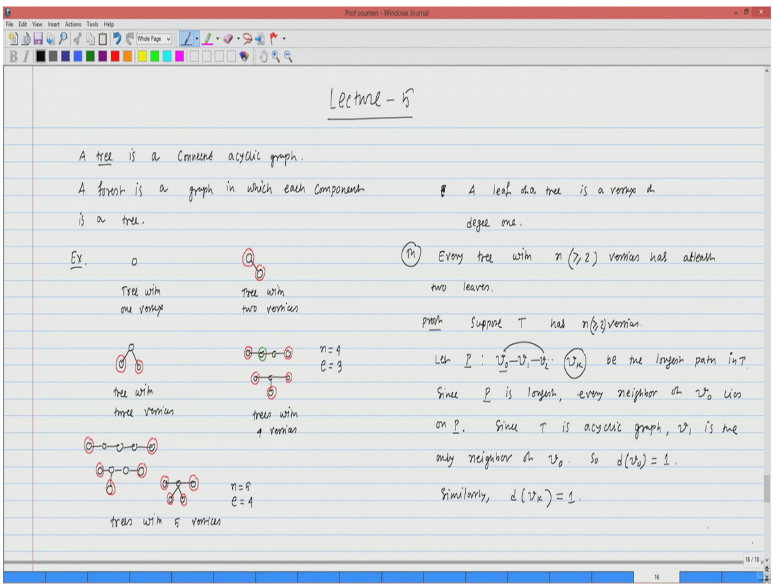

So, a tree is connected acyclic graph and a forest is a graph in which each component is a tree. So, acyclic means the graph has no cycles. So, let me give some example of trees now. This is a tree this is a tree with one vertex this is a tree because this is connected graph and there is no cycle here this is tree with 2 vertices. This is a tree because it is a connected graph and there is also no cycle. So, it is a acyclic connected graph and this is tree with 3 vertices.

So, this is a tree with 4 vertices and this is another tree with 4 vertices. So, the trees with 4 vertices and, this is 1 tree with 5 vertices and this is another tree with 5 vertices they are not isomorphic. So, this 2 trees are not isomorphic and, this is these are all trees with 5 vertices and every tree and let me first defined what is a leaf; a leaf of a tree of a tree is a vertex of degree 1.

These are the leaf nodes this is a leaf this is a leaf this is a leaf node is a leaf nodes location. So, vertex of degree 1 entry is called leaf. Now we prove a theorem related to tree every tree with n vertices n greater than equal to 2 has at least 2 leaves. So, that sort of you can you can observe that this is correct that every tree with at least 2 vertices has at least 2 leaf nodes.

So, this is a tree with 4 vertices it has 2 leaf nodes this is a tree with 2 vertices and both are leaf node this is a tree with 3 vertices. So, it has 2 leaves so here both are leaf node because both are of degree 1. So, this graph or this tree has 2 leaf node this tree has 3 leaf node and this tree has 4 leaves. So, this is quite clear, but we need to formally prove this one proof suppose tree T has n nodes vertices.

And, let P be the longest path in T and the longest path is this 1 v naught v 1 up to v k be the longest path in T. Now since P is longest every neighbor of say v naught lies on P why this is true let me well. So, this is the longest path, now, suppose v naught has you know what is neighbor right first of all you know what is neighbor so the neighbor of this vertex.

So, the neighbor of this vertex are this node and this node are the neighbor of this vertex, what it says is that all the neighbors of v naught are on this path, if there is another neighbor of v naught which is not in the path, say something v star which is a neighbor of v naught, but it is not in this path then you can get a longer path than this path because you can sort of add v star here to get a longer path than this one and since we have assumed that this is the longest path.

So, that is why all the neighbors of v naught lies on P, and since T is acyclic graph acyclic graph v 1 is the only neighbor of v naught because this is the acyclic graph. Of course, this 2 are there in the path these 2 are adjacent and then v naught cannot have this also as a neighbor then this will form a cycle, but the graph T is a acyclic graph. So, v 1 is the only neighbor of v naught.

So, that is why the degree of v naught is equal to 1 because it has only one neighbor that is v one and using the similar argument you can also prove that v k is also of degree v k has degree 1. Similarly degree of v k is equal to 1. So, what we have proved is that a tree with at least 2 vertices has at least 2 leaves and next we prove another result related to a tree.

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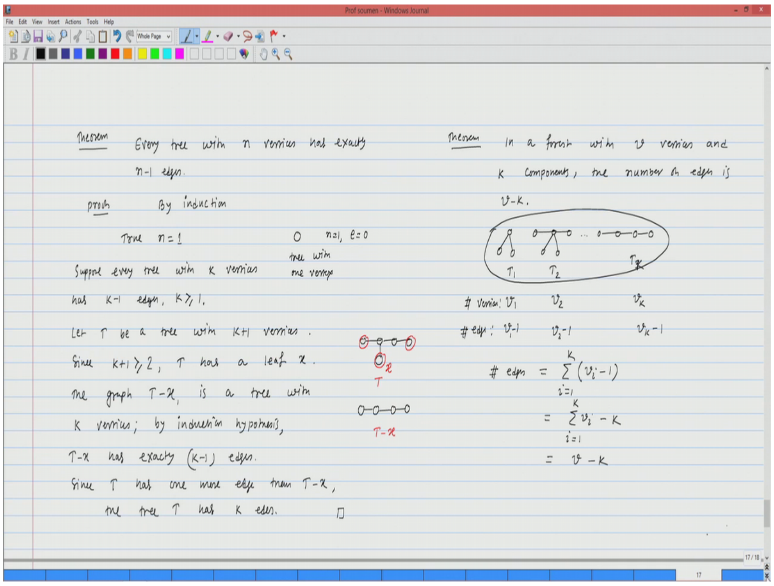

This is another Theorem every tree with n vertices has exactly n minus 1 edges because of the property of the graph that the graph is the graph is acyclic and connected because of that if we tree with n nodes has n minus 1 edges let me just illustrate this thing using the example that we have in the previous page of show here you can see that. So, this is a tree with 5 vertices, n equal to 5 here and the number of edges is e denotes the number of edges.

So, number of edges is 4, it has n minus 1 edges, what about whichever tree you take this is true this is tree with 4 vertices. So, n is equal to 4. So, the number of edges is equal to n minus one I can see that there are 3 edges, but we need to prove this result that every tree with n vertices has n minus 1 edges. So, will prove this formally and will prove this by induction we have proved some theorem using contradiction and different cases this is perhaps the first time we proving a result in this course by induction.

So, induction goes in this way whether this result is true for true for n equal to 1. So, n equal to one means a tree with only 1 vertex and this is the tree with 1 vertex, n equal to

1 here and the number of edges is equal to n minus 1 0 it has no edge. So, this theorem is true for n equal to 1 and then this is the induction hypothesis suppose every tree with k vertices has k minus 1 edges for k greater than equal to 1.

This is the induction hypothesis that every tree with k vertices has k minus 1 edges and let T be a tree with k plus 1 vertices and then we have to prove that the T has k edges. Now since k plus 1 is greater than equal to 2 this T has at least 2 leaves, here I am just T has a leaf node for sure T has a leaf say X. Now I will denote this thing that the graph T minus X; that means, from the tree I am removing this leaf node X.

So, how this operation goes suppose I have this tree for example, and this is a tree with 5 nodes and this is a leaf node, this is the leaf node, this is a leaf node. There are 3 leaf nodes and suppose this node I denote by X and this is my current tree T, now what I mean by T minus x is that you remove this node from this graph. So, when you are removing a node you have to remove the adjacent edge also. So, after removing this vertex x and the T minus x is this tree this is what the T minus x is when T is this 1 and x is this node.

The graph T minus x then is a tree with k vertices and by induction hypothesis induction hypothesis this is my induction hypothesis I had assumed that every tree with k vertices has k minus 1 edges. So, by induction hypothesis T minus x has exactly k minus 1 edges because T minus x is a graph is a tree with k vertices. So, by induction hypothesis T minus x has exactly k minus 1 edges.

Since T has one more edge then T minus x, here you can see that this is your T minus x and this is your T. So, T has one more edge then T minus x that since T has one more edge then T minus x, the tree T has k edges. So, this is how you prove using induction hypothesis that you first proof to prove a theorem using induction hypothesis first you prove that the base case is true; that means, you prove that statement is true for n equal to 1.

And then you assume that the statement is true for k; that means, here the statement is true for tree with k vertices, that every tree with k vertices has k minus 1 edges and then finally, you have to assuming that this is true you have to prove that a tree with k plus 1 edges, a tree with k plus 1 vertices has k edges and that is complete the whole proof. We will talk about another theorem in a forest with v vertices and k components the number of edges is v minus k.

So, you understood what is forests right, this is forest consists of several trees each component is a tree, let me just draw a forest of this is an example of a forest because this is one graph the whole thing is one graph and this is a forest and it has 3 components and let me call this one the first component is T 1, second component is T 2, and the third component is T 3.

In general I am assuming that this tree has v 1 vertices these trees has v 2 vertices and say suppose there are k components in general and this tree has say v k vertices, if tree T 1 has v 1 vertices then it has v minus 1 v 1 minus 1 edges. So, this is the number of edges, this is the number of vertices and second tree has v 2 minus 1 edges and the k th tree has v k minus 1 edges.

So, the total number of edges in the tree in the forest is equal to summation of vi minus 1 i is from 1 to k, which is equal to summation of vi i equal to 1 to k minus k and summation of vi is from 1 to k is equal to the total number of vertices this is v minus k. We have proved that of course, this is you know I am using the previous theorem to claim that tree with v 1 vertices is having v 1 minus 1 edges and from this theorem follows from the previous theorem.

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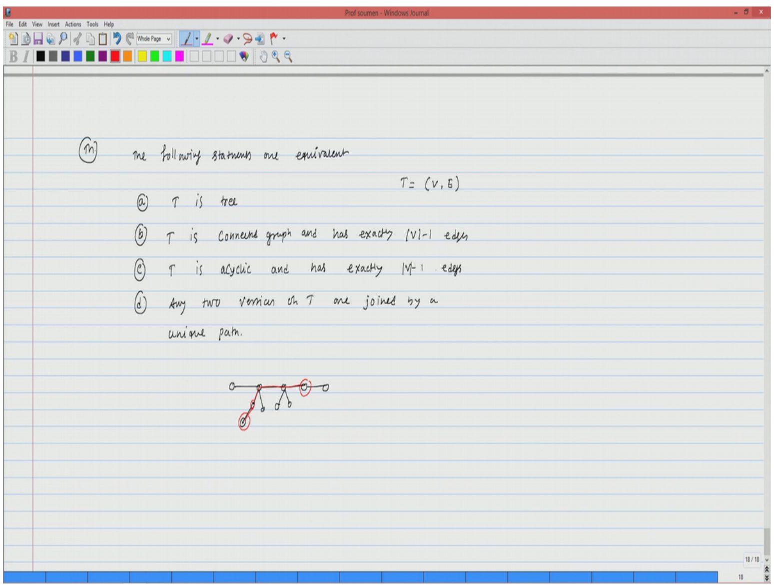

We have proved that the number of edges in this forest is v minus k. Next we just state another theorem related to tree the following statements are equivalent the first statement is T is a tree. I will not prove this theorem just for your information these are easy you can refer some book for the formal proof T is a tree which is equivalent to say that T is connected graph and has exactly v minus 1 edges.

So, T means it is a T v is the vertex set and e is the edge set. So, a is equivalent to v and the third statement is T is connected T is acyclic and has exactly v minus 1 edges and d is the another statement any 2 vertices of T are joined by a unique path, this is very much helpful because there is no cycle as you can see let me just draw a tree quickly.

This is a tree and now you consider any 2 vertices say this vertex and this vertex there is a unique path between these 2 vertices. So, the path here clearly you go from here to here and this is the unique path between these 2 vertices. And since tree is a connected and acyclic graph any 2 distinct vertices are joined by a unique path. We have learnt what is a tree, what is a forest and some basic results related to trees and forest. In the second part of this lecture we learn what is a minimum spanning tree in a weighted graph.

Thank you very much.