Graph Theory
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Lecture - 04
Part 2
Diameter of a graph; Isomorphic graphs

Welcome to the second part of lecture 4 on Graph Theory. Now, we will learn graph isomorphism. Sometime two graphs they may look different, but they are actually the same graph. So, then those two graphs or several graphs are said to be equivalent. So, formally we will learn; what is Graph Isomorphism of Isomorphic Graphs.

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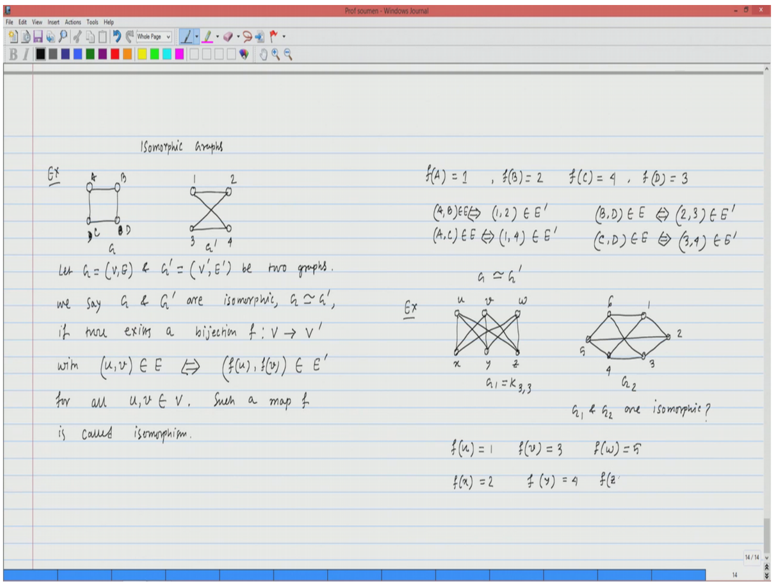

Let me give let me start with an example this is one graph. Let me call the vertices A B C D. And I draw another graph with the same number of vertices and the same number of edges of course, and this is the other graph, and I label the vertices by 1 2 3 4.

Now, these 2 graphs they may look different, but they are basically the same graph, this is very easy to observe because this graph is very small here. Let me give the formal definition of a graph isomorphism. Let G V E and G prime V prime E prime be 2 graphs. We say G and G prime are isomorphic. Or notation is G isomorphic to G prime, if there exists a bijection from the vertex set of graph G to the vertex set of graph G prime with this condition. That if u v is an edge in the graph G this is an edge in the graph G, if and only if f u f v is an edge in the graph G prime, for all u v belongs to v, and such a map f is called isomorphism, ok.

Now, if I consider this example here what is that map I say that this 2 graphs are isomorphic and this is a visually also it is very easy to see that, now if I map this one say if I map the vertex A to 1 vertex B I am giving the map here. F of B, B maps to 2 f of C, C maps to 4 and f of D; that means, D maps to 3, sorry this is this is C, this is D otherwise have to change here. Now I have to see that whether the adjacency is preserved. Now A and B their adjacent in G; so this is in E so 1 2 is also a is adjacent in G prime. So, 1 2 is an edge in G prime.

Similarly, A C is an edge in the graph G. So, A C 1 4 is an edge in graph G prime. And similarly let me just complete this one B D is an edge in E. So, 2 3 is an edge is in graph G prime. And C D is an edge in E. So, C D 3 4 is an edge in E prime. So, all the adjacency relations are preserved. So, this map is an isomorphism that is why let me call this as G and G prime. So, G and G prime are isomorphic. This is one example; let me give another example of suppose this is my first graph G 1. And this is a complete bipartite graph. So, this is nothing but G 1 is nothing but k 3, 3 I draw another graph with 6 vertices again. You can see that these 2 graphs are having the same number of vertices same number of edges everything same G 2.

Now, the question is whether G 1 and G 2 are isomorphic. So, this is the question of let me label them u v w x y z and 1 2 3 4 5 6. Now this is not so easy to realize that whether these 2 graphs are isomorphic or not. But actually they are isomorphic if you can see that this is also the G 2 is also a bipartite gap and G 2 is in fact k 3 3. So, what you have to do is that you have to see. See 1 3 and 5 these 3 vertices are non adjacent.

So, they will go to one part and then 2 4 6 that will go to the other part, and then you will see that this graph is actually k 3 3. And formally here is the isomorphism f of u is 1. F of v is equal to 3; that means, 1 3 and 5. I am keeping in one side of a bipartite graph. And f of x is equal to 2 f of y is equal to 4, and f of z is equal to 6.

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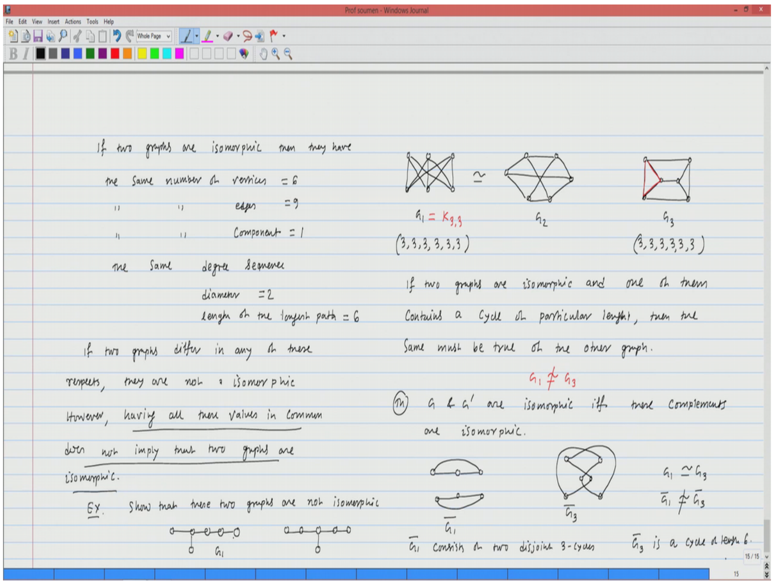

So, next we learn few conditions if 2 graphs are isomorphic, then they have the same number of vertices they have the same number of edges. They have the same number of components; they have the same degree sequence. You do not know; what is degree sequence I just explained this one. They have the same diameter and the same length of the longest path.

You can see that all these things are true in G 1 G 2 in my previous example. If 2 graphs differ in any of these respects they are not isomorphic. However, this is important you know they have to be the same in all this respect this is necessary condition, but this is not enough; however, having all these values in common does not imply that 2 graphs are isomorphic, ok.

So now to explain that even if all these conditions are satisfied all these values are in the same, that does not imply that the graphs are isomorphic, let me take a another a example and that is G 3 maybe I will just go back to the previous slide. Let me continue here. So, this was my G 1 I will just redraw it quickly. So, this is my G 1 and a G 2 was this is a beautiful example to explain isomorphism. This is G 2 let me consider another graph G 3 and here is the graph, this is my G 3.

Now, the question is whether G 3 is we know that G 1 and G 2 are isomorphic. My question is whether G 3 is isomorphic to G 1 or G 2 the same thing if I look at all these conditions here a G 3 has 6 vertices and this graph G 1 or G 2 they are also having 6 vertices same number of vertices equal to 6. They have the same number of edges 9 this graph G 3 has 9 edges G 1 has 9 edges. They have same number of components G 1 has one component G 3 is also having one component.

Now, the degree sequence I said that the degree sequence probably you do not know the degree sequence is simple just you write down the degree of all the vertices. So, here the degree of every vertex is thing 3. So, the degree sequence is this one. And you can see that the degree sequence for this graph is also this is a 3 regular graph. So, the degree sequence is 3 3 3 3 3, 6 vertices that is why it is a sequence of 3's. The degree sequence is also the same, the diameter of this graph is 2 and you can check that the diameter of this graph is also 2, because you compute the distance between any 2 non adjacent vertices.

For example this and this they are non adjacent the distance is 2 this and this, this distance is 2. A length of the longest path you can check that the length of the longest path between 2 non adjacent vertices, 2 vertices basically, it truly 6 you can check and this is true here also. Now the question is everything is same, but that does not imply that that 2 graphs are isomorphic. So, what is the difference between G 1 and G 2? You need to look at the other factors also like if, if 2 graphs are isomorphic and one of them contains a cycle of particular length particular length.

Then the same must be true of the other graph. So, here is the difference if 2 graphs are isomorphic and one of them contains a cycle of particular length. So, G 3 has a cycle of length 3. So, this is a cycle of length 3, but as we have you as you have learned that G 1 this is a bipartite graph. So, a graph is bipartite, if and only if it does not have any odd cycles. So, G 1 since it is the bipartite we do not need to check and it is obvious that G 1 is not having any odd cycles where G 3 has a odd cycle, ok.

So, that is the reason that G 1 G 1 and G 3 they are not isomorphic. So, this is one way of checking sort of identifying that or proving that a 2 graphs are not isomorphic. There is another theorem I will state now, that G and G prime are isomorphic if and only if their complements are isomorphic. So, this is also a theorem which is helpful sometime to prove that 2 graphs are isomorphic or not isomorphic. Now G 1 and G 3 will be isomorphic, if and only if their complements are isomorphic, I am trying to prove the same result that G 1 and G 3 are not isomorphic in a different way.

So, what I will do is that I compute G 1 complement. So, G 1 complement is this one is not difficult to observe, that this is G 1 complement. So, my G 1 is this k 3 3, and G 3 complement is this graph of you can verify that. Well, this is adjacent to this, this is adjacent to this one this one is adjacent to this; this one is adjacent to this. Now these 2 vertices are adjacent in G complement, and these 2 vertices are also adjacent in G complement.

Now you can see that G 1 complement consists of 2 disjoint 3 cycles. And G 3 complement is a cycle of length of length 6. So, or it is a 6 cycle basically 3 cycle means cycle of length 3. So, you can see that G 1 complement if I now look at this conditions G 1 complement is having 2 components. Whereas, G 3 complement is having only one component. So, that is why G 1 complement and G 3 complement G 1 complement and G 3 complement are not isomorphic. And since they are not isomorphic G 1 is also not isomorphic with G 3, because of this theorem.

And the other way to claim that they are not isomorphic because G 1 complement has a 3 cycle, whereas G 3 complement has a 6 cycle. So, that is also you do not have a 6 cycle in G 1 complement. So, I will just talk with one problem you just this is an exercise So that these 2 graphs are not isomorphic. So, the first graph is 5 vertices here, then one vertex here 1 2 3 4 5, and one vertex here. So, this is my G 1 and this is my G 2 see they look very similar, but they are not equivalent basically. And you try to prove formally that these 2 graphs G 1 and G 2 are not isomorphic.

So, in lecture 4 we have learned how to compute diameter of a graph and also we have learned some techniques to verify whether two given graphs are isomorphic or not. That is all for today.

Thank you very much.