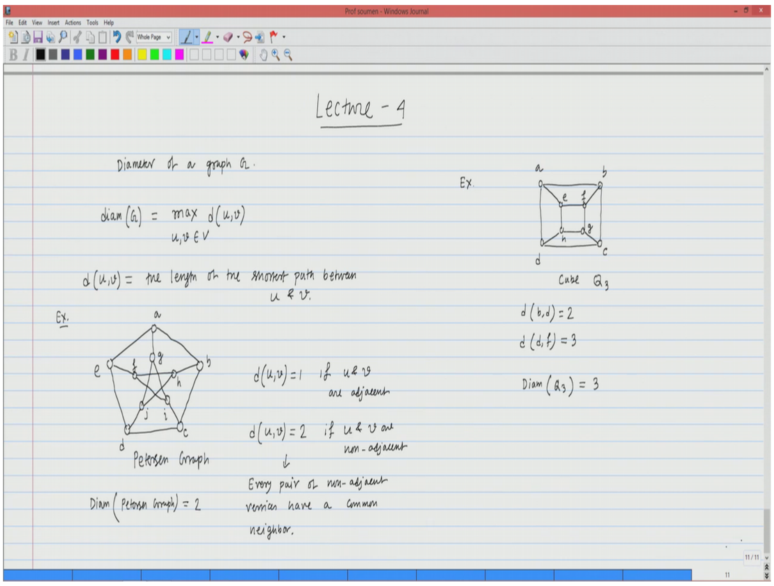
Graph Theory

Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture - 04
Part 1
Diameter of a graph; Isomorphic graphs

Welcome to the 4th lecture on Graph Theory. Today we will learn Diameter of a Graph and Graph Isomorphism. So, the diameter of a graph is computed as follows. You compute given a graph you compute the distance between every pair of vertices. By distance I mean you compute the length of the shortest path between every pair of vertices. And then once you have the distance between every pair of vertices, the maximum of all these distances is the diameter of the graph.

(Refer Slide Time: 01:23)



So I will formally define; what is a diameter of a graph and give many examples to understand the definition of diameter of a graph. So, let us start with diameter of a graph G. So, notation for this one is diam G. So, this is the maximum of the distances d u v and u v are from the set from the set v.

So, d u v is the distance between vertex u and vertex v. So, this is computed as follows. So, d u v is equal to the length of the of the shortest path between u and v. So, I will give an example to illustrate this definition. Let me start with the Petersons graph that we learned in the previous class in lecture 4.

So, it has 10 vertices, and we know how to construct Peterson graph. So, these are the edges of this graph. And this edge is adjacent to these 2 edges. Similarly these 2 vertices are adjacent, and these 2 vertices are adjacent. So, this is what the Peterson graph is Peterson graph. Now I can label them by a b c d e f g h i and j. And now you can see that. So, what I could do is that I have to compute the distance between every pair of vertices.

Now, the distance between u v when u v are adjacent like b c the distance between b and c there are many paths between b and c you can come to see from b using this path also, like you go to a a to e e to d d to c that is a longer path, but you have to consider the shortest path between b and c. So, if this is equal to 1 if u and v are adjacent. And now you consider 2 vertices say u and v and these 2 vertices u for 2 non adjacent vertices u and v are non adjacent.

So, for example, d and b are 2 non adjacent vertices, and there are many paths between d and b. So, for example, this is a path d j h b this is a path of length 3 there are many other path you from d j g a b this is the path of length 4, but I have to consider the shortest path the shortest path is d c b. And this is equal to 2 and this is true for any 2 non adjacent vertices ok.

So, if 2 vertices are adjacent and then; obviously, the distance between those 2 vertices is equal to 1 and if 2 vertices are non adjacent then their distance is 2 specially for the Peterson graph, because in case of Peterson graph this is true because every pair of non adjacent vertices have a common neighbor, in case of Peterson graph of course.

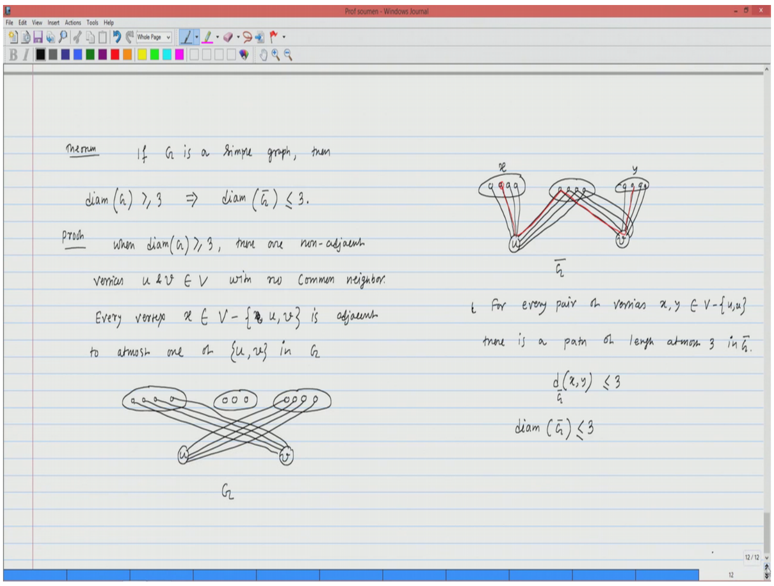
So, what I mean by neighbor the meaning of neighbor is that. So, the neighbor of a vertex a are e g and b. So, this is these are 3 neighbors of the vertex a. So, if you take any 2 non adjacent vertices for example, d and a they have a common neighbor that is why the distance between 2 non adjacent vertices is 2.

So, you compute the distance between every pair of vertices and you see that for 2 vertices which are adjacent the distance is one, and for any 2 vertices non adjacent vertices the distance is equal to 2. So, the diameter of Peterson graph is equal to 2. So, let me take this is one example, let me take another example. So, that you do not have any doubt this is called cube it has 8 vertices. So, I label them by again a b c d e f g h.

Now, you compute the distance between any pair of vertices. For example, the distance between b and d the distance between b and d is 2. Now the distance between say d and f d and f is equal to 3 because this is the shortest path d h e f, there are many paths of length 3. So, this is one the other one is d a b f. So, the distance is equal to 3.

So, this way you can see that the distance between you can compute distance between every pair of vertices and you can see that the maximum distance is equal to 3. So, this is denoted by q 3. So, the diameter of q 3 is equal to 3. Next we prove result theorem related to the diameter of graph.

(Refer Slide Time: 11:22)



If g is a simple graph, then diameter of g greater than equal to 3 implies diameter of G complement less than equal to 3, this is a beautiful result. So, you know; what is the complement of a graph the complement given a graph G. We know how to compute compliment of that graph 2 vertices are adjacent in the complement graph if and only if they are not adjacent in the original graph G. So, this is what the compliment of a graph.

Now, we will prove this theorem. So, so when diameter of a graph G is greater than equal to 3, there are non adjacent vertices u and v in the graph G with no common neighbor. So, when I say the graph has diameter greater than or equal to 3; that means, there exist a pair of vertices u and v such that the distance between these 2 parties is greater than or equal to 3.

Since the distance is greater than equal to 3; that means, there is no common neighbor of these 2 vertices if there is a common neighbor of these 2 vertices say a, then there could the shortest path could have shortest path between u and v could have been u a and t. So, so this is what this is the reason why there are there is no common neighbor between the vertices u and v.

Now, every vertex every vertex x in this set p minus except these 2 vertices u and v we consider all other vertices. So, every vertex x is adjacent to at most one of u v in G. So, I now we considered we look at, we think about the other vertices except u v. So, suppose these are the other vertices and u and v is here.

So, this is u this is v and we look at the other vertices with respect to the relation of other vertices with u and v. So, there are some set of vertices here, and there are some set of vertices here. As we said that these are like this could be x this could be x. So, these are the vertices like other than u and v. So, the statement says that every vertex x is adjacent to at most one of them.

So, suppose that these vertices these are the set of vertices which are adjacent to v only. And these are the set of vertices which are adjacent to u only and these are the set of vertices which are neither adjacent to u nor adjacent to p. So, you can see that there is no vertex which is adjacent to both u and v in the graph G. So, this is how the graph of graph G with respect to the relation or adjacent relation with u and v.

Now, what will happen in G complement? This is again the vertex u and the vertex v. So, these are the set of vertices which are neither adjacent to u nor adjacent to v in the graph G, these are the set of vertices which are adjacent to say only v. These are the set of vertices which are adjacent to only u in G. Now the in G complement you see that these vertices are neither adjacent to u nor adjacent to v in the graph G.

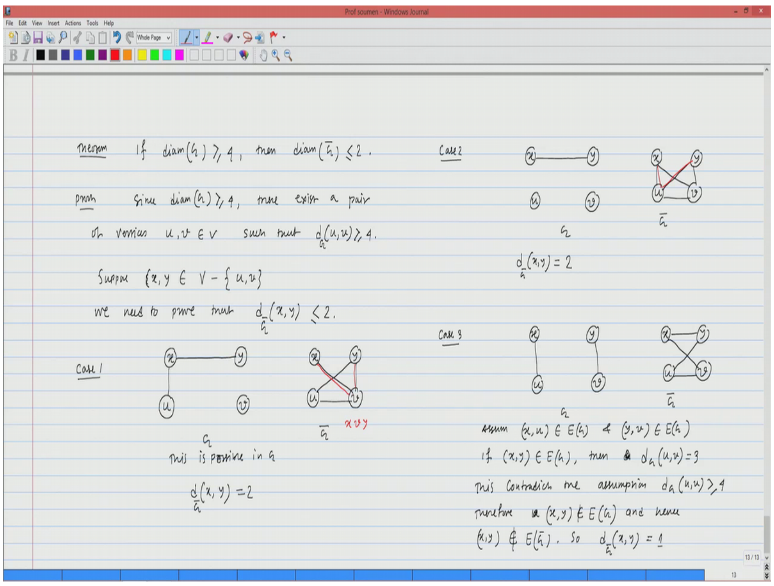
So, they will be adjacent to both u and v in graph G complement right. This is from the definition of complement of a graph. Now this set of vertices are adjacent with v in graph G. So, in G complement they will be adjacent to u only. And this set of vertices are adjacent to u in the graph G. So, in G complement they will be adjacent to v only. Now you consider any pair of vertices from here.

Now we will be able to prove that for every pair of vertices x and y in say v minus u v, there is a there is a path of length at most 3 in G complement. So, if I take a vertex x here and y from this set. Then I can see that there is a path from x to y of length 3 that is this is the path.

If this is my x then this is a path of length 3 from x to y. And if both x and y are here then I can get a path of length 2 like that. So, that is why you take any pair of vertices there is a path of length at most 3 in G complement. So, so the distance between every pair of vertices in G compliment is less than or equal to 3. So, the diameter of the graph G complement is less than or equal to 3, ok.

So, next we prove another result of related to the diameter of graph theorem another theorem.

(Refer Slide Time: 22:04)



If this is slightly complicate complicated compared to the previous 1 if diameter of the graph G is greater than equal to 4, then diameter of G complement is less than or equal to 2.

So, the diameter of a graph G is given to be greater than or equal to 4; that means, there exist a pair of vertices say u and v, such that the distance is 4 and that is why the diameter is 4 or more than 4. So, the proof of this theorem is since diameter of a graph G is greater than equal to 4. That means, there exist a path there exist a pair of vertices, there exist a pair of vertices u and v in v, such that the distance is distance between u and v is greater than greater than or equal to 4.

So, I will use the notation that is the distance between u u and v in the graph G that is what the notation for this one. Again you considered a pair of vertices suppose suppose x and y are 2 vertices in v minus u v. So, I am not considering the vertices u v we need to prove, prove that the distance between any pair of vertices in G complement is less than or equal to 2. In order to prove that the diameter of G complement is less than equal to 2.

Now, we consider different cases probably. So, case one similarly we look at the other vertices with respect to their adjacency with u and v adjacency relation with u and v. So, u and v are 2 vertices which are at distance 4 or more than 4, and x and y are any 2 arbitrary vertices. Now x cannot be adjacent to both u and v in G. So, this is the situation in G, if x and sorry if x is adjacent to both u and v then there is a path of length 2 between u and v in the graph G ok.

So, but the but we have, but the distance between u and v in the graph G is greater than or equal to 4. So, x cannot be adjacent to both u and v this is one observation. X can be adjacent to only one of them right. This is one situation case 2 again how x and y are related with u and v in terms of the adjacency relation. The other case is that of course, y adjacent to v only v that is the same as this case one. So, this case we will assume that x is not adjacent to u and v similarly y is also not adjacent to u and v.

The case 3 is x is here y is here and u v are here. X is adjacent to y and y is adjacent to v. So, these are the 3 3 cases basically you can think about it. So, we consider 3 cases which cover almost everything I mean everything not almost. Now what else we can assume here is that. So, this is a possible situation this is this is this is possible this is possible in G.

Now, what else we can assume is that x and y are also adjacent in G. Otherwise if they are not adjacent in G they will be adjacent in G complement and then there will be path of length one between x and y, but what we are trying to prove is that the distance between x and y is less than or equal to 2. So, in case one I will assume that x and y are adjacent in case of case 2 also I will assume that x and y are adjacent.

So, in case 3 I cannot assume that x and y are adjacent. If I make x and y adjacent in G then there will be a path of length 3 in g between u and v, but I have assumed that the distance between u and v is greater than equal to 4 in G. So, in this case what we are assuming is that assume x and u they are adjacent in G, this is the a set in the graph G and y and v are adjacent in G

Now, as I said that this can not be adjacent if x and y are also adjacent in G then the distance between u and v in the graph G will be 3. These contradict the assumption that the distance between u and v is greater than equal to 4. Therefore, u and sorry therefore, x and y are non not adjacent in Graph G. And hence they are adjacent x and y are adjacent in G complement right.

So, the distance between x and y in G complement is equal to 1. So, in this case in case 3 we have proved that the distance between x and y is equal to 1, because this is the situation in graph G in graph G complement it will be like this x and y will be adjacent, because they are not adjacent here. So, in graph G this will be something like this.

So, we proved that distance between x and y is equal to 1 in G complement. So, here in case 2 this is the situation in G. So, in G complement of course, this edge also will be there. In G complement what will happen is that x y u v. So, x is neither adjacent to u nor adjacent to v in G. So, x will be adjacent to u and v both in G complement, and similarly y will be also adjacent to u and v in G complement.

So, here in this case in case 2 will get a path between x and y in G complement of length 2. And the path is obviously, the path is x u y or x v y. In this case also in case one also this is g and in G complement it will be like this x y u v. So, y is neither adjacent to u nor adjacent to v in G. So, y will be adjacent to u and v in G complement and x will be adjacent to v and of course, this will be there.

So, in this case also we have distance between x and y is 2 in G complement, because here I can see a path of length 2 between x and y and here is that path x v y is a path of length 2. So, in this first part of lecture 4 we have learnt what is diameter of a graph. So, we can now compute given any graph you can compute the diameter of that graph. And we have learnt about learnt two results related to diameter of a graph.

Thank you very much.