Graph Theory
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Lecture - 03
Part 2
Bipartite Graph

Welcome to the second part of third lecture. So, in this part we will talk about special type of graph called Bipartite Graphs. So, a graph is said to be bipartite if and the vertex set of the graph can be partitioned into 2 parts such that every edge joins a vertex in one part to a vertex in another part.

So, let me start with the formal definition of bipartite graph.

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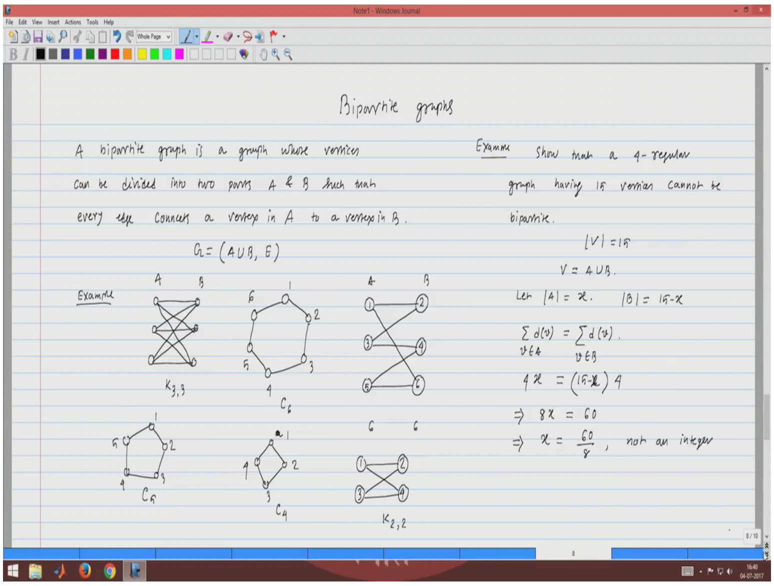

So, a bipartite graph is a graph whose vertices can be divided into 2 parts; A and B such that every edge connects A joints vertex into A vertex in B.

So, usually a bipartite graph is will be denoted in this way G is A union B. So, A union B is the vertex set and it is in 2 parts and E is a set example; let me draw some bipartite graphs this is one bipartite graph. And so this is the part A and this is the part B and these are the edges you can see that every edge connects the vertex in a with the vertex in B and this is an example of a complete bipartite graph. So, this is denoted by K 3 comma 3 because all possible edges are there note that there cannot be an edge between a vertex in A with another vertex in A, they cannot be an edge between the vertices in the same part.

Let me draw another graph again this is say C 6 I label the vertices by number 1, 2, 3, 4, 5, 6, note that this is C 6 cycle of length 6, but this is also a bipartite graph because you know if you redraw it. For example, you put the vertices labelled with odd numbers 1, 3, 5, in one part and all the vertices labelled with even numbers 2, 4, 6. This is in the other part. Now we can see the edge 1; 2 is corresponds to this edge 1, 6 is this edge; now 3, 2 and 3, 4, 5, 6 and 5, 4. So, you can redraw this graph and then you can see that this is basically a bipartite graph this is a bipartite graph.

Let me give one more example; just to understand the definition of bipartite graph clearly you can solve this problem. So, that 4 regular graph having 15 vertices cannot be bipartite well. So, we know that what is 4 reg regular graph or 4 regular graph; graph is said to be 4 regular if every vertex is having degree 4.

Now, there are total 15 vertices in the graph. So, we need to partition these vertices into 2 parts A and B list 2 that say I partition this graph has total 15 vertices. Now I have to partition this V into 2 parts A union B; let the number of vertices in the part a is equal to x.

Now, one thing to observe here is that you can see that the degree sum in one part. So, here the degree sum in the part A is 2 plus 2 plus 2 6 the degree sum in the other part is 2 plus 2 plus 2; it is 6. So, this is always true for the bipartite graph that the degree sum for the vertices in A; part A is equal to the degree sum for the vertices in part B. This is easy to observe.

Now, if in part a there are x many vertices and each vertex has degree 4 because this is a 4 regular graph, then this has to be equal to; so, the number of partisan number of vertices in the part B, then 15 minus x; so, 15 minus x sorry; 15 minus x into 4. Now if you solve this equation these 2 numbers should be the same if you solve this; what you get is that 8 x is equal to 60 which implies x is equal to 60 by 8; this is not an integer. So, the number of vertices in part A is not an integer. So, you cannot find integer solution for this equation. So, there is A; you do not know the number of vertices in part A. So, for this combination that they are full regular graph having 15 vertices cannot be a bipartite graph. So, that is what the proof of this one.

So, next we characterize bipartite graph. So, in the previous example of we saw that this C 6; C 6 is a bipartite graph, but similarly says C 4; C 4, this is C 4, this is also a bipartite graph A; sorry, say 1, 2, 3, 4; this is C 4; similarly you can put 1 and 3 in 1 part and 2 and 4 in the other part. So, 1 is adjacent to 2 and 4; 3 is adjacent to 2 and 4. So, C 4 is a bipartite graph and this is also complete bipartite graph K 2; 2, but such thing is not possible if you are given a cycle of odd length suppose C 5; 1, 2, 3, 4, 5. So, this is C 5 and let me label the vertices 1, 2, 3, 4, 5. So, here you cannot partition the vertices into 2 parts such that the edge is every edges from one part to another part.

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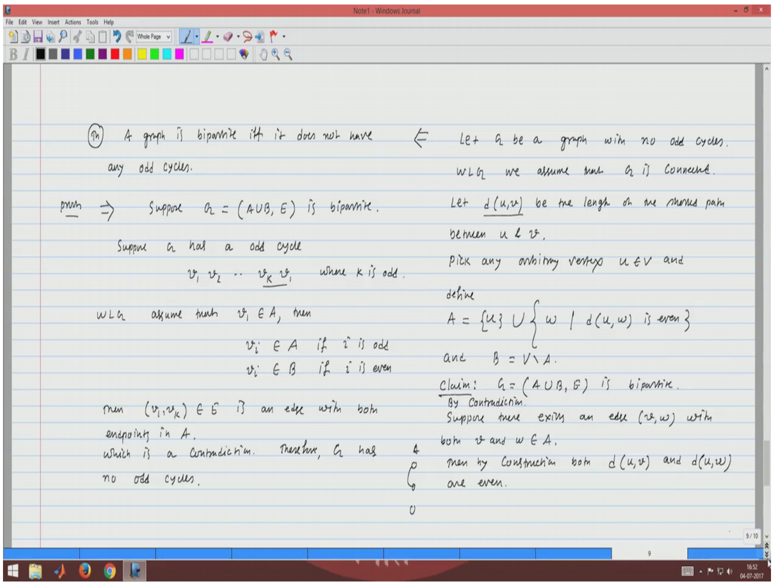

So, next we move to the characterizing of bipartite graph this is theorem a graph is bipartite if and only if; it does not have any odd cycles. So, a graph is bipartite if and only if it has no odd cycles what is odd cycles odd cycles means cycle of odd length for example, C 5 is a cycle of length 5. So, it is a odd cycle C 5, C 3, C 7; you cannot partition the vertices of odd cycle into 2 parts such that it forms a bipartite graph. So, that is this also the odd cycle also characterized the bipartite graph a graph is bipartite if and only if it does not have any odd cycles.

Let us prove this one. So, we will prove this part first that. So, this is the proof suppose G is equal to A union B E is bipartite and then we have to prove that G has no odd cycle. The proof is again by contradiction suppose G has a odd cycle and let me denote that cycle by B 1, B 2, B K and then again B 1 where K is odd. So, without loss of generality;

without loss of generality; assume that assume that V 1 is in set in the part A then; obviously, you know if V 1 is in part A, V 2 will be in part B and V 3 will be in part A again. So, then V i is equal to is in part A if i is odd and V i will be in part B if i is even.

So, you have you have observed in my previous example that I am putting all the odd vertices into one part; I mean odd means odd index into one part and even indexing another part then you see that K is odd and K is adjacent to V 1, then V 1 V k is an edge with both end points in A, but which is a contradiction; therefore, G has no odd cycles.

Next we prove the other part that let G be a graph with no odd cycles; they need to prove that this is Hamiltonian; sorry, G is bipartite without loss of generality we assume that G is connected even if it is a disquieted graph no issue, let d is notation; let d be the length of the shortest path between u and v, this is the notation to denote the length of the shortest path between u and v. Now you pick any arbitrary vertex u belongs to V and define the following.

So, A is equal to the vertex; u union some more vertices w such that the distance between u and w is even and B is V minus A; the other vertices and what we claim is that. So, so this gives a technique to partition the vertices into 2 parts; A and B and the claim is that the graph G which is now we have found an way to partition the vertices into A and B and we got a bipartite graph; I mean the claim is that this G is bipartite.

So, the second part of that G has no odd cycles, then we have to prove that G is bipartite and this proof is sort of a constructive proof you are given a graph which is not having any odd cycles and then you give a technique or an algorithm to construct bipartite graph to prove that G is bipartite. So, what you do is that you pick one vertex arbitrarily and then that arbitrarily picked vertex is denoted by u here and then a is the set of all vertices which are at distance even distance from u distance mean shortest distance length of the shortest path and B is the set of all vertices which are at odd distance from u and v; claim that this is bipartite.

This is our claim, but we need to prove that. So, to prove this that this is a bipartite graph what you have to prove that suppose, there exist an edge v w with both v and w using a this is not allowed. So, this is a we are trying to prove that this is bipartite using sort of a contradiction technique by this claim we are proving by contradiction by contradiction means that is this is not bipartite. That means, there exist an edge between v and w where v and w both are in the part a see in case of bipartite graph there cannot be an edge between 2 vertices in one part, but we are assuming that this is this is there, then by the construction technique, then by construction by construction both of they are in the same part in the part A.

That means, the distance from u to v and the shortest length of the shortest path or the distance between these 2 vertices u and w because both v and w are in a; that means, this 2 quantity are even that is why they are in the same part in part A.

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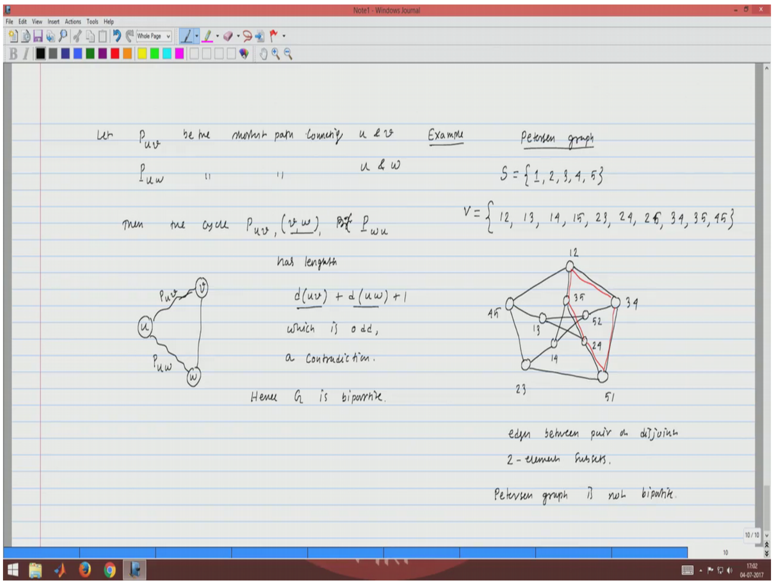

Next let P u v be the shortest path connecting u and v and P u w be the shortest path connecting u and w, then we form a cycle then the cycle. So, u is here and this is u to v and this is the path shortest path P u v and this is of length even length there is another path from u to w. And this path I denoted by P u w and this is also of even length and we have assumed that these 2 are adjacent.

Then this form a cycle then the cycle P u v then the edge P u v in the edge v w and then P sorry; in the path P w u; P w u; u w are the same this cycle has length has length d u v plus d u w plus 1 and this is even; this is even. So, this is odd which is odd. So, it is a contradiction; it is a contradiction, this is the contradiction to the fact that this is there is an edge between v and w where v and w both are in part a hence. So, this is not possible. So, hence the construction technique that we have given is correct. So, hence G is bipartite.

So, let me give one example of a very special graph called Peterson graph. Peterson graph and this will be used frequently in this class let us say S be a set of the S be the set of 5 elements 1, 2, 3, 4, 5 and the vertices of this graph is the vertices of this graph are all 2 subsets of this set S; that means, say 1, 2 for simplicity I am writing 1 2, 1 2, means it is a set basically 1 comma 2. So, I just write simply 1 2. So, 1 2, 1 3, another set I am considering all 2 subsets of the set S 1 4, 1 5, 2 3, 2 4, 2 5, 3 4, 3 5, 4 5.

So, these are the possible vertices of the graph and let me draw this graph. So, this is one vertex this is corresponds to 1 2. So, this is corresponds to 3 4; this is corresponds to 5 1, 2 3, 4 5 and these are the remaining vertices. So, this is corresponds to 3 5; this is corresponds to 5 2 or 2 5, 2 4; this corresponds to 1 4; this corresponds to 1 3.

So, I have drawn all the ten vertices here and there will be an edge between these 2 vertices because the union of sorry the intersection of this set and this set is null then only you add an edge between the corresponding vertices. So, see there will there will not be an edge between these 2 vertices because the intersection of these 2 sets is not null here, but there will be an edge here, here, here, here and also 1 2, 4 5 are disjoint sets; this is disjoint 2 3, 5 1 disjoint 1 5, 3 4 disjoint 1 2; 3 4 disjoint. Now 3 5, 5 2, 1 4, 3 5, 2 4; there will be an edge; there will be an edge between 1 4 and 5 2; there will be an edge between 1 3; 2 4 and 1 3, 2 5.

So, this is what the Peterson graph is. So, the edges between pair of disjoint 2 element subsets and you can see now that this graph has the reason I gave this example just to illustrate that I can find a cycle of length 5. So, if I take this cycle 1, 2, 3, 4, 5. So, I can find a cycle. So, this graph Peterson graph contains a cycle of odd length. So, that is why by using the previous theorem; I can say that that Peterson graph is not bipartite.

So, what we have learned regarding the bipartite graph is that a graph is bipartite if and only if does not contain and odd cycles does not contain odd cycles and we have proved that characterization and also we learnt what is Peterson graphs. And we noticed that the Peterson graph is not a bipartite graph.

Thank you very much.