Graph Theory
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Lecture - 03
Part 1
Bipartite Graph

Welcome to the third lecture on Graph Theory. So, in this lecture we will talk about Hamiltonian graph and bipartite graphs. So, before I start I just want to recall the theorem that we proved in second lecture.

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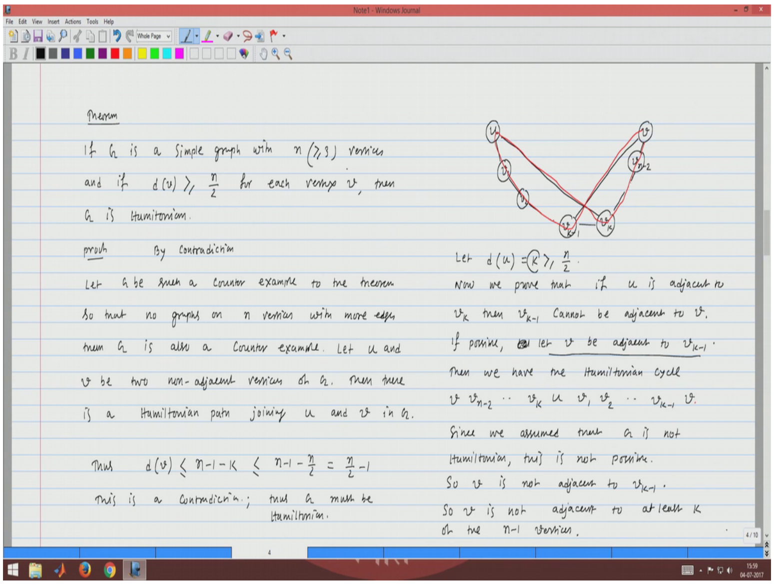

That if G is a simple graph with n vertices and if the degree of every vertex is greater than or equal to n by 2 then G is Hamiltonian. So, in this proof this is a proof by contradiction we considered a maximal counter example to this theorem; that means, G is a graph this is the counter example having n vertices. And it satisfy for all the conditions the conditions that n is the degree of every vertex is greater than equal to n by 2.

But it is not Hamiltonian. And this is a maximal counter example in the sense that if you add one more edge to this graph the gap will become Hamiltonian. So, thoroughly this is a little difficult part let u and v be 2 non adjacent vertices of G, then there is a Hamiltonian path joining u and v. This is because see u and v are non adjacent, and if you add this edge if you add this edge u v in the graph. And the graph will become Hamiltonian that will be; that means, there will be a Hamiltonian cycle. And if you remove the edge u v from the graph there will be a Hamiltonian, Hamiltonian path or a spanning path connecting u and v. So, this is I think I should I wanted to mention once more this one, and this theorem is due to dirach ok.

So, next we move to one more example.

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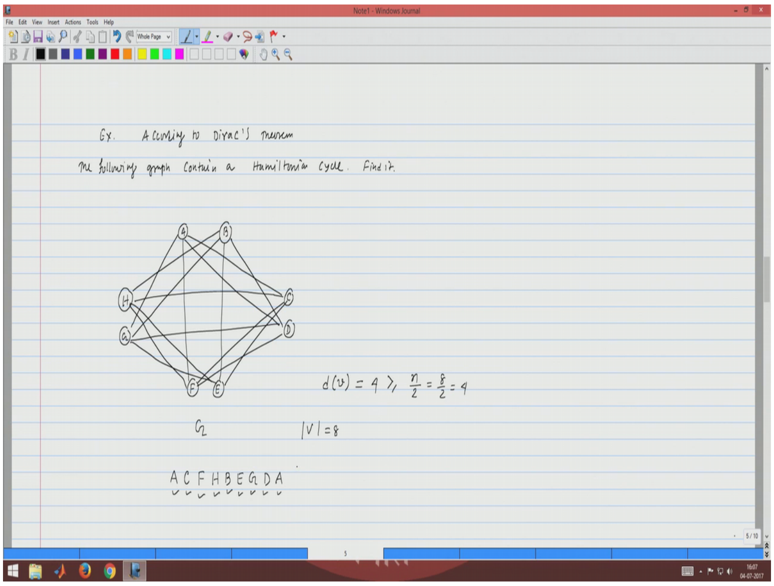

According to diracs theorem the following graph contains a Hamiltonian cycle, find it. Let me draw the graph it is slightly complicated graph A B C D E F G H. So, there are 8 vertices and A is adjacent to G F C and D. B is adjacent to H G D and E. C is adjacent to A adjacent to H F and E. And D is adjacent to F G and A and B. E is adjacent to B C G H, G and H. And F is adjacent to H A C D and G is taken care already H is taken care. So, this is a graph I just I am giving this example to illustrate the diracs theorem.

So, this graph has 8 vertices and you can see that the degree of every vertex here is equal to equal to 4. So, 4 is greater than or equal to n by 2, which is 8 by 2 equal to 4. So, this satisfy the graph satisfies the condition of diracs theorem that every vertex has degree greater that than or equal to n by 2. So, the gap is a Hamiltonian graph, you find the Hamiltonian graph. You find the Hamiltonian cycle in G he spend some time to find the Hamiltonian cycle of this graph.

Let me just give the answer, here this is one Hamiltonian cycle A C F H B E G D A. You can check that this is Hamiltonian cycle. And it is Hamiltonian cycle because it includes all the vertices A B C D E F G and H, and A bigger than because it is a cycle. So, this is what the Hamiltonian cycle for the graph given here.

Next we talk about another theorem.

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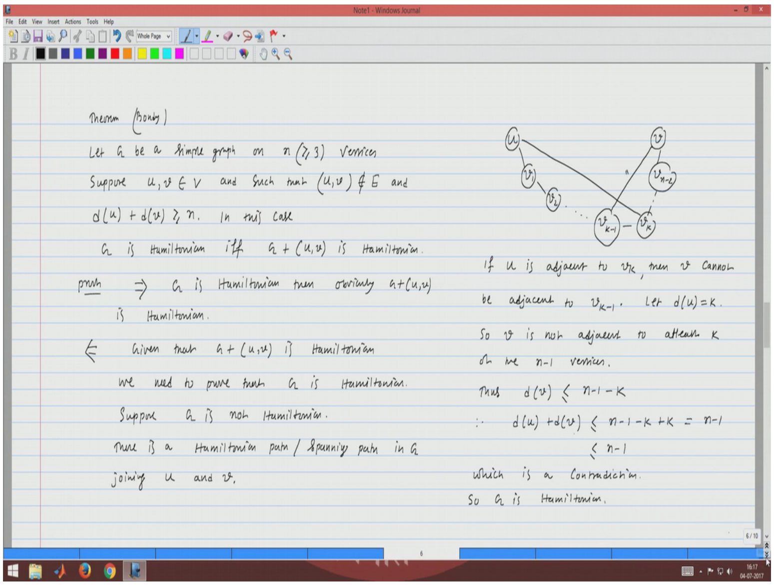

This is due to bondy, let G be a simple graph on n vertices. Suppose u and v are 2 vertices in the graph, such that u and v are not adjacent they are not in edge in the graph. And degree of u plus degree of v is greater than equal to n. So, in this case G is Hamiltonian if and only if G plus u v is Hamiltonian. By the meaning of G plus u v means this u v was not an edge in the graph G.

Now, you add this edge u v in the graph G, that is the meaning of G plus u v. So now, in G plus u v, u v is an edge let me prove this one. G is Hamiltonian and I have to prove that G plus it is given that G is Hamiltonian, then I have to prove that G plus u v is also Hamiltonian this is obvious, because if you add one more edge in the graph in the Hamiltonian graph the graph will remain Hamiltonians Hamiltonian So, there is nothing to prove here. So, G is Hamiltonian then; obviously, G plus u v is Hamiltonian, ok.

Now, the other part that it is given that G plus u v is Hamiltonian. We need to prove that prove that G is Hamiltonian. So, this is this part is not obvious it is you are given that G plus u v is Hamiltonian, and then G is also Hamiltonian. So, you are removing a specific edge from the graph and then after removing that after removing an edge in the graph might not remain Hamiltonian. So, you need to prove this is non trivial part. So, we will prove this part now.

So, we need to prove that G is Hamiltonian. So, this proof is again by contradiction, suppose G is not Hamiltonian suppose G is not Hamiltonian. Then there is a Hamiltonian path or spanning path or spanning path in G joining u and v. This is obvious because a G plus u v is Hamiltonian. Now you are removing one edge u v from that graph G plus u v then there will be a spanning path joining u and v in the graph G.

So, in the very similar argument; so u and v and there is a spanning path joining them. So, v 1 v 2 exactly the same proof v k minus 1 v k v n minus 2; so this is the spanning path joining u and v. And then again we do the same thing we have seen the proof of this one that if u is adjacent to v k, then v can not be adjacent to this is not possible v can not be adjacent to v k minus 1. If u is adjacent to v k, then v can not be cannot be adjacent to v k minus 1. We should have agree with this one because of because otherwise there will be Hamiltonian cycle here, assuming that this is correct.

Let the degree of u v equal to k, we do not know what is the value of k. So, so v is then is not adjacent to at least k of the n minus 1 vertices, because if u is adjacent to something v can not be adjacent to it is neighbor. So, so v is not adjacent to at least k of these vertices, because u is adjacent to k vertices. So, v can not be adjacent to it is k neighbours thus the degree of v is less than or equal to n minus 1 minus k. And what we get is that degree of u plus degree of v is less than equal to n minus 1 minus k plus k, which is equal to n minus 1. So, this is basically less than or equal to n minus 1, which is a which is a contradiction. So, so G is Hamiltonian. So, this was a proof by contradiction.

We assume that G is not Hamiltonian, then we came to a contradiction that degree of u plus degree of v less than or equal to n minus 1. So, finally, we proved that then this assumption is not correct G is actually Hamiltonian. So, next we talk about a concept called closure of a graph the Closure C G of a graph of order n is obtained, obtained from G by recursively joining pair of non adjacent vertices whose degree sum is at least n until no such pair edges.

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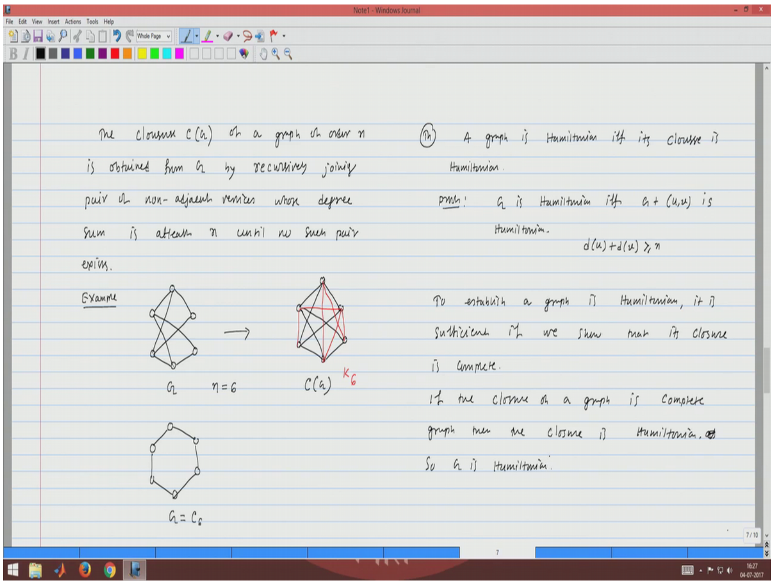

Well, so the closure of a graph is a you start with the graph and look at the non adjacent vertices and you compute their degree sum.

If the degree sum is greater than or equal to n, where n is the number of vertices of the graph you add an edge between these 2 non adjacent vertices. So, let me give an example to illustrate how to compute closure of a graph example. This is a nice example maybe, this graph is having 6 vertices. And these are the edges of this graph. So, this is the graph G given and the number of vertices is equal to 6. Now I compute the closure of this graph that is cg well. So, the closure of this graph is obtained as follows, this is the same graph I am drawing again.

These are the edges of the original graph G. Now I looked at these 2 vertices these 2 vertices are non adjacent. And the degree of this vertex is 2 the degree of this vertex is 2. So, the degree sum is 4. So, the degree sum is not greater than or equal to 6. So, I do not do anything here at this moment. Now you see that these 2 vertices are for these 2 vertices the degree sum is equal to 6. So, I add an edge between these 2 vertices.

Now, I look at these 2 vertices for example, these 2 vertices now. So, here the degree sum this is degree 2 this is the degree 4. So, the degree sum is equal to 6. So, I can I add an edge between these 2. Now this has degree 2 and this has degree 4. So, I can add an edge between these 2. What about these 2 vertices? This is of degree 3, this is of degree 3 now. So, I can add an edge between them of this is of degree 4 this is of degree 3. So, the degree sum is 7 I can add an edge here, this is of degree 3 this is of degree 4. So, degree sum is 7 greater than or equal to 6. So, I can add an edge.

So now this graph has become a complete graph I believe. So, this is a complete graph with 6 vertices. So, this is this is the closure this is the closure of this graph. Now if I take this graph for example. So, this graph is also having 6 vertices the graph G which is a cycle of length 6 C 6. Now you can see that there are many non adjacent vertices here, but the degree sum is 4, which is not greater than or equal to 6 for any non adjacent vertices.

So, you can not basically add any edge here the closure of this graph is the graph itself. So, we have explained what is closure of a graph. Now it has some consequence it is a theorem. A graph is Hamiltonian if and only if it is closure, if and only it is closure is Hamiltonian. The proof is very simple because you know when we add the at a new edge this is coming from the previous theorem that G is Hamiltonian, if and only if G plus u v is Hamiltonian. Where we know that u and v are 2 non adjacent vertices in G and they satisfy the condition that degree of u plus degree of v is greater than or equal to n. Where n is the number of vertices in G, and this is the condition for adding a new edge u v enclosure.

So, this is the proof of this theorem comes from here that G is Hamiltonian if and only if G plus u v is Hamiltonian. Now to establish a graph is Hamiltonian it is sufficient if we show that it is closure is complete. If the closure of a graph is complete is complete graph, then the closure is Hamiltonian. As the complete graph is always Hamiltonian. So, G is Hamiltonian ok.

So, this concept of closure is somewhat important, because to prove that a graph is Hamiltonian or not. One way to find check whether the graph is Hamiltonian or not is to is to find whether the group contains a Hamiltonian cycle or not. The other way of proving that is- that you find the closure of the graph if you see that the closure of the graph is a complete graph, then the closure is Hamiltonian and then the graph is also Hamiltonian. And you must have noted or it is easy to note that complete graph is a Hamiltonian graph.

Thank. Thank you very much for your attention.