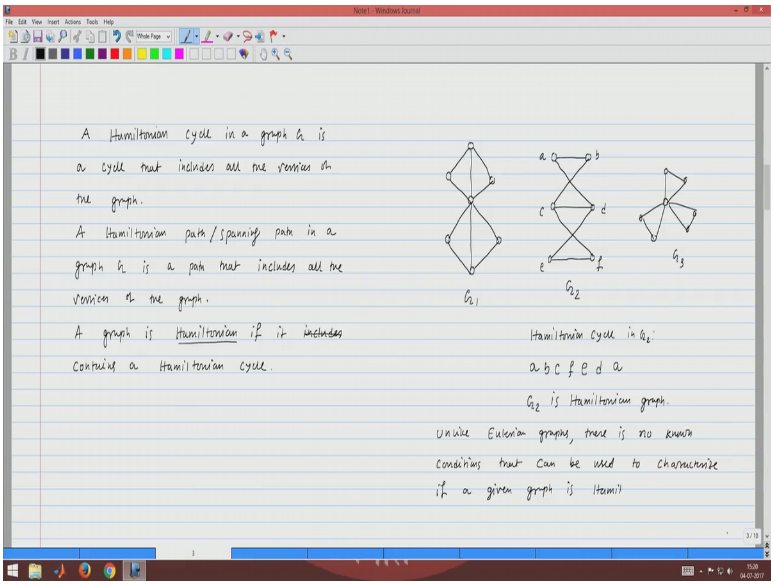
Graph Theory
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture - 02
Part 2
Eulerian and Hamiltonian Graph

Welcome to the second part of second lecture. And in the first part we have learnt about Eulerian graph. Now we will talk about Hamiltonian graph. So, again you recall that that a path is a walk in which all the vertices are distinct. And once all the vertices are distinct all the edges are also distinct. Now a closed path is a cycle.

(Refer Slide Time: 00:54)



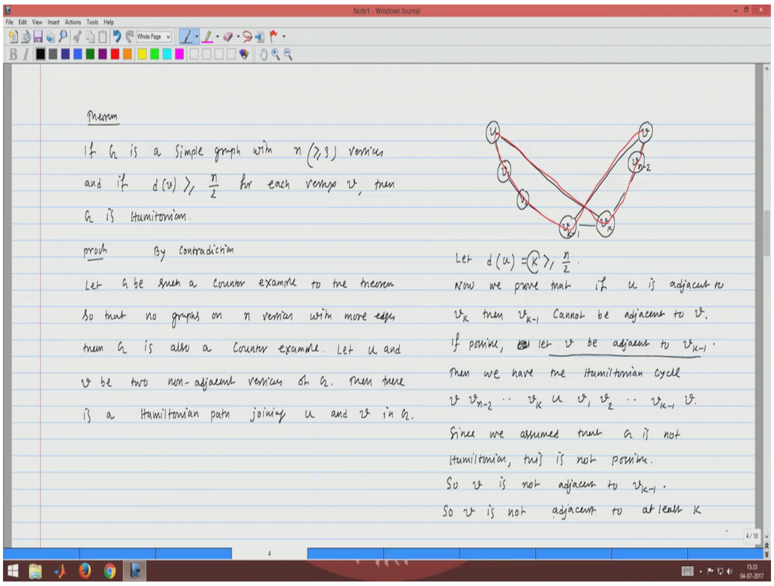
A Hamiltonian cycle in a graph G is a cycle that includes; all the vertices of the graph. And similarly a Hamiltonian path a Hamiltonian path also called spanning path in a graph. G is a path that includes; all the vertices of the graph. And a graph is Hamiltonian if it includes; if it contains a Hamiltonian cycle. So, this is the definition of Hamiltonian graph. Now let me just give some example of graph: say this is one graph with of these vertices. Now these two are adjacent.

Now, the question is whether, let me call this graph as G 1; whether this graph is Hamiltonian or not. So, you try to find Hamiltonian cycle in this graph. That you can try on your own. The question is whether G 1 is Hamiltonian or it is not Hamiltonian? Let me draw another graph with 6 vertices. I am giving many examples, so that the definition of a Hamiltonian graph is clear to you. There is no doubt about the definition of Hamiltonian graph. Let me call this graph G 2 and the third graph is this one, say G 3.

Now if I label them a, b, c, d, e, f, I will talk about the graph G 2 only. Can I find a Eulerian sorry a Hamiltonian cycle here. Yes, probably I can find one. So, Hamiltonian cycle in G 2 could be a, then go to b. From b you go to c. And from c you go to f. And go to e. And then d, and then a. Again I think I am not repeating any vertex here and so it is a it is closed path. So, a cycle and it includes all the vertices of the graph. So, this is a Hamiltonian cycle.

I can say that G 2 is a Hamiltonian graph. Unlike Eulerian graphs there is there is no non test or conditions that can be used to characterize or to determine if a given graph is Hamiltonian. Next we talk about a theorem.

(Refer Slide Time: 08:39)



If G is a simple graph with n vertices, And if degree of every vertex is greater than equal to n by 2, for each vertex v, then G is Hamiltonian. In the case of Eulerian graph which a there is a characterization to characterize and there are some conditions to characterize Eulerian graph, like a graph is Eulerian if and only if; every vertex is of even degree, but there is no such condition in case of Hamiltonian graph. And the theorem that I wrote just now is of course, it is a old theorem it says that, if G is a simple graph with n vertices and if the degree of every vertex is greater than equal to n by 2 then the graph is

Hamiltonian. One thing to note that, this is not a necessary condition; this is a sufficient condition; so proof of this theorem.

So, the proof is: By contradiction this is by contradiction. Let G be such a counter example to the theorem, so that no graph on n vertices with more edges, then G is also a counter example. Let G be such a counter example to the theorem, so that no graphs on n vertices with more edges than G is also counter example. Let u and v be two non adjacent vertices of G. So, you need to understand this is probably the first non trivial proof that you are giving the proof is slightly difficult.

This is the proof is using by the technique of contradiction so; that means, suppose there is a graph G which satisfy graph g with n vertices. So, it satisfies a the condition that a every vertex has degree greater than equal to n by 2, but the graph is not Hamiltonian and the counter example that we considered for this for the proof of this theorem is sort of a maximal counter example; that means, it has the maximum number of edges in the graph G.

If you add one more edge to the graph G, then the graph become a Hamiltonian. So, the graph the counter example, that we consider is sort of a maximal counter example: let you u and v be two non adjacent vertices of G. Then there is a Hamiltonian path joining u and v in G, because a u and v are a non adjacent. Now, if you add that this edge u, v in the graph, then the graph will become Hamiltonian. So that means, u and v of u and v this two vertices are a joint by a spanning path for Hamiltonian path.

So, besides u and v there are n minus 1 n minus 2 vertices. Let me call them v 1, v 2, v k minus 1, v k v n minus 2, because total number of vertices are n. So, n minus 1 vertices are here sorry n minus 1 vertices are here and then u and v. So, this is a spanning path joining u and v; now let the degree of u is equal to k. And we know that the degree of every vertex is greater than or equal to n by 2. Now, we prove that if u is adjacent to v k, then v k minus 1 cannot be adjacent to v.

So, if u is adjacent to v k, then v k minus 1 cannot be adjacent to v. So, this we prove just to determine the degree of v and will come to a contradiction. So, if u is adjacent to v k, then v cannot be adjacent to v k minus 1, because let me prove this one also. This is again by contradiction; if possible, let v be adjacent to v k minus 1. If this is adjacent to v k minus 1, then we have the Hamiltonian cycle v n minus 2, v k, u v minus 1 sorry v 1, v

2 up to v k minus 1 and v; that means, if v is adjacent to v k minus 1, then we get this Hamiltonian cycle v be n minus 1, v k and then you go to u and then v 1, v 2, v k minus 1 and then you go to v. So, you get this Hamiltonian cycle here.

Since, we assumed that G is not Hamiltonian, this is not possible. That means, see we assuming that G is a counter example so G is not Hamiltonian, but here it is becoming that G is Hamiltonian. So that means, v cannot be adjacent to v k minus 1. So v is not adjacent to v k minus 1. And you know that see u is adjacent to k vertices and since u is adjacent to k vertices v cannot be adjacent v cannot be adjacent to at least k many vertices.

So v is not adjacent to at least k vertices k of the n minus 1 vertices. Thus, the degree of v, so ultimate goal was to compute the degree of v and come to a contradiction. So, since v cannot be adjacent to at least k of this n minus 1 vertices, you can see that the degree of v should be less than or equal to n minus 1 minus k and k is greater than equal to n minus sorry n by 2. So, this is less than equal to n minus 1 minus n by 2, so which is equal to n by 2 minus 1.

Thus, what we got is that the degree of v is less than equal to n by 2 minus 1. So, this is a constant this is a contradiction; thus G must be Hamiltonian. So, this is the end of this proof. So, we started with the started with the contradiction that G is a maximal counter examples to the theorem; that means, G every vertex of the G has degree greater than equal to n by 2, but the graph is not Hamiltonian.

And then finally, by different of different steps what we realized observed that, the degree of v is less than or equal to n by 2. So, sorry less than equal to n by 2 minus 1 which is a contradiction, because the original graph G every vertex has degree greater than equal to n by 2? So, so we came to a contradiction and that is why the graph must be the graph g must be Hamiltonian. That is all for today’s class.

Thank you very much.