Graph Theory
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Lecture – 20
Part 2
Colouring of Planar Graphs

Welcome to the second part of a lecture 20. So, in this lecture we learn one more characterization of planar graph using edge contraction and we will also prove that every planar graph is 6 colourable of course, we know that every planar graph is 4 colourable which is the 4 coloured theorem, but we are not going to prove that, but we will be proving that every graph is 6 colourable.

So, I will start with another characterization of planar graph using edge contraction. So, this is a theorem due to Wagner 1937 a finite graph; that means, it has finite number of vertices is planar if and only if it contains no subgraph that is edge contractible to K 5 or K 3 3.

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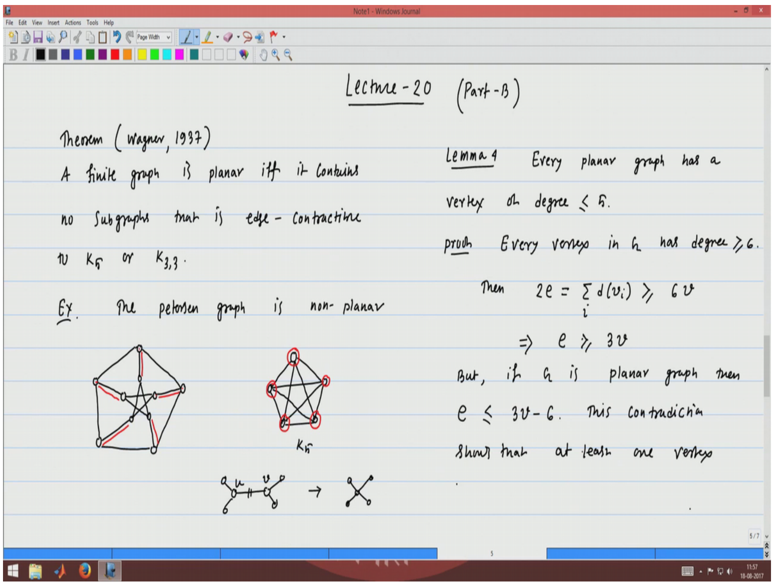

So again I am not going to prove this theorem I will just give an example to say that how do you to show that how to prove graph is non planar using this theorem. So, the Petersen graph is non planar. So, here is the Petersen graph right. So, the edges are like this.

So this is what the Petersen graph is and we know what is the meaning of edge contraction right contracting this edge please recall that this is what the edge contraction is suppose this is the edge u v if I want to contract the edge u v, so I will get this graph after contraction of the edge u v. So, this edge u v will be removed will be deleted and then this to vertices will be merged into one vertex.

Now you can prove that this graph is a edge contractible to K 5 K 5 is complete graph with the 5 vertices. Now you can see that you can get this K 5 from in fact, the Petersen graph is edge contractible to K 5 because if you say, if you contract this edge you contract this edge apply edge contraction on this edge, this edge, this edge, this edge and this edge you will get this graph K 5.

So, when you apply edge contraction on this edge this edge will be deleted and this two vertices will be merged and you will get something like this right and after edge contraction these two edges sorry these two vertices will be merged into one vertex this is what you will get similarly these two vertices will be merged into one vertex here and these two vertices will be merged into one vertex these two vertices will be merged into one vertex. So, it is easy to see that Petersen graph is edge contractible to K 5. So, Petersen graph is non planar.

Now, we will talk about colouring of planar graph and we will prove that every planar graph is 6 colourable. So, for that we need some more result here is one of them lemma 4 may be this is that every planar graph has a vertex of degree less than 5.

Suppose every vertex in G has degree greater than equal to 6. So, what we are going to prove that every planar graph has a vertex of degree at most 5. So, we first assume that every vertex in graph G has degree greater than equal to 6 and then will come to our contradiction.

Then the degree sum which is equal to 2 E, 2 E is the degree sum we know that and what we are doing is that I am replacing. So, each degree is greater than equal to 6. So, it can be more than 6 I am replacing them by a smaller quantity. So, 6 only and there are v many vertices. So, which is 6 v which implies v is which implies e is greater than equal to 3 v, but we know that if G is a planar graph then from the first lemma that then another number of edges is less than equal to 3 v minus 6 and this contradiction, but why what we got here is that e is greater than 3 v. So, this contradiction shows that at least one vertex has degree less than equal to 5; that means, every vertex in the graph G cannot have degree greater than or equal to.

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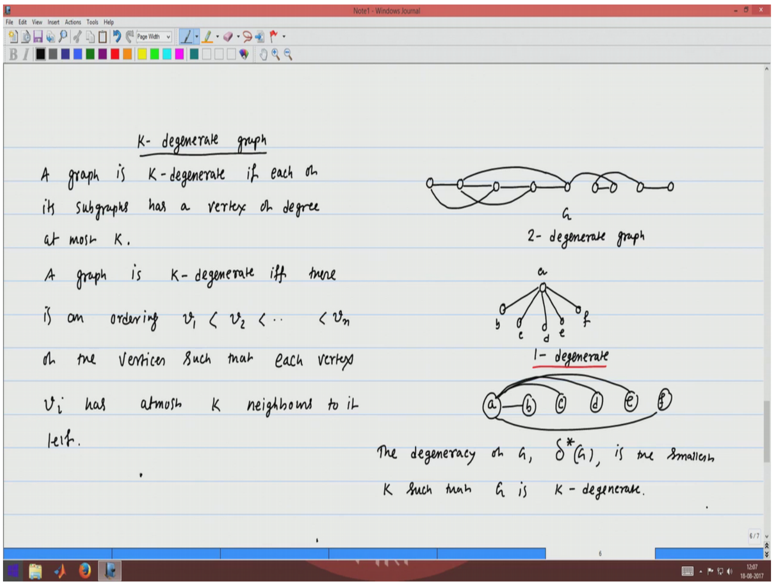

Next we prove the concept of K degenerate graph. So, a graph slightly from be complicated a graph is K degenerate if each of its subgraphs has a vertex of degree at most k. So, a graph is K degenerate if each of its subgraph has a vertex of degree at most K we will try to understand this one. Let me write down the characterization of K degenerate graph and then I will explain. So, a graph is K degenerate if and only if there is an ordering of the vertices v 1 less than v 2 this is not less than its a ordering v n ordering of the vertices such that each vertex v i has at most K neighbours to it is left then we say that the graph is K d j chi.

Let me just give an example of a two degenerate graph. So, this graph, so this is the linear ordering of the vertices and then now I draw some of the edges. Well, these two are adjacent and that is nothing specific with this graph of course, but what I am trying to make sure that every vertex in this linear ordering has at most two neighbours on its left.

Now, you see that this vertex has one neighbor to its left this vertex has two neighbour this one and this one on its left, similarly this vertex is also have been two neighbours on it is left this vertex has no neighbour at most two neighbours. So, this vertex has two this vertex has one this vertex has one. So, this is a two 2-degenerate graph because of the fact that now you take any sub graph of this graph then you can always see suppose I take I remove this edge I take this sub graph of this graph now it is always true that the right most edge sorry the right most vertex has a degree at most two.

So, that ensured that the graph is 2-degenerate because each of its sub graph has a vertex sub degree at most 2. So, this linear ordering what it does is that it make sure that whatever subgraph you take of this graph the right most note has at most two degree right I showed before that if you remove. So, this is subgraph of this graph G and you can see that the right most node has degree at most two. So, and this is true for any subgraph its not that particularly for this subgraph. If you take for example, any other subgraph of course, yeah. So, this a example of for 2-degenerate graph.

Let me just give another example of one degenerate graph this graph is 1-degenerate, why that is true because if I take this ordering I put this vertex say a b c d e f i put the vertex a at the beginning and then b c d e and f. So, what I get is that I get this c is adjacent to a everyone is adjacent to a and the f is also adjacent to a. So, you can see that in this linear ordering every vertex in this linear ordering has at most one edge to its left, sorry at most one neighbour to its left. So, that is why this graph is one degenerate graph.

I hope that now the definition is clear and the characterization is also clear while it will be while a graph is K degenerate if and only if there is a ordering of the vertices such that each vertex v i has at most K neighbours to it is left.

Now, the degeneracy of a graph of the graph G which is denoted by delta star G is the smallest K such that K is such that G is K degenerate right. So, this is the degeneracy of the graph G. Now if you remember the greedy algorithm for graph colouring if the graph is K degenerate; that means, in this linear ordering it has at most for a vertex I there are at most K vertices K neighbours to it is left. So, when the graph has degeneracy delta prime g; that means, the graph has for if you have a linear ordering of the vertices and then every vertex v i has at most delta star G neighbours to it is left. So, that is why using the greedy technique you need the number of colours required to colour the graph which has degeneracy delta star G is delta star G plus 1.

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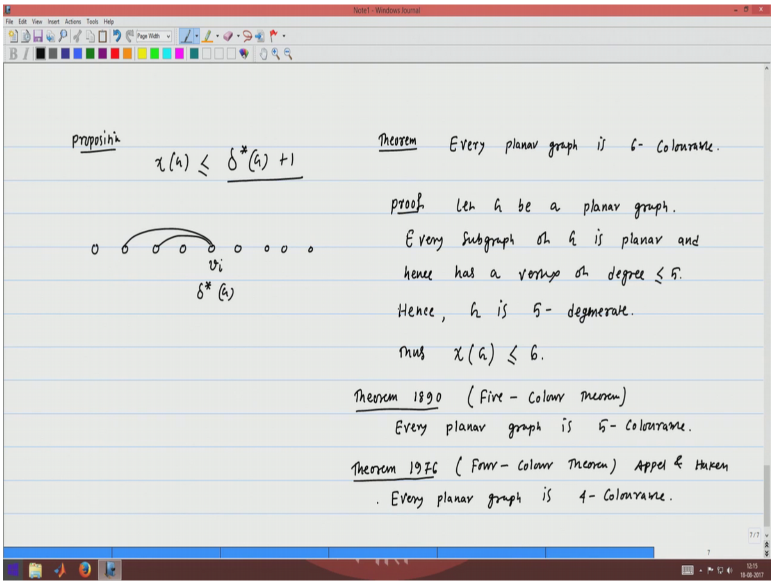

As I told just now that suppose this is the linear ordering of the vertices of a graph which has degeneracy delta star G the meaning of this one is that the vertex v i has the vertex v i in this linear ordering has delta star G at most delta star G neighbours to it is left. So, using the greedy technique you need at most delta star G plus 1 colour to colour the graph G.

Well, so we are sort of ready to prove the theorem that every planar graph is 6 colourable. So, this is almost we have to combine many results previous results together to prove this first of all let G be a planar graph then this is easy to observe that every subgraph of G is also planar right and hence has a vertex of degree at most 5. This is what we proved just now that every planar graph has a vertex of degree at most 5.

So, G is a planar graph. So, every subgraph of G is also planar and. So, every subgraph of G has a vertex of degree at most 5. Hence in terms of degeneracy of the graph G is 5 degenerate right. So, once we know that G is 5 degenerate we can use this proposition that 5 degenerate means in the linear ordering of the graph G every vertex v i has at most 5 neighbours to it is left right. So, you need at most 6 colours to colour the vertices of this graph right, hence that is used in the greedy algorithm thus chi of G is less than equal to 6. So, we have proved that every planar graph is 6 colourable and since the graph every planar graph is 5 degenerate we have a linear ordering of the vertices and we know that the ith vertex just recall the greedy technique of colouring the vertices of a graph then as we know that v i has at most 5 vertices to it is left you need at most 6 colour to colour the vertices of the graph. So, that is why every planar graph is 6 colourable

Now, there is there are results more stronger results theorem which is 1890 result is called 5 coloured theorem and it has been proved that every planar graph is 5 colourable again we are not going into the detail of the proof of this theorem 5 coloured theorem, and the final result the stronger result that we have tilled it is that 4 coloured theorem which is 1976 is called 4 coloured theorem.

So, the 4 coloured theorem says that every planar graph is 4 colourable again this is very long proof. So, we are not going into the detail of proving 4 coloured theorem. So, what we have proved in this last lecture is that we have proved that every planar graph is 6 colourable and then we have stated 5 coloured theorem and 4 coloured theorem that is all about colouring planar graph and this is the last lecture on graph theory.

I hope that you will enjoy this course and I would like to thank the NPTEL team in IIT Madras, more specifically my special thank to Bharathi and my cameramen Ravi for giving me all the support I required to record this courses.

Thank you very much.