Graph Theory
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Lecture – 20
Part 1
Characterization of Planar Graphs

Welcome to the first part of lecture 20 on Graph Theory. So, in this lecture will prove that complete graph with 5 vertices is a non planar graph and also will prove that K 33 that is the complete by partite graph with 3 and 3 vertices is non planar graph and also will talk about one characterization of planar graph.

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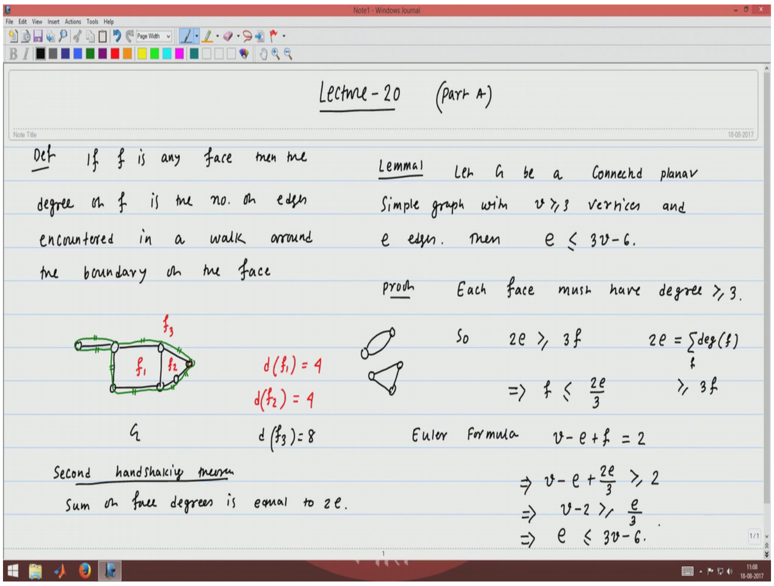

Let me start with definition of phase degree we know what is a phase in a graph if f is any face then the degree of f is the number of edges encountered in a walk around the boundary of the face.

So, let me explain this using an example concentrate this graph G and we can see that it has 3 phases this is one face, this is another phase, second phase and this is another phase f 3. Now the degree of this phase f 1 the degree of f 1 is equal to 4 because the number of edges encountered in a walk around the boundary of the face. So, this is if the walk starts here then the walk visit this 4 edges and come back to this vertex again. So, that is why this face f 1 has degree 4, f 2 has a degree 4 because the walk starting here visit 4 edges and come back to the same vertex. So, the degree of f 2 is also 4.

Now, what is the degree of f 3? If you start from here start from this vertex for example, start from this vertex and then consider this walk you visit this edge and again come back to this vertex and come back to the starting point here. So, you see that the number of edges encountered in a walk around the boundary of this face how many edges you encountered here visit edge 1 2 3 4 5 6 7 8. So, the degree of this face 3 is equal to 8 this is how you compute the degree of a face.

Now, this the second handshaking theorem says that the sum of face degree degrees is equal to twice e the first handshaking theorem is the sum of vertex degrees is equal to twice of e and here is the sum of face degree and you can see that this is true because here in this example you can see the face sum of face degree is 12 and there are of 6 sorry the sum of face degree is 16 and there are 8 edges in the graph. So, this is twice of e ok.

So, next we move to another lemma, lemma 1 let G be a connected planar simple graph with v vertices be greater than or equal to 3 vertices and e edges then this is true the number of edges is less than equal to 3 v minus 6. Since this is a simple graph, so I can say that each face must have degree greater than or equal to 3 because you see it is not a it is a simple graph. So, such multiple edges are not allowed. So, if this is allowed the multiple edges are allowed then a face can have degree 2 also, but since it is a simple graph 1 face will have at least this is the smallest face smallest degree face that is possible a triangle. So, each face must have degree greater than or equal to 3.

So, can we write that twice of e which is equal to the sum of face degree, but I am replacing the face degree for every face by smaller quantity 3. So, this quantity is greater than equal to 3 f, f is the number of faces right because of the fact that twice e is sum of face degrees sum of all faces and each of them I am replacing by 3 a smaller quantity because this because the face degree could be greater than 3 also that is why twice e is greater than or equal to 3 and the number of faces is f. So, 2 e is greater than equal to 3 f which implies that f is less than equal to twice e by 3.

Now, from Euler formula what we know is that we know that v minus e plus f is equal to 2, now I replace f by a bigger quantity twice e by 3. So, this one is larger than 2 and from here you can check that will v minus 2 is greater than or equal to e by 3 which implies that e is smaller than 3 v minus 6. So, this proves that if G is a connected planar simple graph then the number of edges is bounded above by 3 v minus 6 it cannot have more than this many edges.

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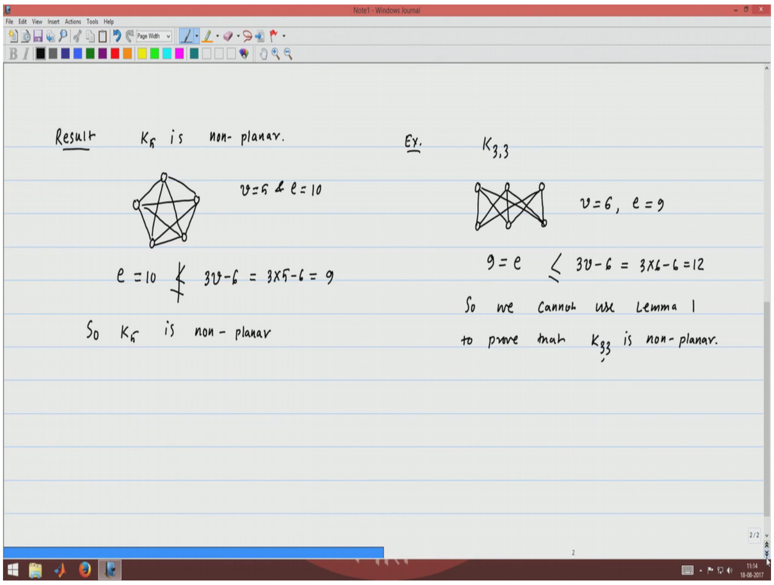

So, this will be helpful to prove the result that K 5 is non planar. So, what is K 5 K 5 is a complete graph with 5 vertices. So, it has. So, it has 5 vertices v equal to 5 and e is equal to 10 right. So, e equal to 10 and what we have for the upper bound for e is 3 v minus 6 which is equal to 3 into 5 minus 6 this is 9 and we know that e the number of edges must be less than or equal to 3 v minus 6, but this is not true. So, the number of edges is more than thee v minus 6. So, K 5 is non planar.

Now, the other result for say we are trying to prove that whether K 33 is non planar with that can be proved using this lemma. So, you consider the graph K 33 and this is what the complete graph complete by partite graph 3, 3 vertices right. So, this has v equal to 6, 6 vertices and the number of edges 9. So, 9 which is the number of edges and we compute 3 v minus 6 which is equal to 3 into 6 minus 6 that is 12. So, this is less than. So, the condition of the lemma true. So, we cannot use we cannot use lemma 1. So, you cannot use lemma 1 to prove that K 33 is non planar.

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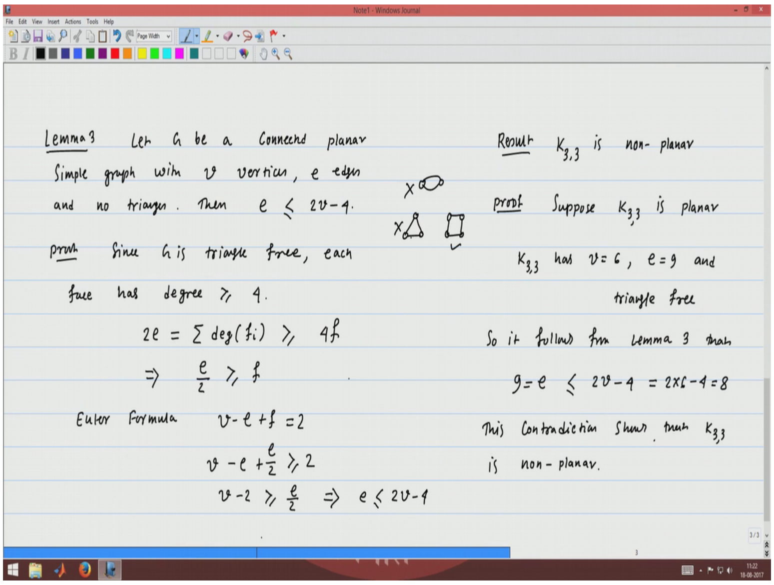

So, we talk about another lemma this K 33 is of by partite graph and then we know that a graph is by partite if and only it has no wart cycles. So, that fact we need to use let G be a connected planar simple graph with v vertices e edges and no triangles then the number of edges must be less than equal to twice v minus 4. So, note that K 33 is a bi partite graph and a triangle is in odd cycle right. So, a bipartite graph cannot have odd cycle because we know that a graph is bi partite if and only if it has no walk cycles. So, K 33 cannot have a cycle of plane 3 which is a triangle.

So, using this result using this result that this fact that the graph K 33 does not have any triangle we get a improved bound for the number of edges. So, this proof of this one here since G is simple and triangle free G is triangle free. So, you do not have a face like this in the graph, so every face and it its simple also right. So, you do not have face like this also. So, the minimum face degree has to be 4 since G is triangle free each face. So, this is not allowed in this graph this is not there in this graph. So, the minimum face size is 4 or face degree. So, each face has degree greater than equal to 4.

Now, again the same thing that twice e is the sum of faces and then I am replacing every face by this is the sum of face degree this is degree of fi and I am replacing each degree by a smaller quantity, so 4 and there are f many faces. So, that is why twice e is greater than or equal to 4 f which implies e by 2 is greater than equal to f. Now if I use Euler formula what I get is that v minus e plus f is equal to 2 this is the Euler formula and I put replace f by a bigger quantity e by 2. So, this is greater than equal to 2 right. So, v minus 2 is greater than equal to e by 2 which implies e is less than equal to twice v minus 4.

Now, we can prove that using this lemma we can prove this result that K 33 is non planar. So, we will prove this suppose K 33 is planar if K 33 is planar then K 33 is triangle free because it is a bipartite graph. So, you can apply this lemma 2 now and it is a simple connected graph.

Suppose K 33 is a planar then lemma 3 says that this would be true let see. So, K 33 has v is equal to 6, e equal to 9 and triangle free right. So, it follows from lemma 3 that 9 is, this is e equal to 9 this is less than equal to twice v minus 4 which is equal to twice into 6 2 into 6 minus 4 this is equal to 8. So, this shows that 9 is less than equal to 8. So, this contradictions, this contradiction shows that what we have assumed at the beginning that K 33 is a planar graph that is not true that K is or K 33 is a non planar. So, using this 2 lemmas we have proved that K 5 complete graph with 5 vertices is a non planar graph and using the other lemma we have proved that K 33 which is simple connected and triangle free this graph is also a non planar graph.

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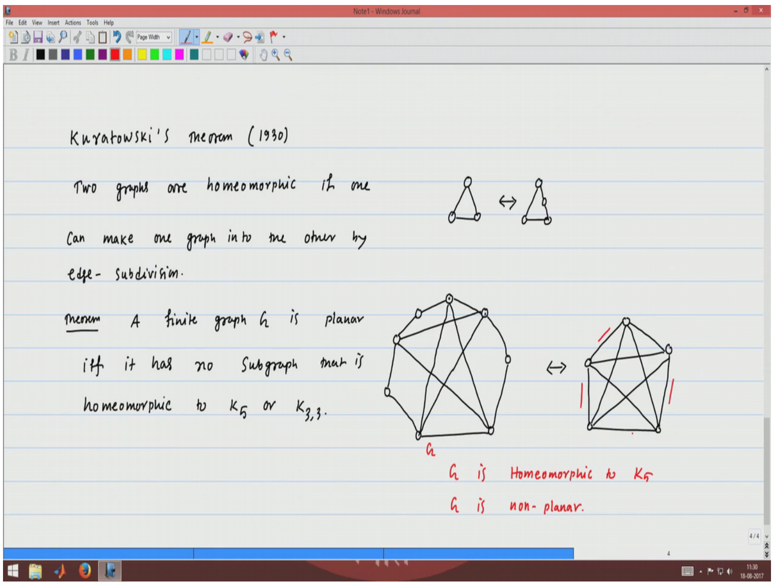

So, next we talk about another characterization of planar graph, this says Kuratowski's theorem 1930, 2 graphs are homeomorphic if one can make 1 graph into the other by edge sub division. So, we know what is edge sub division. So, this 2 graphs are homeomorphic because I can get this graph sub dividing an edge. So, this graph and this graph this 2 graphs are homeomorphic because I get this graph from this graph by sub dividing this edge right and this theorem says that a finite graph G is planar if and only if it has no sub graph that is homeomorphic to K 5 or K 33 right this is one characterization of planar graph.

So, this theorem can be just elasted I am not going to prove this theorem. So, I can this theorem can be used to prove that this graph is non planar. So, this will take some time to draw this graph, now for this 2 vertices this vertex is adjacent to this vertex and this 2 vertices are adjacent this 2 vertices are adjacent and this 2 vertices are adjacent. Now, we will show that this graph is very clear now that you know this graph is homeomorphic to K 5 because you can see that this is K 5, now you can see that you can get this graph from this graph by subdividing this edge, this edge, this edge and this edge. So, by subdividing this edge you will get this 2 edges or a path of length 2 by subdividing this edge you will get this path of length 2 and by subdividing this edge you will get a path of length 2 like this. So, since this graph G is G is homeomorphic to K 5 that is why G is non planar this is my G here. So, G is non planar.

So, just we have elasted this theorem a finite graph G is planar if and only if there is no sub graph that is homeomorphic to K 5, but you can see that this graph you just take this graph as a sub graph of this graph itself. So, this graph G is homeomorphic to K 5, so the graph G is non planar. So, we talk about, we have seen the formal proof for K 5 and K 33 to be non planar graph and also we have seen one characterization of non planar graph. So, will stop here.

Thank you very much.