Graph Theory
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Lecture – 19
Part 2
Planar Graphs & Euler's Formula

Welcome to the second part of lecture 19 on graph theory. So, today we will talk about planar graph. A graph is planar, if it can be drawn on plane such that the edges meet only at the vertices. And at the end of this lecture we talk about how to characterize planar graph and coloring of planar graph ok.

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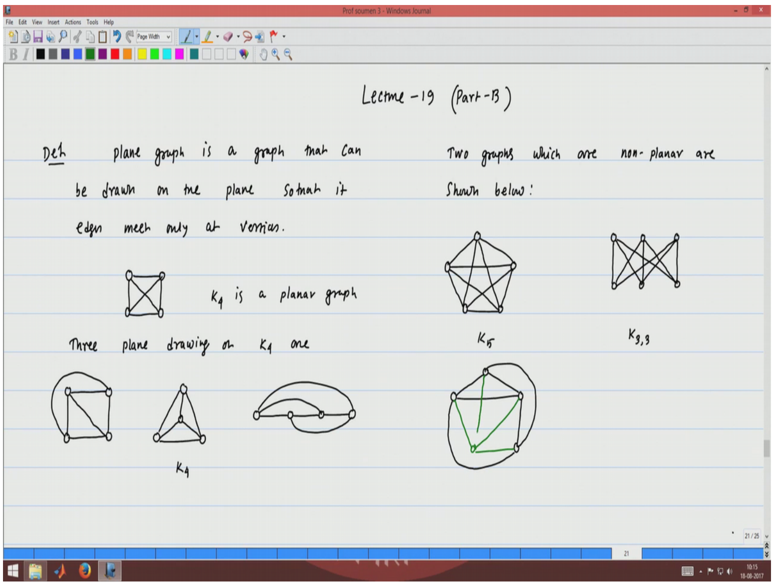

Let me start with the formal definition of planar graph definition. Plane graph is a graph that can be drawn on the plane so that it edges meet only at vertices ok.

So, let me give an example of a planar graph. So, this is a planar graph, though you can see that the 2 edges cross here, but we can redraw it. So, k 4 so that the edges meet only at vertices. So, o well k 4 is a planar graph and 3 plane drawing of k 4 are.

So, we need to draw the edges in such a way that the edges meet only at vertices. So, this edge instead of drawing in this way I can redraw it in this oil. So now, you can see that the edges all the edges meet only at vertices. And another drawing of the same graph k 3 without any edge crossing. So, this is an edge crossing. And so, this is not allowed in planar graph. So, k 4 can be also drawn in this way.

So, this is also k 4. And you can see that there is no edge crossing the edges meet only at vertices. And the other drawing is this one 4 vertices. Well so, this has now degree 3. So, every vertices has degree 3. So, this has degree 3 this has degree 3, this has degree 2 and this has degree 2. So, I join them by an edge. So, this is another drawing of planar drawing of k 4. So, k 4 is a planar graph, 2 graphs which are non planar are shown below. So, one is complete graph with the 5 vertices. So, this is a complete graph with 5 vertices k 5.

And you can try to draw this graph on plane without edge crossing. It does not matter how hard you try you can not draw. So, it will also try to do that, but you will not be able to do that. So that is why k 5 is non planar and also will prove formally that k 5 is non planar graph. And other non planar graph is this is also a non planar graph complete graph complete bipartite graph k 3 3.

So, this is also non planar, well as I said that let us just convince our self that we can not draw this graph without edge crossing. So, I am trying to draw all the edges without edge crossing. Now I can draw this 2 edges no problem still now there is no edge crossing. Now this vertex is adjacent to this vertex. So, I can draw this way to avoid edge crossing, and this vertex is adjacent to this edge sorry, this vertex is adjacent to this vertex right.

Now, what we are left is that, this vertex is also adjacent to this vertex right. So that one we can not draw without any edge crossing. So, one way to draw this edge is directly you just draw this way, but this is this is the edge crossing here. The other way you can draw is this way, but here also you have to cross this edge. So, you can try and convince yourself that it does not matter how hard you try there will be edge crossing.

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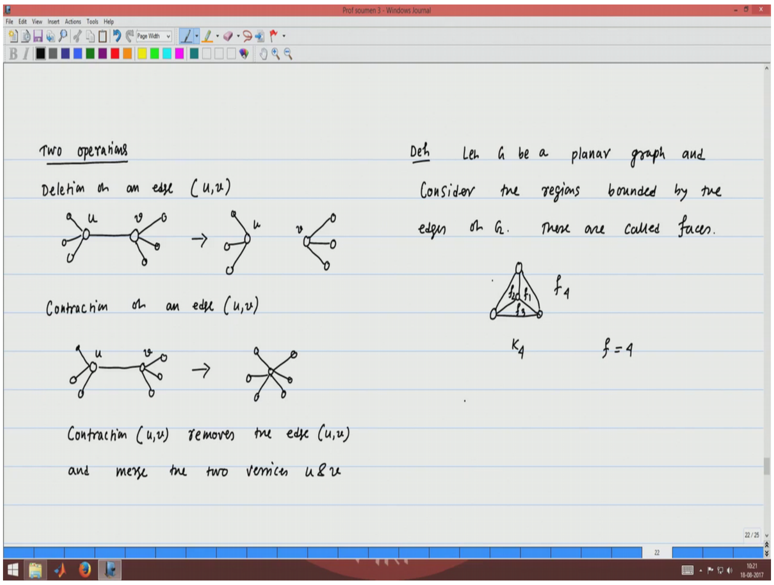

Now, we talk about 2 operations. One is deletion of an edge u v. So, we suppose this is the edge u v. U v and this vertex is adjacent to some other vertices also. And similarly u is also adjacent to some vertices and the deletion of this. Edge u v will result in this graph; you will be left with this graph. The other one is important which is called contraction of an edge u v. So, contraction of this edge u v results in this graph. So, what it does is that this contraction operation it removes the edge u v and merge 2 vertices, u and v into one vertex. So, contraction u v removes or delete the edge u v, the edge u v and merge the 2 vertices u and v. That is what the contraction operation.

So, we learned about 2 operations one is deletion of an edge and contraction of an edge. And these 2 operations will be used frequently in planar graph to prove some results ah on characterizing planar graph and some other related result in this area. So, next we learn what is a face of a planar graph. So, we will start with this definition. Let G be a planar graph. And consider the region bounded by the edges of G. These are called faces. Let me explain this one. I draw k 4 it is a complete graph with 4 vertices. And so, as I said that considered the regions bounded by the edges of G. So, this is one region which is bounded by these 3 edges. So, this is one face let me call this as face one. This is another face, face 2 face 3 and this region is also called a face, this is a face 4 ok.

So, this is what the definition of face, and this if denote the number of faces. So, k 4 has 4 faces basically. So, next we approve very famous result due to Euler, which relates the number of faces number of edges and number of vertices of a planar graph.

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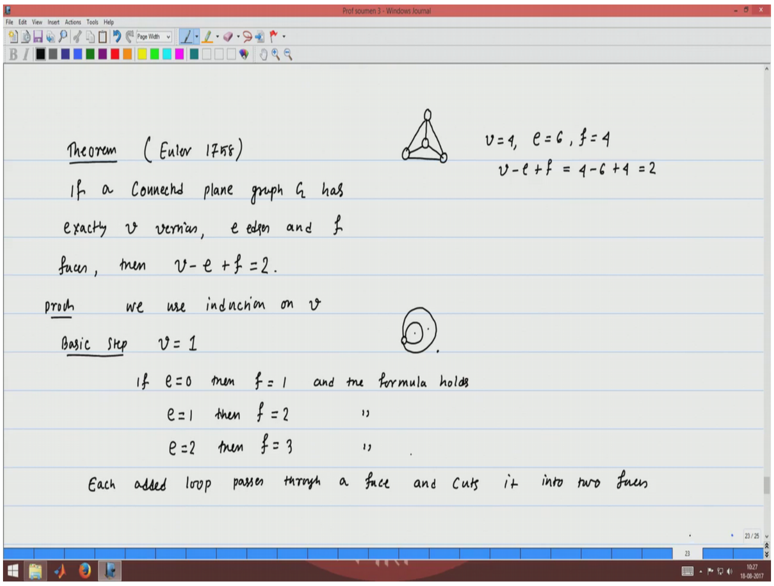

So, here is the theorem. And this is due to Euler, if connected plane graph G has exactly v vertices e edges and f faces, then v minus e plus f is equal to 2. So, let us understand that what do you mean by this one. So, I will draw the k 4 again. And we know that this graph has 4 vertices, and 6 edges e equal to 6 and it has 4 faces f equal to 4.

So, you can see that v minus e plus f is equal to 4 minus 6 plus 4 which is equal to 2. And this is true for every connected planar graph G. We are going to prove this one. And to prove this we use the induction technique. We use induction on v. It is a beautiful proof. So, basic step is. So, induction on the number of vertices. So, we will prove that this result is true for all graph all planar graph with one vertex. So, only one vertex; that means, this graph could be simply this one, if e equal to 0, then the number of phases of this graph f is equal to 1 and you can put the form you can check the formula is correct because v equal to 1 f equal to 1 e equal to 0. So, this is equal to 2, and the formula holds right. If e equal to 1; that means, it has one edge that is one edge one vertex so that edge has to be a loop.

Now, you can see that this graph has 2 faces, then f is equal to 2. So, e is equal to 1 v equal to 1 and f equal to 2. So, the formula holds. If e equal to 2, 2 edges that can be like this now you can see that this graph has 3 faces one face here one face here and one face here, then f is equal to 3 and the formula holds. And similarly you can check you can you just keep on adding the edges once you add one edge the number of the number of faces also increase increases by 1.

So, the formula will be true always. So, each added loop passes through a face right. Once you add a new edge it passes through a face and cuts it into 2 faces right. So, so once you increase e by one the face number of face also will increase by 1 and the formula will be true always. So, what we have proved is that this Euler formula is true in the base case when the number of vertices is equal to 1.

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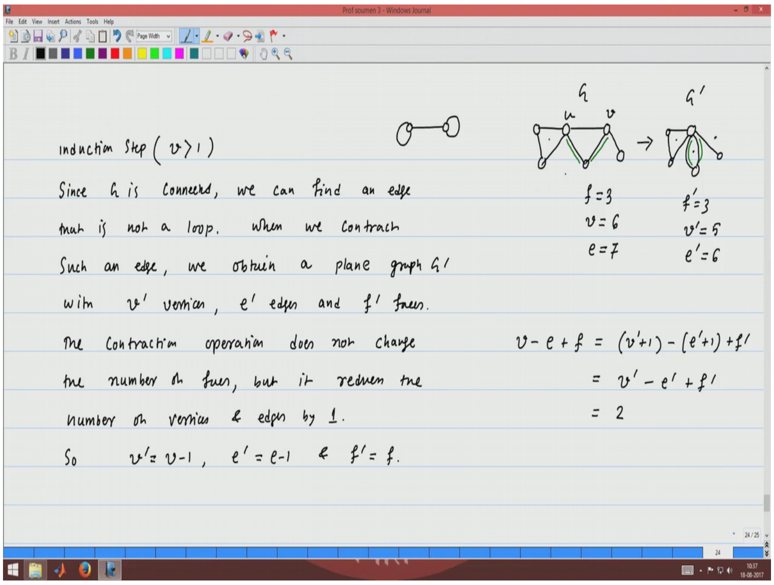

Next we prove in general that the induction step here v greater than 1. So, the idea is that if it is true for a graph with d minus 1 vertices then it will be true for a graph with vertices also.

Since G is connected we can find and edge that is not a loop. See the number of vertices is strictly greater than 1. So, suppose at least 2 vertices are there in the graph. And then all the edges can not be just loop because the graph is connected. So, there has to be an edge which is connecting 2 vertices 2 distinct vertices. So, we can find an edge that is not a loop. And then when we contract such an edge, we obtained a plane graph G prime.

Let me just give another example of contraction I wanted to give before. So, suppose this is an edge u v. And I want I want to contract the edge u v the contraction of edge u v will result in this graph. So, what it does is that it delete this edge, delete the edge u v and merge these 2 vertices u v into one vertex. So, the result of this contraction will be this. So, you can see that these 2 edges, these 2 edges now become the parallel edges here right. So, this is also a result of contraction.

And we obtained a plane graph G prime with v prime vertices, e prime edges and f prime faces. So, what we are doing is that you are we are just contracting an edge of a planar graph. And then in the after contracts contraction we get the graph G prime, which has v prime vertices e prime edges and f prime faces.

Then one thing to observe that the contraction operation does not change the number of faces. But it reduces the number of vertices and edges. We will try to understand let me just add one more edge here. So, the contraction will result in this graph. So, at this moment I can see that this graph has 3 faces. This is one face, this is another face and this is the other face. For this graph has number of faces of this graph is 3, let me call this graph as G and this is G prime. After contracting the edge u v I get another graph, and this graph also has 3 faces.

So, that is what it says the contraction operation does not change the number of faces, but it reduces the number of vertices and edges. So, the number of vertices of this graph the graph G was 1 2 3 4 5 6, and this graph has number of vertices 5. This graph has number of edges 3 6 7, and this graph has number of edges 6. So, it is very clear that the contraction operation does not change the number of phases, but it reduces the number of vertices and edges right. Number of faces and edges by 1.

So, in general v prime is equal to v minus 1. E prime is the number of edges in the graph G prime is equal to e minus 1, and f prime is equal to f. I want to know what is the value of this one. V minus e plus f. And I can write this thing in terms of v prime e prime and f prime, which is v equal to v prime plus 1, e equal to e prime plus 1 plus f prime. So, which is equal to v prime minus e prime plus f prime. Now see this graph G prime has one less vertex. The graph G prime has v minus 1 vertices, and by induction hypothesis the Euler formula is true for G prime right. Because it has v minus 1 vertices and we are trying to prove that the Euler formula is true for a graph with v vertices.

So, since the Euler formula is true for a graph with v minus 1 vertices, and v prime is exactly v minus 1. So, this is the Euler formula true is true for G prime. So, I can write that this is equal to 2. And I have proved that this is also true for G. So, this is how we proved the Euler formula for planar graph using the induction technique.

Thank you very much.