Graph Theory
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture - 19
Part I
Edge Colouring

Welcome to the first part of lecture 19 on Graph Theory in this lecture we will talk about Edge Colouring. So, like vertex colouring in the edge colouring we will colour the edges of the graph in such a way that no 2 adjacent edges have the same colour and of course, the optimisation problem again is we need to colour the edges with minimum number of colours. So, I will start with formal definition of edge colouring.

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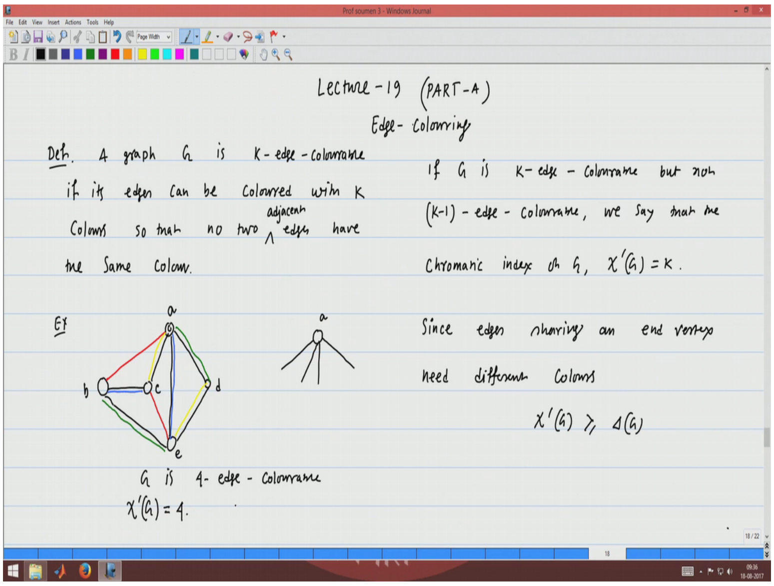

A graph G is K-edge-colourable if it is edges can be coloured with K colours so that no 2 edges have the same colour. Let me a give an example of a graph hence it edge colouring so I will considered this graph again.

Now, we need to colour the edges so, that no 2 adjacent edges, no 2 adjacent edges have the same colour. So, by adjacent edges means this 2 edges are adjacent edges, they will get different colour let me colour this edge say red and so, I coloured this edge red then I cannot colour this edge by red colour, I can colour this edge by red colour and of course, I there was edge here.

Now I colour this edge by blue colour and this edge also by blue colour I am not removing the black thing and so they are not 2 adjacent edges, let me colour this edge and this edge by green colours. So, and this 2 edges are left, they can be coloured with say yellow colours. So, I have coloured this graph with 4 colours and no 2 adjacent edges have the same colour, this graph G is 4 edge colourable.

Of course the optimisation problem is to colour the edges of the graph with minimum number of colours, now we will see whether we can colour this particular graph with less number of colours than 4 colour. At this moment we have used 4 colour the question is whether we can colour this graph with less than 4 colours if G is K H colorable, but not K minus 1 edge colorable then we say that the Chromatic index of G, denoted by chi prime G is equal to K. Now it is easy observation, this is the chromatic index denotes the minimum number of colours required to edge colour a graph.

Since edges sharing and end vertex need different colours. So, chi prime G the chromatic index of a graph G is always greater than equal to delta G; delta G is the maximum degree of the graph G. So, here you can see that this node has highest degree of course; this node is also having a highest degree that is 4. Let me call label this node as a b c d and e. So, look at nod a it has 4 edges; 4 edges are incident to this vertex a and so this edges are adjacent edges. So, to give different colours to all these 4 edges that is why the chromatic index of this graph will be at least the degree of this vertex which is of course, the highest degree. This is a lower bound for the chromatic index and here you can see that we can colour G with 4 colours which is the lower bound that is why the graph G has chromatic index 4.

I hope that this is clear to you.

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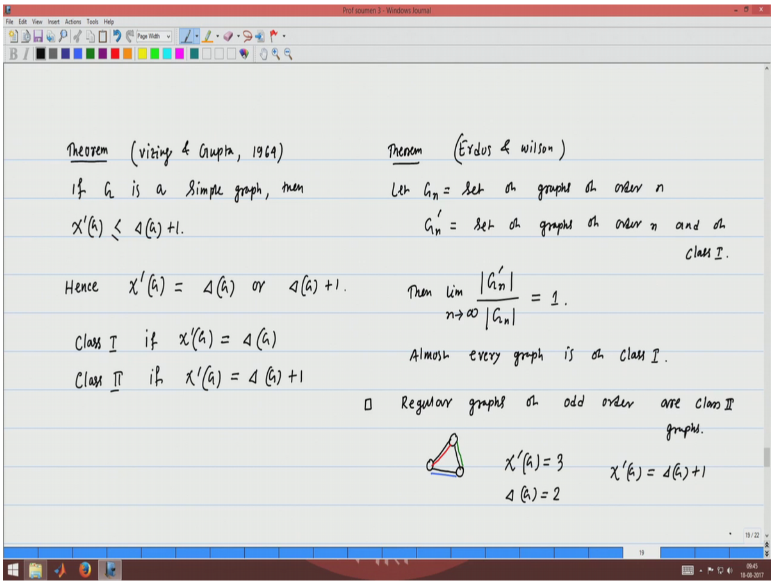

Next we talk about a Theorem which gives the upper bound on chromatic index this is due to Viting and Gupta in 1964, it says that if G is a simple graph, then chi prime G is less than equal to delta G plus 1; that means, the maximum number of colours required to egde colour the graph is delta G plus 1 maximum degree plus 1 and just now you have observed that the lower bound from for chi prime G is delta G this is the minimum number of colours required to edge colour a graph and this theorem says that the upper bound is delta G plus 1.

So, chromatic index will take value either delta G or delta G plus 1. Hence chromatic index of a graph is either delta G or delta G plus 1, now we will classify the graphs into 2 classes class 1, a graph is in the class 1 if the chromatic index of that graph is equal to delta G and a graph is in class 2 if the chromatic index of the graph is equal to delta G plus 1. What we have observed is that the chromatic index of a graph can be either delta G or delta G plus 1. So, on the basis of this one we classify the graphs into 2 classes class 1 if the graph has a chromatic index delta G and class 2 if the graph has chromatic index delta G plus 1.

Now, there is a Theorem due to Erdos and Wilson, let Gn be the set of graphs of order n order n means the number of vertices is equal to n and Gn prime is set of graphs of order n and of class 1. Then the number of graphs of order n and of class one that is cardinality of this set by the number of graphs of order n this ratio tends to 1 as n tends to infinity.

So, the meaning of this one is that of almost every graph very powerful theorem that almost every graph is of class 1; that means, you pick a graph of order n and it is highly probable that it will be of chromatic index delta G right for most of the graph can be coloured with delta G edge colour with delta G colours. I give some examples of graphs which are in class 1 and class 2. So, regular graphs of odd orders. So, regular means every vertex has the same degree and odd order means it will have odd number of vertices. This is a small example of a regular it is a 2 regular graph because every vertex has degree 2 and it has odd number of vertices, this is a 2 regular graph with 3 vertices.

And what is the chromatic index of this graph. If I colour this edge by red colour I cannot use red colour for this edge or for this edge, I have to use a different colour for this edge say green colour, if I use green for this 1 and red for this edge then I cannot use red or green for this edge I have to use a different colour. Let me use blue colour so we can see that the chromatic index of this graph chi prime G is equal to 3 and you see that the delta G is equal to the maximum degrees equal to 2, for this graph chi prime G is equal to delta G plus 1. So, regular graphs of order of odd order are class 2 graphs this is this is true you can pick any regular graph of odd order their of class 2 graph.

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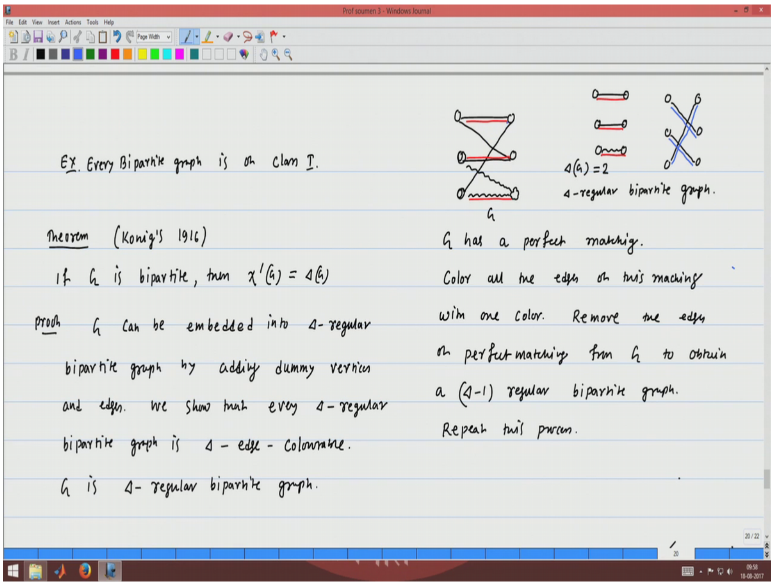

And the other example this is for say Bipartite graph, it says that every bipartite graph every bipartite graph is of class here is the Theorem this is due to Konig's 1916 that if G is bipartite then chi prime G is delta G. This theorem will prove that every bipartite graph is of class 1. So, here is the proof this is interesting result, G is a bipartite graph, G can be embedded into delta regular bipartite graph by adding dummy variable sorry dummy vertices and edges.

So, what I mean by this one is that you are given a arbitrary bipartite graph and it can be embedded into delta regular bipartite graph. Suppose you are given this graph for example, this bipartite graph. So, and the edges of this are this edge this edge this edge and this edge sorry this is the graph that is bipartite graph that you are given initially. Now this bipartite graph has maximum degree delta G equal to 2. So, what it says that this graph can be embedded into delta regular bipartite graph; that means, you can convert it to 2 regular bipartite graph by adding new vertices and edges. So, you add a new vertex say this one and you add new edge this is a new edge which you are adding to make the graph 2 regular bipartite graph and this is another new edge.

Now, you can see that this graph has become 2 regular this has become delta regular bipartite graph. So, you can always embed a bipartite graph into delta regular bipartite graph by adding dummy variables, dummy vertices, and dummy edges. Then we show that every delta regular bipartite graph is delta edge colourable. So, G is delta regular bipartite graph and we know that if a graph is delta regular or irregular then the halls condition is satisfied and graph has a perfect matching, that we have done in the during our matching classes.

So, this graph since it is a delta regular graph, G has a perfect matching. I can show that this graph has a perfect matching. So, one perfect matching is this one simply this one, this is one perfect matching. Now, to colour all the edges of this matching of this matching with one colour, you can do that because matching is a collection of disjoint edges. So, you can colour them with the same colour all this is edges you can colour red; that means, you are colouring this edge red this edge red and this edge red.

And then what you do is that you remove the edges of perfect matching from G to obtain a delta minus 1 regular bipartite graph and repeat this process. Once you have a perfect matching for the delta regular bipartite graph you remove that perfect matching from the graph and then you will get a delta minus 1 regular bipartite graph and delta minus 1 regular bipartite graph has again a perfect matching, you get a perfect matching to colour all the edges of the perfect matching by 1 colour and you just repeat this process say delta minus 1 times.

So, here you can see the graph was initially to regular bipartite graph and once you remove this edges from this one, from this graph G you will get one regular bipartite graph. So, you will be left with this graph once you remove this edges from this graph you will be left with this graph which is one regular.

It is this itself is a perfect matching. So, you colour all this all the edges of this graph with 1 colour. In the original graph you are colouring this edge by blue, this edge by blue and this edge by blue. So, you can see that you can in this way you can colour delta regular bipartite graph by delta colours. So, that is why if G is a bipartite graph it is chromatic index is delta G. So, this proves that every bipartite graph is of class 1.

Thank you very much.