Graph Theory
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture – 18
Part 1
Vertex Colouring

Welcome to the first part of lecture 18 on Graph Theory. So, in this lecture we will learn graph colouring. More specifically today we will talk about vertex colouring. So, we need to colour the vertices of the graph with different colours, such that no 2 adjacent vertices get the same colour. And the optimization problem here is to colour the vertices with minimum number of colours. So, let us I am start with the formal definition of vertex colouring.

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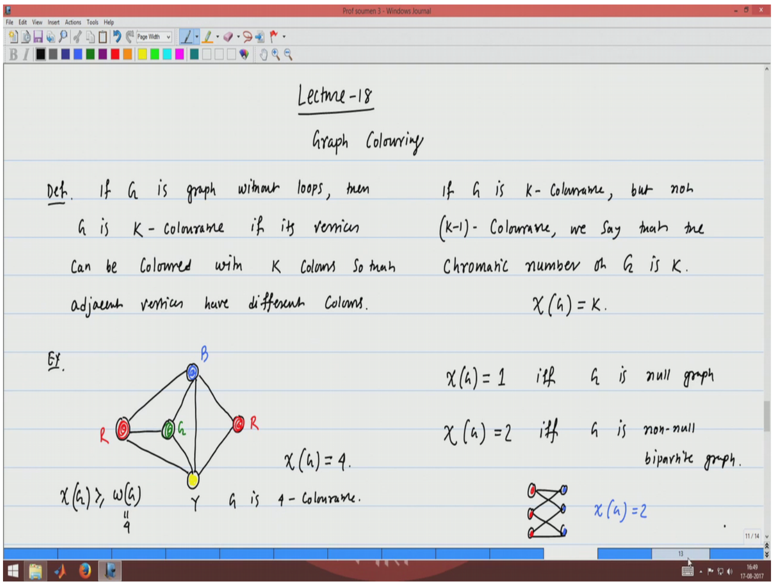

So, definition if G is a graph without loops then G is k colourable if it is vertices can be coloured with k colours so, that adjacent vertices have different colours. So, this is what the vertex colouring is and the graph is k colourable, if you can colour with colour the vertices with k colours subject the condition that adjacent vertices have different colours.

So, let me give one example small example. So, this is a arbitrary graph. And I want to colour them with minimum number of colours of course that is the optimization problem here. So, if I colour this vertex say red I cannot colour this vertices red I cannot colour this one red, this one red, and this one red, because this 3 vertices are adjacent to this vertex. And adjacent vertices must have different colours.

So, I can only colours, but I can colour this vertex by red colours by red colour. Because this 2 are not adjacent. Let me take then another colour, say blue colour for this vertex. Just write blue in case the colour is not clear. So, for this vertex then I cannot use blue or red, I have to use some other colour let me use the green colour here. So, this is green this is red and red and now it look at this vertex this vertex is adjacent to a red vertex a blue vertex a green vertex. So, I cannot use blue red or green for this vertex I have to use some other colour say yellow colour y ok.

So, I have used 4 colours to colour this graph. So, this graph definitely then G is 4 colourable than the question is can we colour this graph with less than 4 colours is it possible to colour this graph with 3 colours that we will check. So, if G is k colourable, but not k minus 1 colourable, then we say that the chromatic number of G is k. So, k is the minimum number of colour that is required to colour the graph G, and notation for this one is chi G the chromatic number of the graph G is equal to k.

The question is I said that whether this graph can be coloured with 3 colours. If it is not possible you need minimum 4 colours then the chromatic number of the graph will be 4 only. And we can you can see that this is small graph you can check that the chromatic number of this graph chi of G is indeed 4 you cannot colour this graph with 3 colours.

So, we characterize the graph with chromatic number one can be characterized. So, the graph having chromatic number G is having chromatic number one, if and only if G is null graph. Then only you can colour with null graph in the sense that it the graph does not have any edge like this is a null graph right. Then you can give the same colour to all the vertices and you need only one colour.

Now, chi of G equal to 2 if and only if G is non null bipartite graph right. Any bipartite graph you get non null of course. That means, there will be some edges then; obviously, if the chromatic number of this graph is 2 because you can colour one side of this bipartite graph by all red and the other side maybe all blue. So, the chromatic number of the graph bipartite graph is 2 ok.

But there is no such characterize as an for 3 chromatic graph; that means, the graph have been chromatic number 3.

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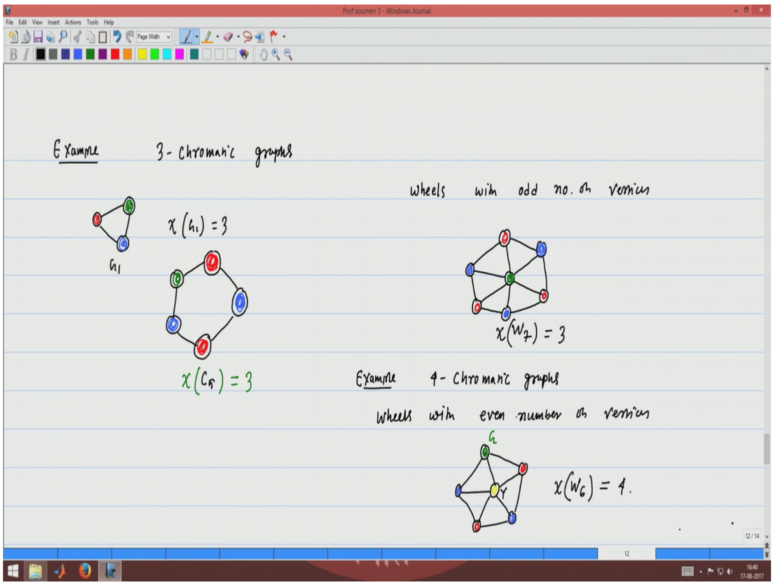

So, I give example of 3 chromatic graph; that means, the graph have been chromatic number 3. Look at this graph, if an a triangle for example. So, you need 3 different colours to colour this graph right. So, if I put red here green here I cannot use red or green here I have to use some other colour blue.

So, the chromatic number of this graph is say call me G 1 which is the chi of G 1 is definitely 3. In fact, a you take any. So, this is a cycle of length 3 or you can say a triangle you trick cycle of length 5, and any odd cycle. So, this is odd cycle this is c 5 cycle of length 5. And you try to colour this graph with minimum number of colours. The red and then you can put red also here and blue you can put blue here, but for this node now you cannot use red or blue you have to use the third colour some other colour it is green.

So, the chromatic number of this c 5 length sorry cycle of length 5 this has chromatic number 3. So, another graph which is a wheel graph wheels with odd number of vertices. So, this is the graph this is say odd number of vertices w 7 7 vertices. So, this w 7 can be obtained from c 6; that means, you start with a cycle of length 6, and then you make every vertex. And join each vertex of this cycle to a central vertex all right.

So, this is what the wheel graph wheel of with 7 vertices. Now you try to colour this graph. Of course, here you can put red, red, red and then you put blue here, blue here, blue here no problem. You can see till now all the adjacent vertices got different adjacent vertices have different colours, but now for the central vertex you cannot use red or blue you have to go for the third colour green. So, that is why the chromatic number of w 7 is equal to 3.

Now, we give example of for chromatic graphs. So, wheels with even number of vertices of vertices or having chromatic number 4. So, since it has even number of vertices you start with the odd cycle say for example, you start with c 5 and then join every vertex with the central vertex. So, this is w 6 wheel with 6 vertices. Now you try to colour this one maybe red here, and then you can use red here also, that is all.

Now, next you have to use another colour blue here you can use blue here no problem, but for this vertex you cannot use red or blue you have to use another colour green. And now for the central vertex you can see that you cannot use red green or blue. So, you have to use the forth colours say yellow right. So, you can see the chromatic number for any wheel of with even number of vertices having chromatic number 4. So, these are the some. So, you have clear idea about what is vertex colouring. So, you have to colour the vertices of the graph in such a way that adjacent vertices get different colours. And you have to colour the vertices with the minimum number of colours that is the optimization problem here.

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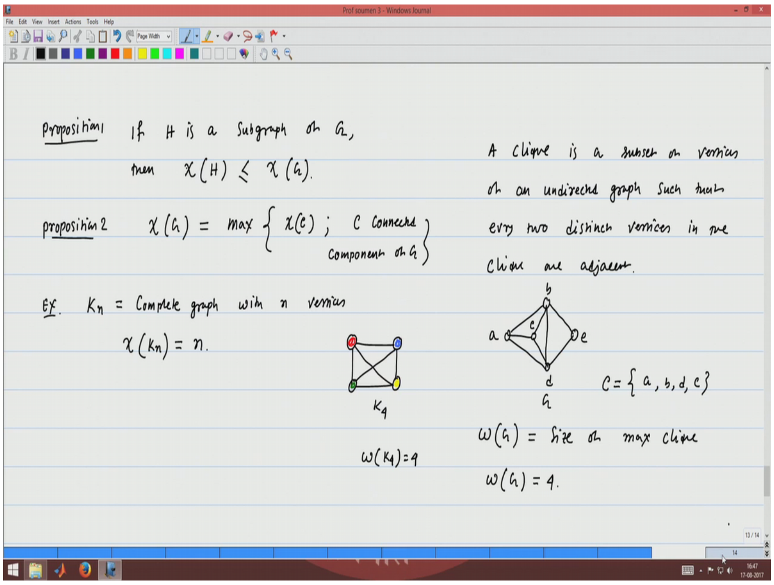

Next we see some result say proposition, they are not So difficult if H is a subgraph of G then the chromatic number of H is less than or equal to the chromatic number of G which is quite obvious if you can colour this graph with 4 colours, then you take any subgraph of this graph that you can colour with 4 or less than 4 colours which is quite obvious. Now proposition 2 that the chromatic number of a graph G is if the graph is disconnected graph you sort of if the graph is disconnected then it has many components, and you colour each component separately. And the maximum number of colours that you need to colour 1 particular component is the chromatic number of the graph ok.

So, chi of G is the maximum of you compute the chromatic number of different components and take the maximum right. So, c is a connected components of G. So, this also is easy we are not going to prove this one. And some special graphs like a k n it is a complete graph with n vertices. Then the chromatic number the chromatic number of k n complete graph with n vertices.

So, suppose you have a complete graph with 4 vertices how many colours you need to colour k 4, it is easy to see that you need 4 colours to colour k 4 like all the colours will be different. So, if it is red we have to use different colour for the other vertex, and similarly for this one you cannot use red or blue and finally, for this one also you have to use for different colour. So, chromatic number of k n is equal to n. Now we learn we talk about basically if bounds. So, we learn what is a clique. So, clique in a graph a clique is a subset of vertices of an undirected graph such that every 2 distinct vertices in the clique are adjacent right.

So, if I take the example that I took at the very beginning. So, in this graph if I label them say a b c d e. A clique is a subset of vertices such that every 2 distinct vertices in that subset are adjacent. So, definitely c which is like consist of a b d and c, this is clique because you can see that this 4 vertices their adjacent to each other. So, that is form a clique, and also b d e is also a clique because this 3 vertices their adjacent to each other. And the notation w or omega G is this is this stands for the size of maximum clique ok.

So, in this graph G you can see that the size of maximum clique is 4 the number of vertices in the maximum clique. So, omega of G for this graph for this G is equal to 4 right. And for this graph also it is itself a complete graph. So, the clique of k 4 is 4 obviously.

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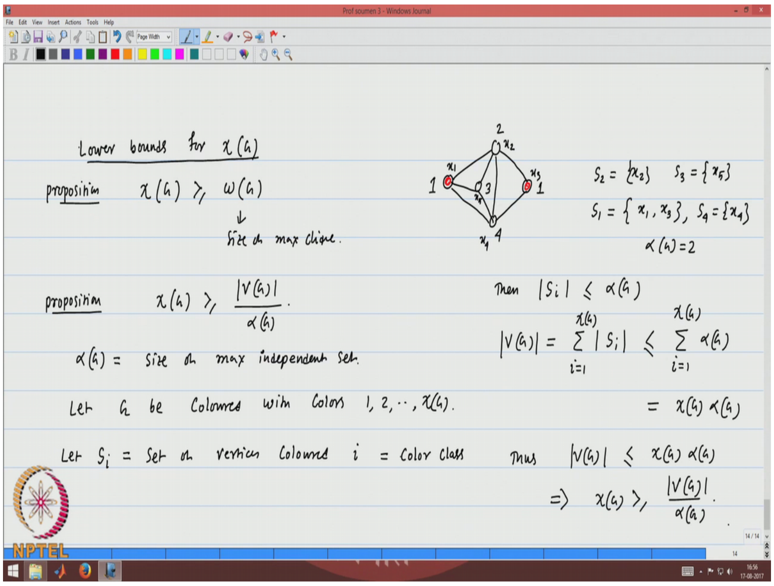

Now, we talk about the lower bounds for chi G; that means, what is the minimum number of colours you need and the first result is that chi G the chromatic number of the graph G is always greater than equal to the clique size the maximum clique size or size of max clique right.

Because you see in the first example here this has the clique of size 4 this is a clique of size 4 this is. In fact, this is k 4 and since every 2 vertices in a clique where adjacent to each other you need 4 colours to colour this part of the graph right. So, that is why for this graph G the chromatic number will be grater than or equal to omega of G, and the chromatic number is greater than equal to 4 because this is equal to 4 for this graph.

But we got a colouring with 4 colours. So, that is why the chromatic number of this graph is 4 right. Now the other result that so, sometimes this bound is good, but it could be very loose in some cases also. There are example now talk about another bound proposition that the chromatic number of the graph G is greater than equal to v G the number of vertices by alpha G. So, alpha G is size of maximum independent set. So, let we will just give outline of this proof of this proposition.

Let G be coloured with colours 1 2 up to chi G. In fact, we do not use colour instead of colours we say that we just assign numbers 1 2 3; that means, this many colours you need to colour the graph G that is why the chi G is the minimum number of colours. And let s i is equal to the set of vertices coloured i because the. So, s i consist of the set of vertices which are not adjacent to each other. Let me repeat the same example again this graph for example, this graph we took at the very beginning. And then we coloured this with red.

Now, instead of colouring with different colours I can colour them with numbers. So, one for this one for this one that mean colour 1. And then I can use colour 2 for this. And then colour 1 2 cannot be used I have to use colour 3 here and for this vertex I have to use the forth colour 4. So, here you can see that s 1 the set of vertices which are coloured one is this 2 vertices say this is x 1 x 2 x 3 x 4 and x 5.

So, s 1 is consist of x 1 and x 3. And you can see that this 2 a form a independent set if you remember the independent set is the set of vertices such that no 2 up them are adjacent. So, x 1 and x 3 are not adjacent that is why you can give them the same colour right. So, this is called the colour class also sometime, sorry then what you can right is that every colour class s i they must be less than or equal to the size of the independent set maximum independent set. And here you can see that the size of the maximum independent set is 2 and the colour class s 2 consist of x 2 colour class s 3 consist of x 5 and the colour class s 4 consist of x 4.

And you can see that the alpha G for this graph is equal to 2. So, every colour class is less than equal to alpha G. And v G I can write v G or the cardinality of all the vertices or the number of vertices is equal to easy observation that sum of the colour classes and i is from one to chi G easy to understand. And then this one is from here I can write this is i to i is from 1 to chi G and I am replacing this cardinality of s i by alpha G by upper bound.

So, this is equal to chi G alpha G thus what we got is that cardinality of v G is smaller than equal to chi G alpha G, which implies chi G is the chromatic number of the graph G is greater than equal to v G by alpha G. So, this is what the statement of this proposition. So, this is another a lower bound. And this lower bound of course, sometime it works good or sometime this is a very use lower bound we talk about one such example in the next lecture.

Thank you very much.