Graph Theory
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Lecture - 17
Max-Flow Min-Cut Theorem

Welcome to lecture 17 on Graph Theory. In this lecture we will learn max flow min cut theorem. This theorem essentially proves that ford Fulkerson algorithm is correct. Please recall that in the previous lecture we learnt what is cut what is the capacity of a cut, what is flow across a cut and we proved a lemma which says that for any given flow in a network and any given cut, the flow value is less than the flow value is equal to the flow across the cut, ok.

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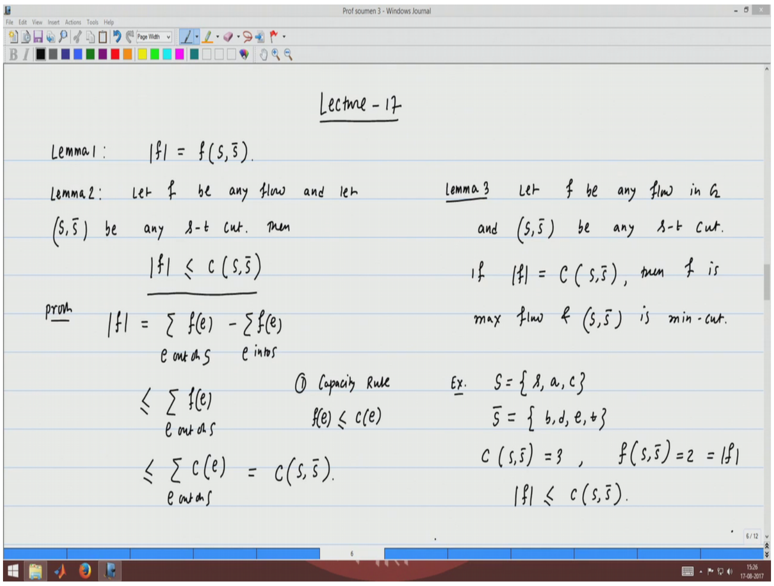

So, we proved the lemma which is we denoted lemma 1 which essentially says that the value of a flow is equal to the flow across the cut s s complement. So, next we prove another lemma, lemma 2 let f be any flow and let s s complement be any s t cut, then the value of the flow is less than or equal to the capacity of the cut s s complement. We will prove this lemma now.

So, what will know is that the value of a flow is equal to f of e, e out of s minus f of e for all e into s right. Now here this is always a positive quantity is the flow is positive because the capacity is positive. So, I can write that the flow value is less than or equal to f of e, e out of s; that means, I am just ignoring this term is positive term right. And also we know that this is the capacity rule that the flow is always less than or equal to the capacity right.

So, from here this is the capacity rule that we learnt this is the first rule for yet to be a flow. Then what he can do is that this term is smaller than or equal to I replace all f by c now capacity. So, this is e out of s and if you recall the definition of capacity of a cut this is the definition. So, that is why this is equal to the capacity of the cut s s complement right. Now we just state another lemma. Let f be any flow in G and s s complement be any s t cut.

Now, what we know is that always see this is this is true, that the flow value is less than or equal to the capacity of the cut. Now if flow value is equal to the capacity of some cut of the cut s s complement, then f is max flow and s s complement is minimum cut right. This is sort of a consequence of this lemma, because for every for any arbitrary flow and arbitrary cut this is true, but when this equality holds here that is the maximum flow and the corresponding cut is the minimum cut.

Also if you remember the previous example I am not drawing it again, where we took s is equal to s a c please refer my example in the previous lecture, and s complement was b d e and t. And the capacity of this cut s s complement was 3 and flow across the cut s s complement was 2 which is also the flow value. So, you can see that this is true that the flow value is less than or equal to the capacity of the cut s s complement, ok.

So, with this with these 3 lemmas, now we are sort of ready to prove the max flow min cut theorem. So now, I will state max flow min cut theorem and prove it.

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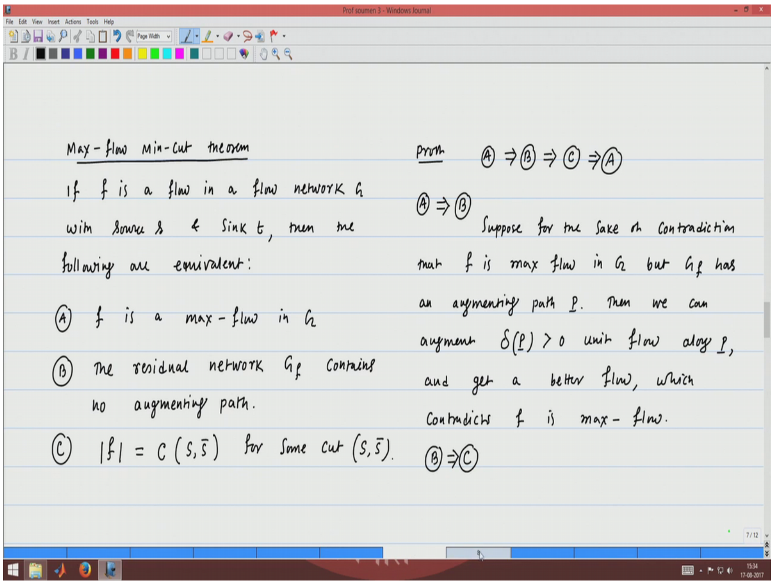

So, this is max flow min cut theorem. If f is a flow in a flow network G with source s and sink t, then the following are equivalent. The first statement is f is a max flow in G.

The second statement the residual network with respect to the flow f that is G f contains no augmenting path, and the condition c is the flow value equal to the capacity of cut s s complement for some cut s s complement. So, the flow is a maximum flow in G if and only if. So, this theorem says that the flow is maximum flow in G if and only if there is no augmenting path in the residual network G f ok.

So, this is sort of the correctness proof of the algorithm ford Fulkerson algorithm. Because there if you remember the stopping criteria was when there is no augmenting path in the residual network then you stop, because we cannot improve the flow anymore. So, A implies B, B implies A this is this sort of proves the correctness of ford Fulkerson algorithm. So, that we will do first we will prove that, that A implies B and B implies A. So, the proof is in this way. So, a will proof that A implies B, B implies C and then C implies A. So that means all this conditions are equivalent.

First we will start with A implies B. Suppose that means, f is a maximum flow in G then will proof that there is no augmenting path in the residual network G f. Suppose for the sake of contradiction that f is max flow in G, but G f has an augmenting path P. If this is true then there is an augmenting path P in the residual net network G f, then we can augment delta P amount of flow this is the capacity of the path P greater than 0, we can augment delta P unit flow along P. And get a better flow, which contradict f is max flow ok.

So, what we have proved is that f is maximum flow. Then we have proved that there can not be augmenting path in the residual network G f. This is A implies B, now will prove B implies C.

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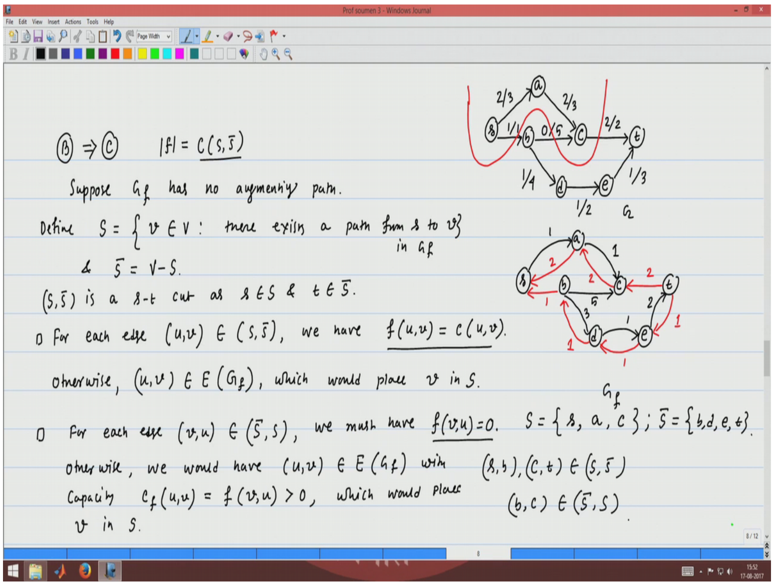

Let us move to the next page B implies C. So, b says that the residual network does not have any augmenting path. So, suppose G f has no augmenting path well. So, what I want to do is that, I want to draw the figure again this will be helpful to explain this proof this is the source node. This says node a b c t d and e. And I know the maximum flow of. So, I directly write the max flow this is the flow I am talking about the flow f along c t flow is 2 capacity is 2.

Along s b flow is 1 capacity is 1 from b to c flow is 0 capacity is 5, b to d flow is 1, and the capacity is 4. D to e flow is 1 capacity is 2. E to t flow is 1 and the capacity is 3, right. Now I know that this flow is maximum flow because this is this is the flow that we obtain from ford Fulkerson algorithm. Now let me also draw the residual network for this one. So, s a b c t d and e these are the vertices of the residual network, the forward edge of capacity 1. So, backward edge of capacity, I draw use red colour for the backward edge. So, the backward edge of capacity 2. Forward edge capacity 1 and backward edge capacity 2, only a backward edge of capacity 2. And here also only of backward edge of capacity 1, and then b to c you have a forward edge of capacity 5 b to d you have a forward edge of capacity 3, and a backward edge of capacity 1. Forward edge capacity 1, backward edge capacity 1, forward edge capacity 2, backward edge capacity 1.

So this is my G f residual network and this is the network and the flow. So, b says that suppose G f has no augmenting path a course this G f has no augmenting path you can not find a path from s to t here. Then you define you define a set s. So, our aim is to find a cut s s compliment, such that for that we will prove that the flow value is equal to the capacity of that a cut s s compliment this is what the c is.

So, we are defining that we are sort of constructing that cut here. So, s is the set of vertices in the network G, such that there exists a path from s to v those edges only. Now let me explain what is the definition of s here. So, s is in this example s is the set of vertices which are accessible from s; that means, there is a path from s to a those vertices only. So, the s consist of the source node itself because you can sort of you know s to s. And then from s you can go to a using this forward edge in the residual network. Sorry I forgot path from s to v in G f this is important in the residual network. So, s consist of s source network and a because a is there is a path from s to a, and also there is a path from s to c.

So, c will be in the set s. Now we can see that there is no path you can not reach to b from s starting from s, you can not reach to b there is no path to b. So, b d e t those vertices will not be in the set s. So, I hope that you understood what is this s is. And obviously, s compliment is equal to b minus s. So, we have defined how to construct this cut. And then what you have to proof is that you have to proof that the capacity of this cut is equal to the flow value in order to proof that b equal to c. Now few things this s s compliment is a s t cut as s is in s and t is in s compliment which is clear. So, here your s compliment is b d e t.

So, the cut we have defined in general this is a s t cut. Or final goal is to proof that the capacity of this cut that we have defined is the flow value. So, will make some observations, first for each edge u v in s s compliment; that means, u is in s and v is in s compliment. So, let me for each edge what is true is that we have flow along this edge u v is equal to the capacity of this edge u v. This is what your claiming, now let see whether this is true for this example. Let me take one edge of this form where u is in s and v is in s compliment. So, one such edge is the cut that we got is that this one right.

So, this is this is your s this is s and this part is s compliment. So, consider an edge say c t, c t, c t is an edge from s to s compliment. And then you can see that for this cut for this edge the flow value is equal to the capacity. And another edge is of this form is s b s b is also from s s path. So, s is in s and b is in s path for s b also the flow value is equal to why this? This should be true in general because if the flow is strictly less than the capacity then there will be a forward edge from u to v or c to t.

Then t will not be in s, sorry then t will be in s itself right. So, this is true otherwise as I said otherwise there will be an forward edge u v in the edge of the residual network. If this is not true if they are not equal if they are not equal there will be a forward edge from s to b, then b will be according to the definition of s b will be in s right. So, otherwise there will be a forward edge which would place v in s.

I hope that this is clear. And we will make another comment about the edge of the form which is from s bar to s. Now for each edge v u which is from s bar to s. So, one such edge is s bar to s is b b c b c is an edge from s bar to s. So, b c is an edge from s bar to s. So, for each such edge we must have the flow v u must be equal to 0. And you can see that the flow b c the flow along b c from b to c is equal to 0. Now let me explain in this example why this this should be always 0, because if this is not 0 suppose this is one.

Then there will be a backward edge in the residual network, from c to b suppose instead of 0 it is say 1, 1 and the capacity is flow capacity is 5. Then this will be 4 the forward edge capacity will be 4 and there will be a backward edge of capacity 1; that means, the b will be in s then right. So, b will not be in s compliment. So, I hope that I tried my best to explain this one using the example. So, this should be true for any edge from s bar to s otherwise, otherwise we would have a backward edge u v in the residual network with capacity c f u v equal to f of v u right which would place v in s.

So, here my v is b and u is c. As you can see from here if instead of 0 if it is one there will be a backward edge with capacity 1, and that will this backward edge will make b or in general v to b in s, because b will be accessible from s then right. So, that is why for each edge which are from s complement to s you this must be true. And for each edge which is from s to s complement this would be true.

Now, let me just remove all they right. So, I hope that you understood this 2 points right.

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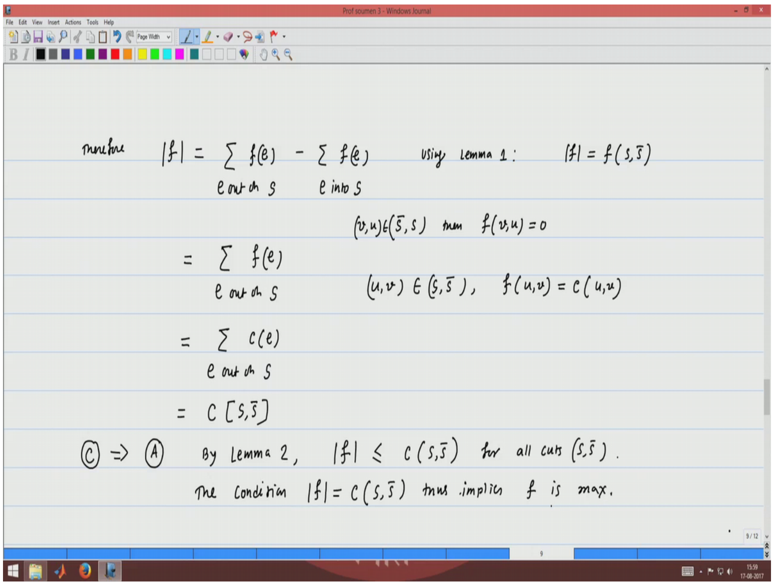

Now, we will prove the therefore, we wanted to prove that this flow value is equal to is equal to the capacity of s s complement. So, let me start with the definition of the flow value with the value of the flow which is as we know that in terms of the cut it is the flow across the cut this is by lemma 1; so f e, e out of s minus f of e, e into s right.

This is by using lemma 1. Because the lemma 1 say that the flow value is equal to the flow across the cut s s complement. And this is the definition of flow across the cut. Now this e into s means these are the all the edges of the form they are the edges of the form s to sorry s bar to s. And we have just prove that if an edge is from s bar to s then the flow value is equal to 0.

So, f e is 0 for all such edges right. So, this is equal to f of e e out of s right. So, just now we prove that if v u is from s bar to s; that means, it is an edge into s then f of v u or f of e is equal to 0. This is just we proved in the previous page. Now the other one is that the e out of s; that means, for all edge u v of the form from s to s complement that is the edge out of s the edge is going from s to s complement. For such edge what we have prove just now is that the flow value sorry the flow of u v is equal to the capacity of u v right.

Then what I can do is that I can replace this flow by the capacity. So, this is equal to c e I am sure that you understood this one e out of s. And this is the definition of the capacity of the cut s s complement if you look at the definition of the capacity of the cut. So, this is c s s complement. So, in this part what in order to prove that B implies C b is the condition that the residual network does not have any augmenting path, and then we described here a technique to find a cut for which the flow value will be equal to the capacity of that cut ok.

So, if there is no augmenting path in a residual network, then you can always find a cut for which the capacity of the cut is equal to the flow value that is what the B implies C, we proved just now the last part that is left is that we have proved that C implies A. So, the last one is C implies A or b is also fine C implies say b oh C implies A that I that is what we have to proved C implies A. Now by lemma 2 what we know is that the flow the flow value is always less than or equal to c the capacity of an arbitrary cut for all cuts s s complement.

Thus the condition c says that there exist a cut for which f is equal to this the flow value is equal to the capacity of the cut. The condition that there exist a cut for which the flow value is equal to c capacity of that cut thus implies f is maximum. Because f is always less than or equal to the capacity of the cut, and when the equality hold that is the maximum flow. So, we have proved the maximum flow min cut theorem. So, given a network we know how to find maximum flow in the network using ford Fulkerson algorithm. And also given network we can find the minimum cut using the concept of residual network and augmenting path technique.

Now I give just one more example if you remember that we started with one network and then I just wrote arbitrary flow in that network. Let me see whether the flow value that arbitrarily we have assigned that is the maximum flow or not refer my first lecture on network flow.

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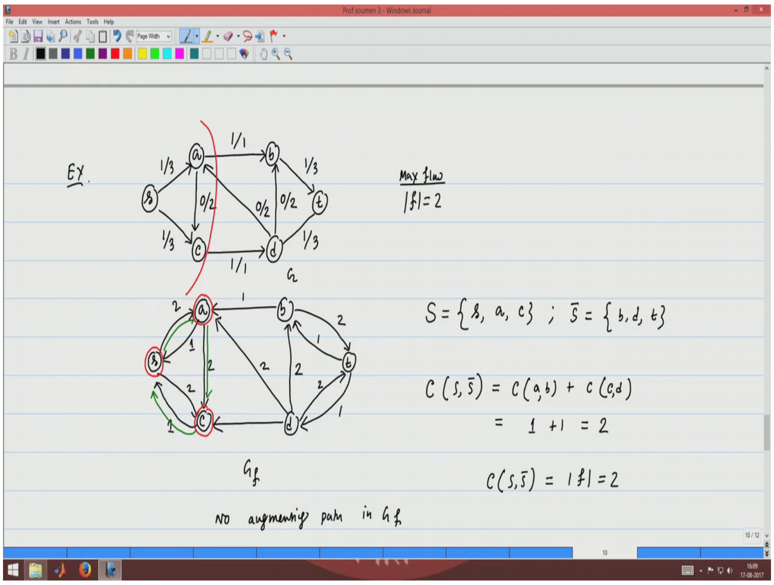

So, the example is this one. We considered this network and at that point of time I did not say I say I said that this is just a flow and the flow value is 2, we could not comment on whether that flow is the maximum flow or not.

Let see this is the example taken from the first lecture on network flow. The flow I assigned was 1 3 1 1 flow 1 capacity 3 and here it is one flow value capacity 3 one flow value capacity 1 1 3. From a to c flow is 0 capacity is 2. From d to a the flow is 0 and the capacity is 2. From d to b flow is 0 and the capacity is 2.

Now, this is the example I took at the very beginning. Now we know that the value of the flow is equal to 2, but it is not clear whether this flow is the maximum flow or not; so to check that what we can do is that we can just. So, this is an arbitrary flow say for example, and we can compute the residual network with respect to this flow right. So, let us draw the residual network of this network with respect to the given flow. Well so, there will be a forward edge capacity 2 and let me draw the backward edge also is in the same color it does not matter. So, the backward edge has capacity 1.

A to b there is no forward edge there is a backward edge of capacity 1. B to t there is a forward edge of capacity 2 and a backward edge of capacity 1. S to c there is a forward edge of capacity 2 and a backward edge of capacity 1. C to d there is no forward edge there is a backward edge. D to t there is a forward edge 2 and there is a backward edge of capacity 1.

A to c there is a forward edge capacity 2 d to a there is a forward edge of capacity 2, and d to b there is a forward edge of capacity 2. So, this is the residual network of the network G with respect to the flow. Now can you find augmenting path here can you find the path from s to t. So, start from s you can go to a, from a you can go to c, and from c you can go back to s only right.

So, start from s you can go to a using this edge forward edge. And then from a you can at most go to c using this forward edge. And from c you can go to s again using this backward edge. This is true even if you start from s to c first and then again go back to s. So, there is no path. So, no augmenting path, sorry no augmenting path in G f. So, then by ford Fulkerson algorithm we can say that the flow value the flow that we have started with which is having the value 2 is the maximum flow. You can not improve the flow value because there is no augmenting path in the residual network.

So, the max flow here this is the max flow. So, this is the max flow and now here can you finds the minimum cut yes. So, by the proof of max flow min cut theorem, the way we find the minimum cut is that you define a set s s is the set of all vertices which are accessible from which are accessible from the source node. So, the source node is s and the vertices which are accessible from s are a and c. So, the a s consist of this 3 vertices s a and c and definitely then; obviously, that s complement is b d and t right. And so, the cut is here. So, this is the cut. So, this is the cut right.

This is s this is s complement. Now what is the capacity of this cut? The capacity of this cut capacity of this cut s s complement is the edges which are going out of s the capacity of a b plus the capacity of c d right. These are the only 2 edges which are going out of s c d. And the capacity of a c is equal to 1 and the capacity of c d is also equal to 1 the capacity is equal to 2, and you can see that since this is the minimum cut the capacity of this minimum cut s s complement is equal to the flow value which is equal to 2.

So, from this lecture on network flow what we know we know that given a network how to find the maximum flow in the network, and also given a network we also can find the minimum cut in the network right. And we know that the maximum flow value is equal to the minimum cut value. This is what the maximum flow min cut theorem is.

Thank you very much.