Graph Theory
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Lecture – 16
Part 2
Max-Flow and Min-Cut

Welcome to the second part of lecture 16 on Graph Theory. In this part or in this lecture we learned what is a cut in network and also we will prove the correctness of Ford Fulkerson algorithm.

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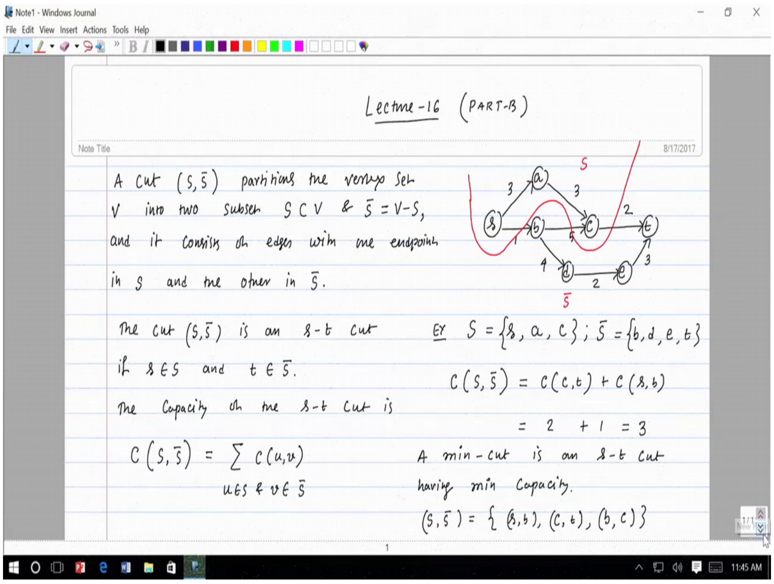

So, let me start with the definition of cut; a cut denoted by S, S complement you have seen this notation before also, partitions the vertex set V into 2 subsets - one is S which is subset of V and S complement which is V minus S and it consists of edges with one end point in S and the other in S complement.

So, let me explain this definition using an example. So, I will consider the same network with a source node s and the other vertices are a, b, c, sink node t, d and e. So, this has capacity 3, this has capacity 3, c t has capacity 2, s b has capacity 1, b c has capacity 5, b d 4, d e 2, e and t, e to t 3. So, as I said cut is partition and it consists of all the edges with 1 endpoint in S and the other end point in S complement. So, if I consider my S to be S is say consist of this sets s, a and c then; obviously, S complement is b d e and t. So, we have partitioned this network into 2 parts one is this part this is the part S and this is the other part S complement and this cu t consist of this edges s b, c t and b c .

The cut S, S complement is an s-t cut if S belongs to S and the source belongs to S and the sink belongs to S complement then it is called an s-t cut and one more definition the capacity of a cut the capacity of the cut of the s-t cut is denoted by C S, S complement. So, you add all the capacities basically. So, C u v where u belongs to S and v belongs to S complement you must be very careful about the definitions here.

So, I will explained I will explain the capacity of this cut this is my example the capacity of this cut C is S complement is equal to, the capacity of the edge c t because c t here c belongs to S and t belongs to S complement. So, that is why we consider the edge c t plus the capacity of the edge s b, so S belongs to S and b belongs to S complement. And this one is capacity of ct is 2 plus the capacity of s b is equal to 1. So, the total capacity is equal to 3. And a min cut that is what the optimization problem that we are going to consider a min cut or minimum s-t cut is an, s-t cut having minimum capacity.

So, there could be several s-t cuts and you need to find the s-t cut which has the minimum capacity. Before I move to another definition you see the cut S, S complement this cut it consists of some edges right. So, it consists of not 2 edges only it consists of 3 edges and the edges are s b this one edge from S to S complement, another is c t, c t the cut, cut consists of edges basically, so c is in S and t is in S complement and there is another edge b c the cut consists of this edge also this is b c, but b c, b is in S complement and c is in s. So, these are the 3 edges in the cut S, S complement.

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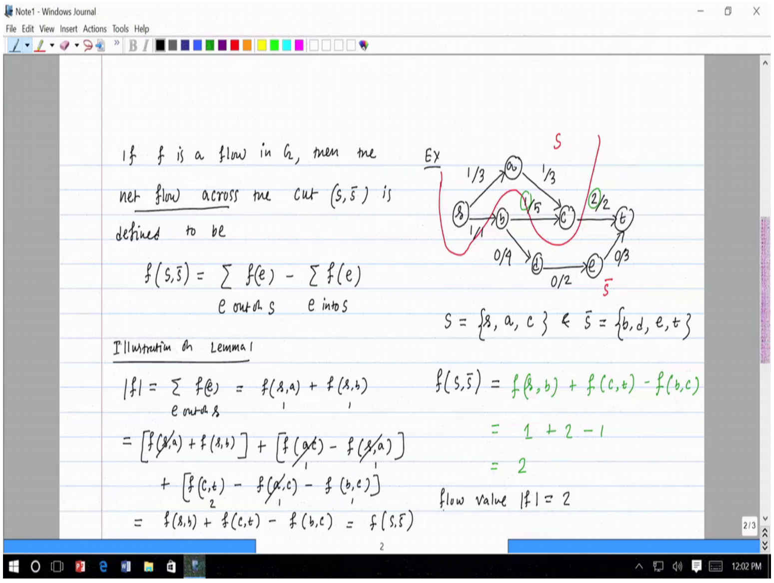

Next we move to the definition of flow if flow across a cut, if f is a flow in the network G then the net flow across the cut S, S complement is defined to be this is another very important definition, defined to be f the net flow across the cut S, S complement is equal to the flow this is the flow value across the edge e and this e out of s minus the flow value where e into s.

Let me again illustrate this definition using an example will consider the same example with source s a b c t d and e. So, this edge has capacity 3, a c has capacity 3, c t has capacity 2, you are sighting this network several times b c has capacity 5, b d 4, d e 2 and this e to t it is 3. Now, consider a flow say this flow for example, consider this flow which we got after the first augmentation in the previous lecture if you remember. So, this is a flow this is a valid flow you can check this is a valid flow the amount of flow coming into c is 2 going out of c 2. So, all the conditions are satisfied for every node.

In this example again I consider the same partition S is equal to s a c and S complement is; obviously, b d e t. Now we are trying to understand the flow across this cut. So, the flow across this cut f S, S complement is the flow that is going out of S let me just partition it this is my S and this part is S complement and let me use 2 different colors. So, the edges which are going out of S going out of S are s b, so the flow across s b plus the flow across ct. So, this 2 edges are going out of S minus the flow for the edge going into S the edge which is going into S is b c. So, minus f b c this is the net flow across this cut and, so here the flow s b is equal to 1 plus the flow c t is 2 minus the flow b c is 1. So, net flow across this cut is 2. So, hope you understood this definition.

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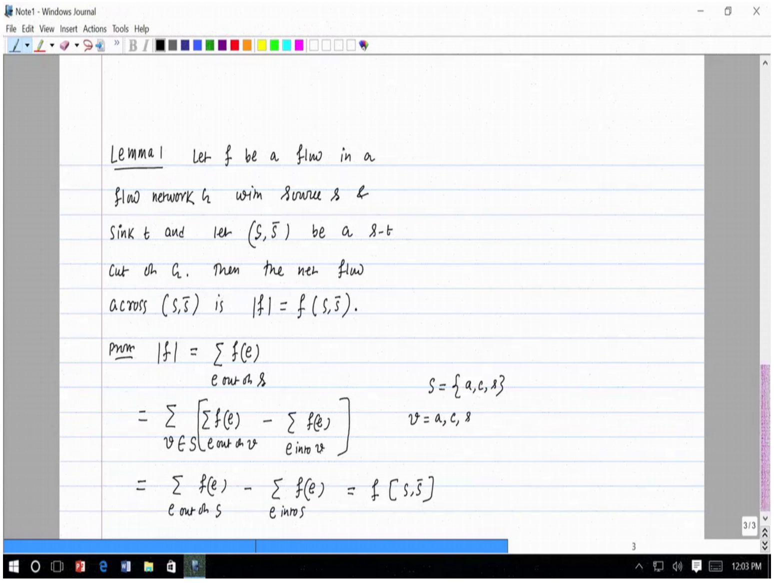

So, now using this 2 definitions we prove a lemma, lemma 1 let f be a flow in a flow network G with source S and sink t and let S, S complement be a s-t cut of G. Then the net flow across S S complement is the flow value I spent I go slow while proving this lemma. Finally, the max cut min min cut max flow theorem. So, this is what the flow across the cut S S complement.

First let me explain what is the meaning of this one I go to the previous page. So, here what I wanted to say is that in the lemma that the flow across the cut S S complement is 2 here and you can see that the flow value here what is the flow value, flow going out of S that is the flow value. So, the flow value flow value that is modulus f is 2 here. So, whatever cut you consider it does not matter for any arbitrary cut the flow across the cut is the flow value that is what the lemma says. So, in the previous example this one is 2 there is total flow value and this is the flow across the cut S S complement and this S S complement can be any arbitrary cut.

So, let us prove this one and I try to illustrate this proof using the previous example again. So, the flow value is the flow that is going out of the source node. So, f of e the flow value is f of e, e out of sink sorry source node it is not the capital S e out of S this is always true because if I go back to my previous example I am now explaining illustrating the lemma of lemma 1. So, the first line that I wrote is that f is equal to f of e e out of source node; that means, f of s a plus f of s b. So, you agree with this one right and so this is in this example this is 1 and this is 1, so this is 2.

Now, what I can write is that I can write this one is equal to summation of for all v belongs to the part s, f e e out of v minus f e e into v. See this quantity if it is for any vertex other than the source node this quantity is 0 because for any other node any other node the flow into a minus the flow out of a is equal to 0 right. So, this one is same as say f of s a, this is the flow plus f of s b flow out of S, there is no flow into S. This is for the nodes S plus f of a c the flow out of a that is f of a c minus f of s a, it is flow out of a flow into a. So, this all this transfers 0 basically this is 1 this is also 1 and I did it for a because a is a vertex in s. And now I do it for c also plus f of c t that is flow out of c minus flow into c f of a c minus f of b c, this is flow into S. Now you can check that this quantity is also 0 because this is 2 this is 1 and this is one because this is from the (Refer Time: 26:29) rule.

So, what you get finally is that, so this will cancel out with this one a c will cancel out this a c right. So, what you have is that you have f of s b plus f of c t minus f of b c; that means, the flow out of S this is these are the 2 edges which are going out of S that is s b is 1 edge which is going out of S capital S and c t is another edge which is going out of capital S and b c is an edge which is going into S. So, this one is nothing, but flow of flow of S S complement. This is exactly what we are doing in the proof of this lemma.

See this is for all different, so here in the previous example v can take value a c and s because my capital S is a c and S, and I have computed explained this term that this is equal to this one basically and this can be viewed as f of e e out of S minus f of e e into S. I explained using the example that why this is finally, this term is all the flow going out of S and all the flow going into S and we know that this is equal to f of S S complement.

So, what we have proved in the first lemma is that given any arbitrary cut S S complement, the flow across this cut S S complement is equal to the flow value. So, this is what the first lemma and we will prove in the second part of this lecture or in the next lecture some more lemmas and finally, the theorem famous theorem max flow min cut theorem, that is all for today.

Thank you very much.