Graph Theory
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Lecture - 02
Part 1
Eulerian and Hamiltonian Graph

Welcome to the second lecture on Graph Theory. In the first lecture we have learnt several definitions, like graphs sub graphs connected graphs, and components of a graph. And also we have learnt; what is a walk what is trail path you know all this things. So, please recall that walk is a sequence of vertices and edges. Starting with a vertex and ending with the vertex, and a walk is said to be a trail if all the edges are distinct not necessarily the vertices are distinct. And a path is a walk in which all the vertices are distinct.

And hence automatically the edges are also distinct. So, we will start with trailed. And in the in this today's class we will be talking about Eulerian trail and Hamiltonian cycle. So, we learn what is a Eulerian graph, and what is Hamiltonian graph today.

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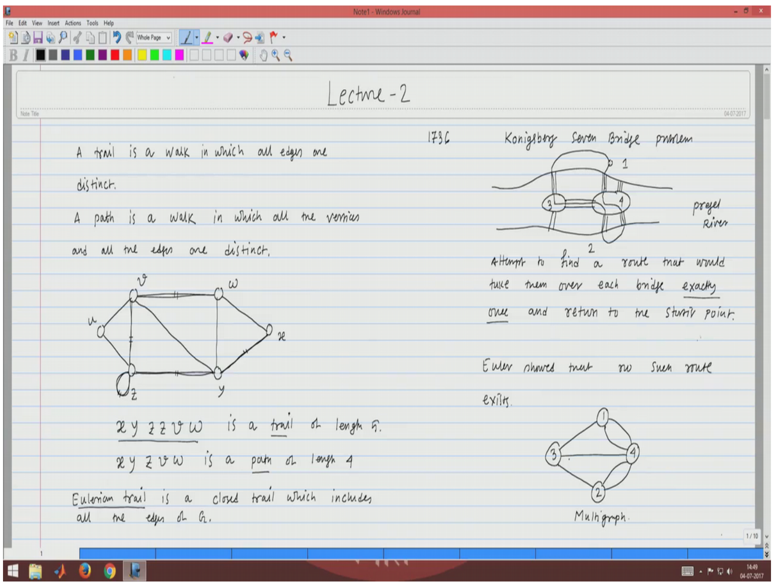

Please recall that a trail, a trail is a walk in which all edges are distinct, but not necessarily the vertices are distinct. And a path is a walk in which all the vertices and all the edges are distinct. So, let me give an example of a graph again.

So, this is a graph with 5 vertices. And here u v w x y z. U and v are adjacent, v and w are adjacent, w and x are adjacent, y x are adjacent z y are adjacent. There is a edge or loop z to z and w z v z w y are adjacent and b to y there is an edge. Now here you can see that x y z, z v w; that means, x y, y to z and then z to z, z to v and v to w, this is this is a trail. Because you can see that all the edges are distinct. So, this is the first edge second edge third edge fourth edge and the fifth edge. So, this is trailed of length, length 5, but this is not a path.

So whereas, x y z v w is a path is a path of length 4. So, this is a path because you can see that all the vertices are distinct here. And hence all the edges are also distinct, but this is not a path this is not a path, because here, here all the vertices are not distinct only the edges are distinct that is why this is a trail and this is a path. And next we move to the definition of a Eulerian trail of Eulerian trailed trail is a is a closed trail which includes all the edges of G. So, this is for the Eulerian trail is, and next to we talk about more about this Eulerian trail. Let me talk about this famous problem which dated 1736. This is called Konigsberg 7 bridge problem.

So, I will draw a river this is this is that pregel river. And there are 2 islands. One is here this is the second island here, and there are 2 bridges which connects this island with this side of the river. And similarly there are 2 bridges here also in this side there is a bridge which connects these 2 islands. And there is one bridge here connecting this small island with this side of the river and there is another bridge here which connects Small Island with this side of the river. And so, what people attempt to find a route that would take them over each bridge exactly once and return to the starting point, ok.

So, what they tried is to you start at some point, and then you find a route which will take you over each bridge each over each bridge exactly once. You can not visit a bridge twice and you have to return to the starting point. So, of but they could not find any route which can take them over every bridge exactly once and returning to the starting point. And finally, Eulerian in 1736 he prove that there is no such route, and the graph theory started with this problem in 1736 ok.

So, let me just give say try out one route. Maybe if I start from here, then what I can do is that I can travel this way I and then come to this breeze, go this way go this way, but I have to see I have to return to the return to the starting point. So, I can go this way and return here, but this route of does not visit all the bridges. So, this 2 bridges are left. So, this is what people try to find a route that would take over take them over each bridge exactly once. This is important you can not visit a bridge twice and return to the starting point. And finally, Euler showed that no such route exist.

So now, if I can redraw this figure in terms of graph, let me level this I will represent this as a graph problem. So, this is say 1, this is 2 this side of the river I label it 2, and this is 3 this island 3 this island 4. And I will represent this side of the river by a node label one this side of the river by the node 2 and 3 represent one island and 4 represent the other island. Now I will present this 2 a bridges by 2 edges here this 2 edges are corresponds to this 2 bridges, and 3 and 4 there will be of there will be an edge because there is a bridge between 3 and 4.

Similarly there will be an edge between 1 and 3 and edge between 3 and 2. And 2 edges between 2 and 4. So, this is this is a multi graph basically, because there are parallel edges here multi graph. So now, this problem is represented by a graph by a multi graph. Now finding a route that would take them over each bridge exactly once, and return to the starting point is equivalent to finding an Euler Eulerian trail in the graph G. So, you look at the definition of Eulerian trail, Eulerian trail is a closed trail which includes all the edges in the graph G graph or multi graphs.

Now the question is how do we characterize the graphs for which the region Eulerian trail or a graph which is having an Eulerian trail is called Eulerian. So, the problem is to characterize the graphs the Eulerian graphs.

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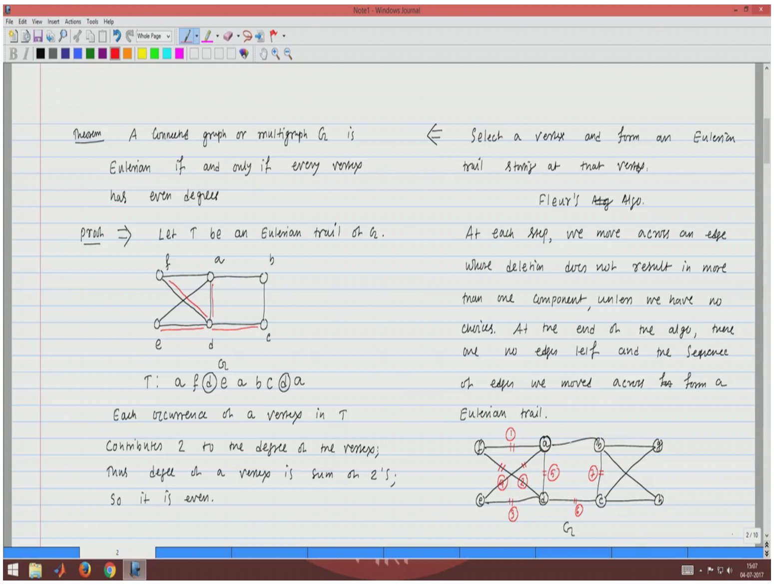

Well, this theorem will characterize Eulerian graph of a connected graph or multi graph. Multi graph means you know parallel edges and loop there allowed, multi graph G is Eulerian, is Eulerian if and only if, if and only if every vertex has even degree even degree. So, this is an if and only if only a only if condition. So, let me prove this theorem. So, I will be proving first this part; that means, the graph G is Eulerian. So, let T be an Eulerian trail of G.

So, we are proving this part that we are assuming. So, the first part is that the if part; that means, the graph is given to be Eulerian, and then we have to prove that the every vertex has even degree. So, the graph is Eulerian; that means, the graph has a Eulerian trail let T be the Eulerian trail of the graph the Eulerian graph given G. Then just I illustrate this part by an example let me just take an example of an Eulerian trail. So, I will concentrate this graph it has 6 vertices.

And I label them as a b c d e and f. A b adjacent, b c adjacent, d e d c adjacent, d a adjacent, e d e a f a and f d their adjacent. Now this is a graph simple graph it is not a multi graph, and this given that G is Eulerian. So that means, that this graph has a Eulerian trail let me considered this Eulerian trail. So, T is a Eulerian trail. So, I have to start at some vertex let me start at vertex a. So, a and then f d e a b c d a. So, it is like a f d b then a b c d a. So, this is you can check that this is this is a Eulerian trail, because it is a closed trail it start from a and ends at a, and all the edges are distinct in between adjacent distinct ok.

Now, what I want to point out here is that, now the claim here that let me write down the claim first that each occurrence of a vertex in T in the Eulerian trail contributes 2 to the degree of the vertex. So, it says that each occurrence of a vertex let me consider the vertex say d. So, each occurrence of a vertex in the Eulerian trail T contributes 2 to the degree of the vertex. So, here d occurred and this corresponds to 2 degree of d because so, while you are in the Eulerian trail. So, you visit d the vertex d using the path, using the edge f d this is you enter to the vertex d and you leave the vertex d using the edge e. So, that corresponds to 2 degree of the vertex d, and again d occurs here and here. You enter to the vertex d using the edge c d using the edge c d and you leave d using the edge d a.

So, that is corresponds to again 2 degree contribution to the total degree counting of the vertex d. So, that is why each occurrence of a vertex of in T contributes 2 to the degree of the vertex. Thus degree of a vertex is sum of twos only so, it is even. So, this is the first part we have proved that if the graph is Eulerian. Then every vertex in the graph has even degree. Now we will prove the other part that will assume that you are given a graph G where every vertex has even degree and you have to prove that the graph is Eulerian; that means, you have to find an Eulerian tail in the graph. So, let me do Prove that part now.

So, this is the only if part we say. So, this part will do in a in a way that you know you are given a graph where every vertex has even degree. And then we will talk about an algorithm to find an Eulerian trail in the given graph, where each vertex has even degree. So, that is a construction proof constructive proof you just give an algorithm to find Eulerian trail in the given graph. So, here is the technique to find Eulerian trail. So, you select a vertex select a vertex.

And form an Eulerian trail, form an Eulerian trail starting at that vertex. So, this is the algorithm is called fleur algorithm. So, this is how we find an Eulerian trail in the given graph, what we do is that at each step we move across an edge whose dilation does not result in more than one component, unless we have no choice. And at the l at the end of the algorithm there are no edges left and the sequence of edges we moved across form a Eulerian trail.

I will explain this technique using an example. So, it says that at each step we move across an edge, whose deletion does not result in more than one component unless we have no choice no choices. At the end of the algorithm there are no edges left and the sequence of edges we moved across form a Eulerian trail. Let me illustrate this algorithm using an example I will take it slightly bigger example now. There are 8 vertices in the graph, this is again a b c d this is e this is f g h: a b, b c, d c, a d their adjacent.

Similarly, here f a e d e a f d they are adjacent b g c h c g and b h they are adjacent. So, you are given this graph with 8 vertices and you can see that every vertex has even degree. So, a b c d their having a degree 4 e f g h they are having degree 2. So, so it says that you start select a vertex. So, let me select a vertex a for example, I select vertex a then we move across an edge whose deletion does not result in more than one component. So, I can move across f I can use this edge to move and I delete this edge from the graph. So, this is my first visit if I delete this edge the number of components does not increase the number the graph remains still connected. From there from f I can go to d only.

So, I move across this. So, this is my second move. And I delete this edge then of course, the number of components increases because now f is a isolated vertex, but I do not have any other choice. So, unless we have no choices we can do that. So, the second move is to d and from d we can go to e, and I remove this edge this is the third visit this is the third edge that the Eulerian trail visits. And then e to a and I remove this edge also this is the fourth. I am just writing the sequence in which we are visiting different edges of the graph.

And at this moment we are at a and then I move across this edge, and then I move across this edge, then I move across this edge. And when I reach to b So, this is where you need to be little more careful like now from b you have many choices you can go to a you can go to G you can go to h, but the algorithm says that you do not go to a because if you go to a you increase the number of components. So, this will be removed and, but you have other choices you have the option of going to b to g b to h.

So, you go there if you go to G and that does not increase the number of components. So, here instead of going from b to a you go to b to G. So, this is the 8th move. And from here you can next go to c this is the ninth move and you are here now in c and then you go to this edge. And then you go to this edge it is eleventh, and now you go to a back, because finally you have to go to a again because you are creating Eulerian trail. So, this is how you know you can and the sequence and finally, you can you see that at the end of the algorithm there are no edges left. And the sequence at the sequence of the edges we move across a former Eulerian trail.

So, this is you know you just write down this sequence like a then f d e a d c b g c h b a. So, this is the Eulerian trail that is given even by the fleur algorithm. So, we have learned how to characterize Eulerian trail, or Eulerian graph a graph is Eulerian, if and only if all the vertices are of even degree. And we have proved of that theorem.

Thank you very much for your attention.