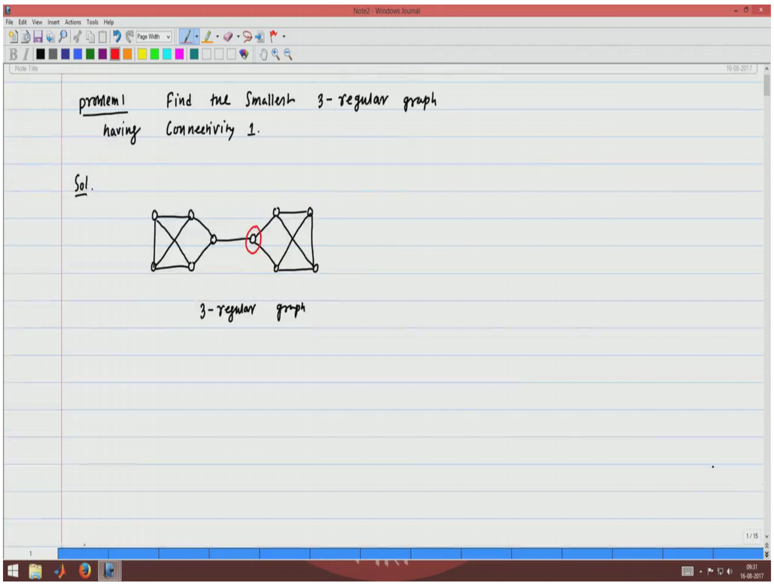
Graph Theory
Prof. Soumen Maity
Department of Mathematics
Indian Institute of Science Education and Research, Pune

Lecture – 14
Part 2
Problems Related to Graphs Connectivity

Welcome to part b of lecture 14 on Graph Theory. In this lecture will solve some Problems related to Graph Connectivity more specifically 2 connected graphs.

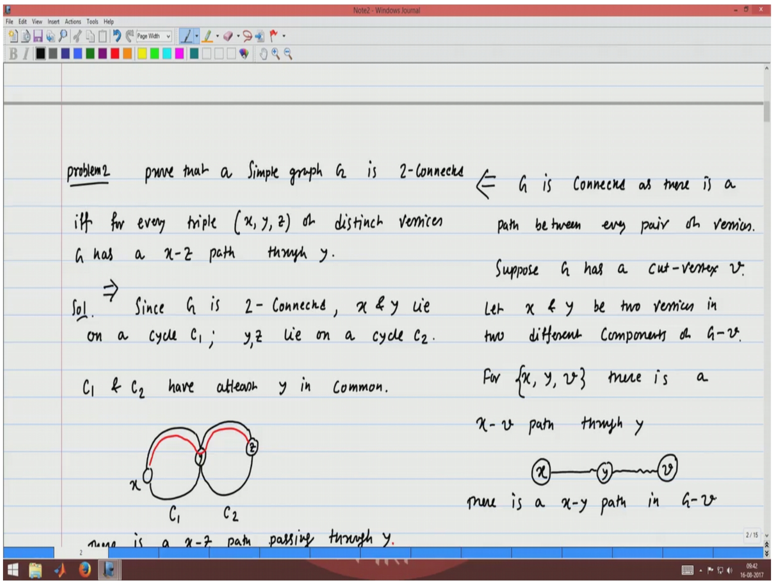
(Refer Slide Time: 00:42)



Let me start with the first problem one find the smallest 3 regular graph having connectivity 1. So, you try to construct the graph which is 3 regular; 3 regular means every vertex has degree 3 and connectivity 1; that means, you need to remove at least 1 vertex from the graph to make the graph disconnected, you try to construct such a graph and then look at the solution. So, here is the graph, it should be 3 regular now you can see that all these vertices are having degree 3 and this side also.

We just replicate the same sub graph here now, you can see that this graph is 3 regular graph because every vertex has degree 3 and you need to remove 1 vertex. This is this is a cut vertex basically, if you remove this vertex or say this vertex then the graph will become disconnected. So, the connectivity of this graph is 1 so this is the first problem.

(Refer Slide Time: 03:37)



Let us consider the second problem 2 prove that a simple graph G is 2 connected if and only if for every triple x, y, z of distinct vertices G has x z path through y. We know that a graph is a 2 connected if and only if between every pair of vertices there are internally disjoined paths, but this is a different characterisation it says that for every triple x y z and x y z are 3 distinct vertices then G has a x z path passing through y.

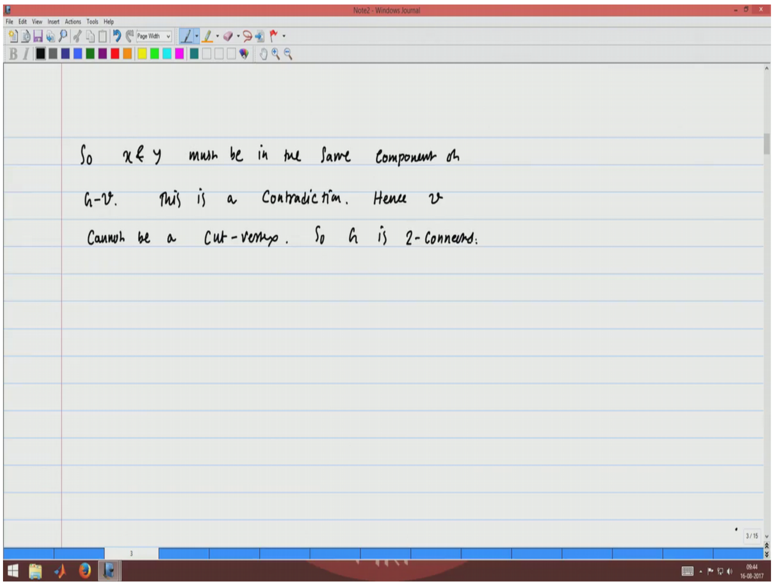
Since G is 2 connected, we know that any 2 vertices x and y lie on a cycle say c 1 and similarly y z lie on a cycle c 2 and this 2 cycles c 1 and c 2 have at least y in common, there could be more than 1 vertices common between these 2 cycles, but at least 1 vertex y is common. Let me draw c 1 c 1 is passing through x and z, this is sorry x and y and c 2 contains y and z. So, this is c 1 this is c 2, then we can say that there is x z path passing through y and you can see that path right.

So, this is 1 possible x z path, x to y and then y to z, this is 1 possible x z path that is passing through y. Now, we have proved that if 2 is graph G is 2 connected then for every triple x y z of distinct vertices; G has a x z path passing through y that is what we prove just now. So, this is the if part the only if part is that G is so you are given that now that for every 3 triples x y z of distinct vertices there is a x z path x z path passing through y; that means, for every pair of vertices x z there is a path which is passing through y which is not required at this moment for my purpose there is a path between x and z.

So, G is connected I can say because between every pair of vertices x z there is a path between x and z. So, G is connected as there is a path between every pair of vertices now supposes that suppose G has a cut vertex G has a cut vertex v; that means, if you remove this G from the graph G then the graph will become disconnected.

So, let x and y be 2 vertices in 2 different components of G minus v so this is the proof this part were proving using contradiction. So, for x, y and v this is a 3 distinct vertices in the graph G, there is x G path through y because that is what it says that for every pair of 3 distinct vertices there is a x z path passing through y. So, is a path from x to v passing through y? So, this implies that if you even if you remove this v the vertex v from the graph there is a x y path in G minus v.

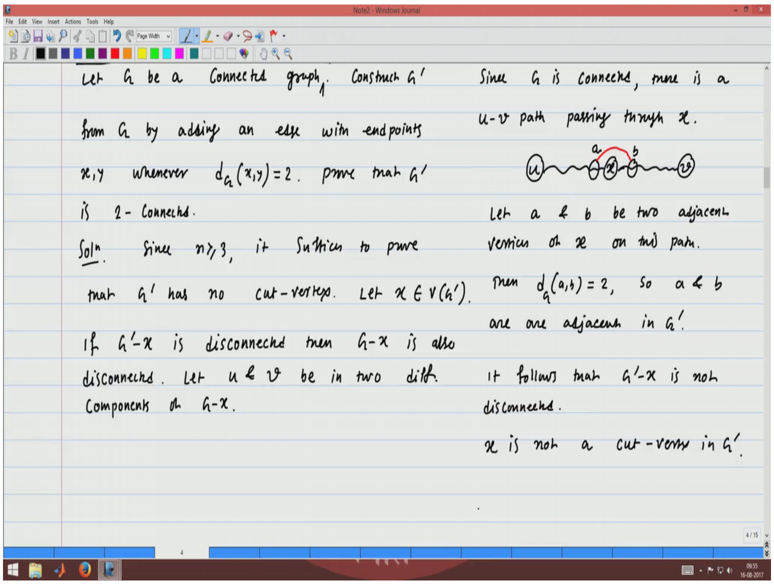
(Refer Slide Time: 14:21)



So, x and y must be in the same component of G minus v, this is a contradiction. So, this is the contradiction to the fact that we assume that v is a cut vertex of the graph G; hence v cannot be cut-vertex so G is 2 connected. So, in the only if part of this proof what you are given you are given that for any 3 distinct vertices x y z there is a x z path passing through y and then you have to prove that if this is true for a graph G then the graph is 2 connected.

So, this is proof by contradiction we assume that there is a cut vertex v, if there is a cut vertex then the graph is one connected and finally, we prove that there cannot be a cut vertex in the graph G. So, the graph is not 1 connected it is 2 connected.

(Refer Slide Time: 16:44)



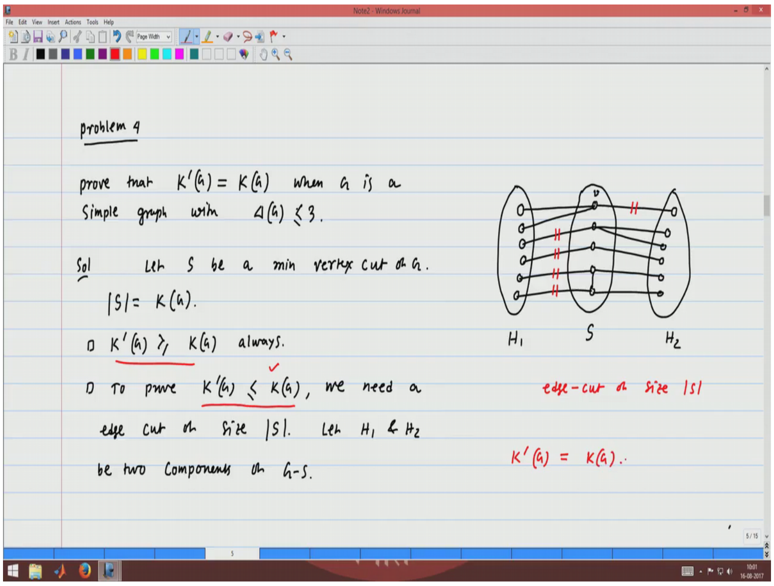
Next we move to the other problem 3 let G be a connected graph then you construct G prime from G by adding an edge with end points x, y whenever the distance between x and y in the graph G is equal to 2. So, this is how you construct a G prime from G then you prove that prove that G prime is 2 connected. So, 2 connected you have to prove that G prime is 2 connected; that means, they are does not does not exist the cut vertex in the graph G prime that does not exist a set of 1 vertex whose removal disconnect the graph.

Let us look at the solution of this problem since this is a connected graph with number of vertices at least 3 since n is greater than equal to 3 it suffices to prove that G prime has no cut vertex. Let x be a vertex of G prime, if G prime minus x is disconnected then G minus x is also disconnected this is true because G prime has more edges than G, G is a basically sub graph of G prime. Let u and v be in 2 different components of G minus x. Now since G is a connected graph G is connected that is what we have given at the beginning, G be a connected graph since G is a connected graph there is a u v path passing through x this is true because of the fact that we can say that there is a u v path passing through x because if you remove x from the graph G it become disconnected that is why.

So, there is a u v path u to v which is passing through x, now let a and b be 2 adjacent vertices of x on this path so; that means, a is here a is adjacent to x this is a and this is b b is another vertex which is adjacent to x on this path. Then you see that the distance between a and b in the graph G is equal to 2. So, a b a and b will be adjacent will be adjacent in G prime. So, in G prime this 2 will be adjacent, it follows that it follows that G prime minus x is not disconnected so; that means, x is x is not a cut vertex in G prime right and x is arbitrarily selected. So, there is no cut vertex in G prime hence G prime is 2 connected.

So, we will talk about another problem.

(Refer Slide Time: 26:59)



Now, problem 4 we have seen the similar problem before also prove that the edge connectivity K prime G that is the number minimum number of edges that need to be removed from the graph to make the graph disconnected is equal to K G K G is the minimum number of vertices that need to be removed from the graph to make the graph disconnected this is the vertex connectivity and edge connectivity.

This is equal to the edge connectivity is equal to the vertex connectivity when G is simple graph with maximum degree less than equal to 3. So, we have proved similar result for when G is simple 3 regular graph, then K prime G is equal to K G and then proving this is true for 3 regular then this is also true for graph simple graph with maximum degree 3, the proof or the solution is the same solution I just outline the solution, let S be minimum vertex called of G; that means, the cardinality of S is K G and what we know that K prime G is always greater than equal to K G this is always true that we have proved at the very beginning.

Now, to prove that there are equal to prove K prime G less than equal to K G we need an we need a edge cut of size this much, and let similarly that H 1 and H 2 be 2 components of G minus S. Now if you remembered what previous proof that so this is what the S is and if I remove this S. So, this consist of some vertices if you remove this S from the graph the graph will be left with 2 components H 1 and H 2, now we know that since S is the minimum S cut every vertex v in S has a neighbour in H 1 and a neighbour in at least 1 neighbour in H1 and at least 1 neighbour in H 2.

Now this graph has degree maximum 3, it can be like this the other situation could be the similar thing we proved for 3 regular graph, this is another situation it could be like this also 1 vertex here; 1 vertex here because it is not a 3 regular graph it has degree maximum 3. So, this vertex has say degree 2 and the other situation could be like this, now the idea is that, suppose the vertex cut has 1 2 3 4 5 vertices now I have to construct a edge cut of size 5.

So, what we do is that we remove this edge we remove this edge and in this case either you can remove this edge or this edge no problem and in this case you have removed these 2 edges. Now you can see that if you remove this edge this 5 edges from the graph there is no path between a vertex in edge 1 to a vertex in edge 2. So, the graph become disconnected so what we got? We got a edge cut of size cardinality of S.

So, this proves that K prime G you can construct starting with vertex cut of size K G you can construct a edge cut of size the same size. So, this is true and then combining this and this combining this and this we finally, prove that K prime G is equal to K G. So, what we have proved in the last problem is that if G is the simple graph with maximum degree 3 then the edge connectivity is equal to the vertex connectivity that is all from graph connectivity.

Thank you very much.