Graph Theory
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Lecture – 14
Part 1
Subdivision of an edge;
2 –edge-connected graphs

Welcome to the first part of lecture 14 on graph theory. In this lecture, we talk about subdivision of an edge and also we talk about we will also solve some problems. So, here is the content of this lecture subdivision of an edge and at the end we talk about some problems.

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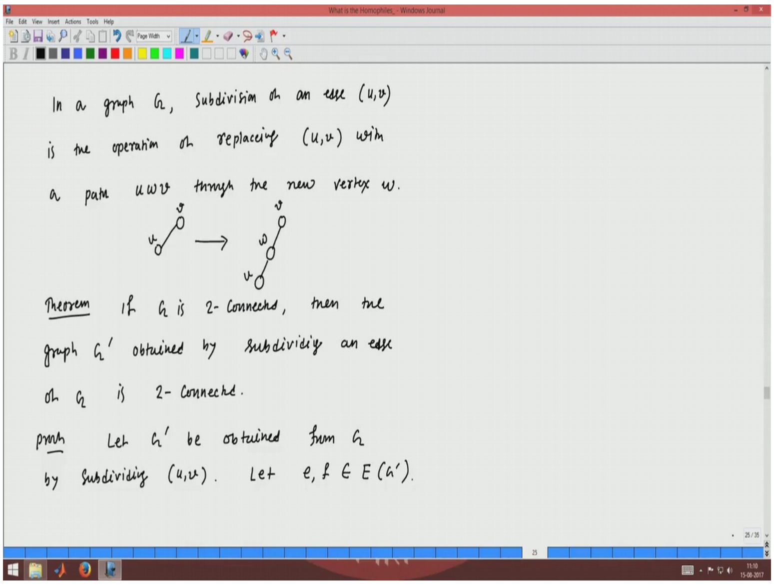

So, in the graph G, subdivision of an edge u, v is the operation of replacing u, v with the path u, w, v through the new vertex v; sorry to the new vertex w. So, u, v is an edge and then you subdivide this edge. So, you replace this edge by a path u, v, u, w, v; that means, you are subdividing the edge u, v by a new vertex w. Using this subdivision operation also from a graph G, you can get a bigger graph and we will study the property of that graph.

So, theorem: if G is 2-connected, then the graph G prime obtained by subdividing an edge of G is also 2-connected. So, G is a 2-connected graph; that means the graph has no cut vertex and G prime is a graph obtained from G by subdividing one edge and then G prime is also 2-connected. So, we prove this theorem.

Let, G prime be obtained from G by subdividing u, v. Let e and f are two edges in the graph G prime see; we have to prove that the new graph that we obtained from the new graph G prime that we obtained from G by subdividing one edge is also 2-connected. So, we know a graph is 2-connected; if and only if for every pair of edges there exists a cycle through those two edges. So, here I considering two edges e and f in G prime and to prove that this G prime is 2-connected, what we have to prove is that? It suffices to find a cycle through e and f. This is the condition d in that equivalent theorem. Now, since G is 2-connected, we know that any two edges of G lie on a common cycle right. This is known.

So, G is 2-connected. So, any pair of edges lie on a common cycle. Now we consider some cases, say case 1: the two edges that we consider e and f of G prime; first assume that both e and f are edges of G. So, there is a cycle c, through e and f in G, because G is also connected. So, there is a cycle c through e and f. And the c cycle c is also a cycle through e and f in G prime. So, here is the e the edge e and here is the edge f.

And these two edges lie on a common cycle c in G, and this c is also a cycle through e and f in G prime; only difference is that if u, v the edge u, v is on the cycle then the c will change in G prime little bit. So, this edge u, v if edge u, v is part of the cycle then after subdivision this edge will be replaced by two edges or will be replaced by this path u, w and v. So, c will slightly in G prime, but still you can see that there is a cycle through e and f in G prime. The c can be the same c or it can be slightly modified cycle in G prime. So, this is one case.

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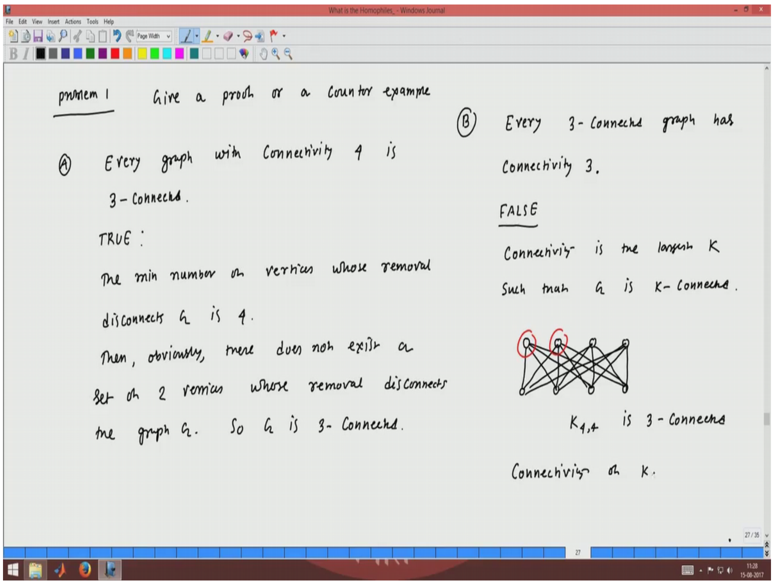

The other case is case 2: where e is an edge of the graph G, and f is one of this edges u, w and w, v. You remember that you know the G prime is obtained by subdividing this edge u, v. So, this is the edge in G and that has been replaced in G prime by u, w, v. So, what we are considering in this case is that if is one of these two edges say f is u, w. So, we know that since G is ah G is 2-connected; there is a cycle containing u, v and e, this is e and this is u, v. These two edges lie on a common cycle c in G. Now we modify a cycle passing through e and u, v in G to get a cycle passing through e and f.

So, what we have to do is that? In G prime this edge will be replace by this path. Now your f if i assume without loss of generality; that f is u, v then this is your f. So, you can get a cycle passing through e and f in G prime by modifying this cycle c in G this is quite trivial; and the other case, case 3 is that the e 2 edges e and f they are these two edges u and w and w, v . So, here we modify a cycle passing through u, v in G. So, G is 2-connected.

So; obviously, there is a cycle passing through u, v the edge u, v, so suppose this is the cycle which is passing through u, v in G. And then the same cycle can be considered as the cycle passing through this is my e and this is my f the cycle passing through u, v, u, w and w, v. So, what we have proved is that; you consider any two edges e and f in G prime and we find a cycle passing through e and f; that means, the graph G prime which is obtained from G by subdividing the edge u, v is also 2-connected; next we prove.

We solved some problems say problem 1. So, here you give a proof for a counter example. The first problem is; every graph with a connectivity 4 is 3-connected is this true or false.

So, the graph with connectivity 4; that means, you need to delete minimum four vertices to make the graph disconnected. And this is that the graph is a graph with connectivity four is 3-connected; 3-connected means you cannot make the graph disconnected by removing two vertices which is true, because you need to remove minimum four vertices to make the graph disconnected. So, this is a true statement. I hope that you understood, still i will write down the argument. The minimum number of vertices whose removal disconnect G is 4, this is what the connectivity.

Then; obviously, there does not exists a set of two vertices whose removal disconnect the graph. So, G is 3-connected. So, this is the true statement; just two make the people confuse is a easy problem. The other problem is; every 3-connected graph has connectivity 3. So, whether this is a true statement or it is a false statement. 3-connected graphs means you cannot disconnect the graph, using a set of two vertices. And the graph has connectivity 3. Connectivity three means it requires at least three vertices to be removed to make the graph disconnected. So, this is a false statement.

You know that the connectivity is the largest K such that G is K-connected. Let me give a counter example, i take a complete bipartite graph K 4, 4. So, this is K 4, 4 is 3connected, because by removing any two vertices you cannot make the graph disconnected.

For example, if you remove these two vertices for example, from this graph the graph will remain connected. So, there does not exist state of two vertices, whose removal disconnect the graph. And we know that the connectivity of K 4 4 of K 4 4 is 4, because the minimum number vertices that you need to remove from this graph to make this graph disconnected is 4. You have to remove all this four vertices to make the graph disconnected. So, what we have proved is that or we have given an example that K 4 4 is 3-connected, but it has connectivity four this is a quite obvious.

Next, we talk about another problem.

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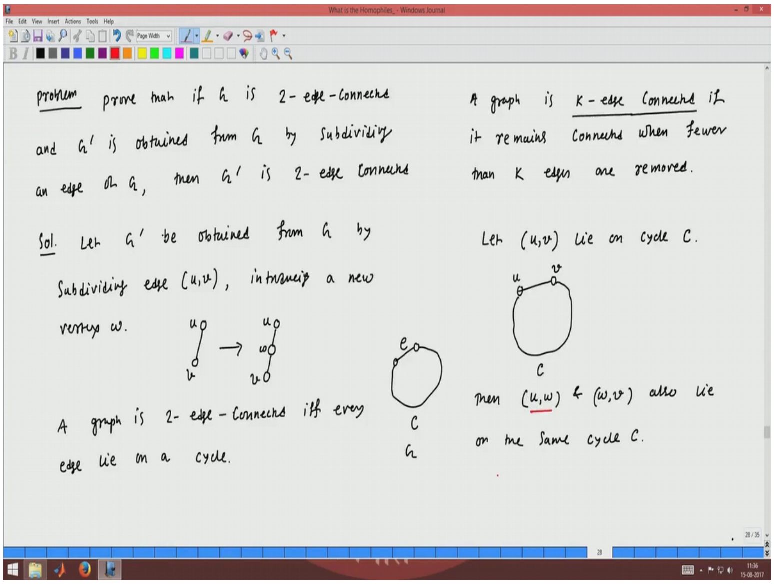

Prove that if G is 2-edge-connected and G prime is obtained from G by subdividing an edge of G, then G prime is also 2-edge-connected. So, probably i did not talk about what is 2-edge-connected graph. Let me just; is very similar to the definition of 2-connected graph. 2-connected graph is two vertex connected graph. A graph is say K connected if the does not exist a set of K minus 1 vertices whose removal disconnects the graph. Similarly K-edge connected graph; a graph is K-edge connected if there does not exist a set of K minus 1 edges whose removal disconnects the graph ok.

So, formally a graph is K-edge connected; if it remains connected when fewer then K-edge are removed. So, this is the K-edge connected graph. So, then 2-edge-connected graph means a graph is 2-edge-connected if they are does not exist a single edge whose removal a disconnect the graph. So, we have to prove that the graph G prime which is obtained by subdividing an edge of G is 2-connected 2-edge-connected.

Let us solve this problem. Let G prime be obtained from G by subdividing edge u, v introducing a new edge, a new vertex w; that means, same like you subdivide this edge u, v and replace this edge by this path that u, v, w. Now a graph is 2-edge-connected, if and only if every edge lie on a cycle. This is quite clear though, we did not prove this thing, but since the graph is 2-edge-connected then does not exist a single edge whose removal will disconnected the graph.

That means every edge is on a cycle C; every edge e will be on a cycle C. Let u, v lie on cycle C. So, u, v the edge u, v lie on cycle C, then u, w and w, v also lie on the same cycle. So, we know that in the graph G every edge e lie on a cycle as the graph G is 2-edge-connected. Now there are two new edges in graph G prime: one new edge is this u, w and v, w and then we have proved that they also lie on a cycle C so; that means, every edge in G prime also lie on a cycle. So, the graph G prime is also 2-edge-connected. So, that is all for today.

Thank you very much.