Graph Theory

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Lecture - 13
Part 2
2 - Connected Graphs

Welcome to the second part of lecture 13 on Graph Theory. Today again we will talk about 2 connected graphs. We know that a graph is 2 connected if and only if every pair of vertices is joined by 2 internally disjoint paths. So, in this lecture we learn some more necessary and sufficient condition for a graph to be 2 connected. And then we so, that all this conditions are equivalent.

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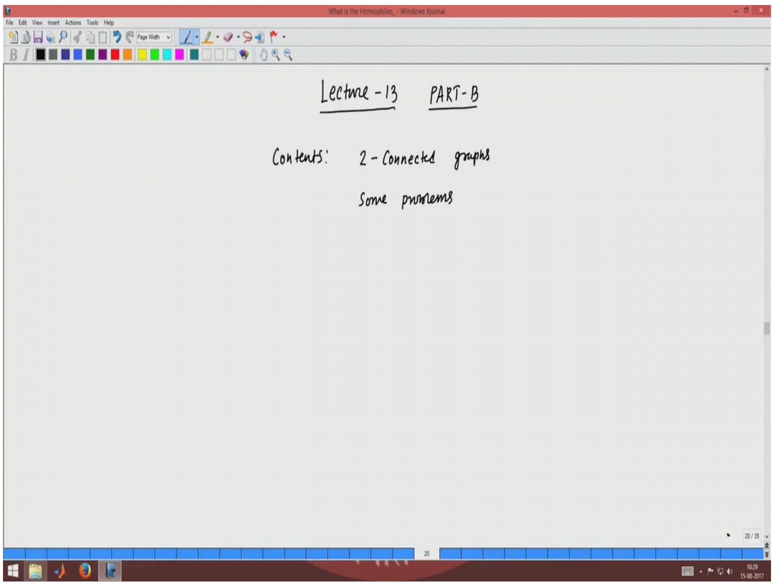

So, let us start with lecture 13. So, contents of this lecture 2 connected graphs.

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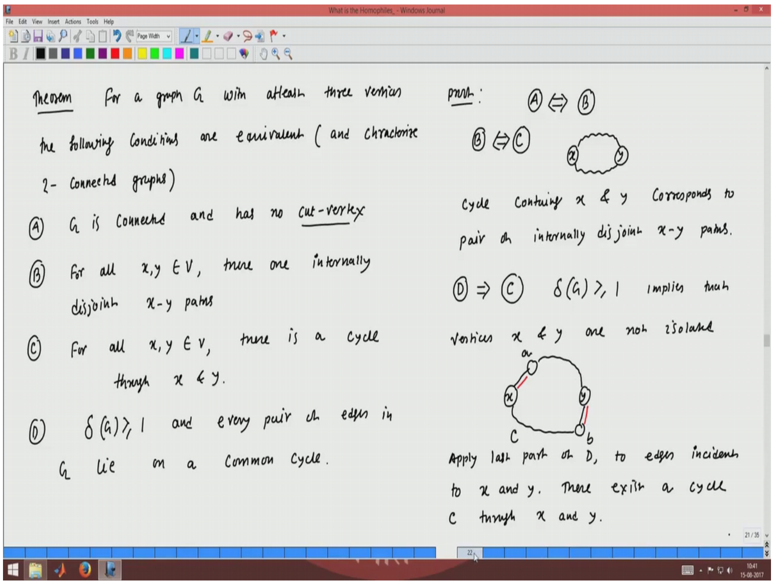

And at the end we will try to solve some problems related to 2 connected graphs. So, we will start with this theorem.

For a graph G with at least 3 vertices the following conditions are equivalent, and characterized characterize 2 connected graphs. The conditions are condition 1 G is connected and has no cut vertex. What is cut vertex? Cut vertex is a vertex whose removable disconnect the graphs. For example, if I consider this graph say this graph, then you can see that this is a cut vertex this is a cut vertex. Because if you remove this graph sorry if you remove this vertex from the graph if you remove this vertex from the graph then the then the graph becomes disconnected.

So, this vertex is a cut vertex the second condition. Condition b for all x y belongs to v there are internally disjoined x y paths. This is what we proved just now that a graph is 2 connected if an only if for every pair of vertices there are internal disjoined paths. Condition C for all vertices for all pair of vertices x y belongs to v there is a cycle through x and y. The condition d is the minimum degree of the graph G is greater than or equal to 1 and every pair of edges in g lie on a common cycles.

So what does this theorem says that all this conditions are equivalent and it characterize 2 connected graph. Let us prove this theorem. This is a very important theorem to characterize 2 connected graph. So, condition one the graph is connected and has no cut vertex; that means, they does not exists one vertex for whose removable disconnect the graph. So, the graph is 2 connected graph. So, condition one says that the graph G is a 2 connected graph because there is no cut vertex and the graph is connected. And then condition one implies Condition 2 condition 2 is for every pair of vertices there exists internally disjoined paths, and that is the theorem we just proved. So, A is equivalent to B. So, this is trivial A says the graph is 2 connected, which implies for every pair of vertices there are internally disjoined x y paths this is the theorem that we proved just now and also B implies A.

So, A is equivalent to B. Now what about B equivalent to C, this is also true because b says you take any 2 vertex x and y, there are internally disjoined path between x and y. And C says that for all x y there is a cycle through x y now you can see that this 2 are the same statement for the cycle containing x and y corresponds to corresponds to pair of internally disjoined x y paths right. So, C implies B, and once you have 2 internally disjoined paths you can get a cycle once you have cycle containing x y you can get 2 internally disjoined paths the same condition.

Now, we will prove that D implies C. So, delta G is the minimum degree delta G greater than equal to 1 implies that vertices x and y are not isolated. So, x and y can not be like isolated vertex, because the minimum then the degree of this x is 0 and degree of y is 0, but the minimum degree given of the graph is greater than 1. So, x is adjacent to at least one edge. And similarly y is also adjacent to at least one edge, or at least one vertex you can say suppose x is adjacent to a and y is adjacent to b.

Now, we apply. So, D implies C we apply last part of D which says that every pair of edges in G lie on a common cycle. So, apply last part of G to edges incident to x and y. So, the edges incident to x and y are this 2 edges. So, we will apply the condition that every pair of edges lie on a common cycle; that means, this pair of edges x a and y be they are on a common cycle. Suppose the cycle is c. So, this is the cycle which contains this 2 edges.

This implies there exists a cycle c through x and y (Refer Time: 12:07) same cycle C which contains the vertex the vertices x and y also.

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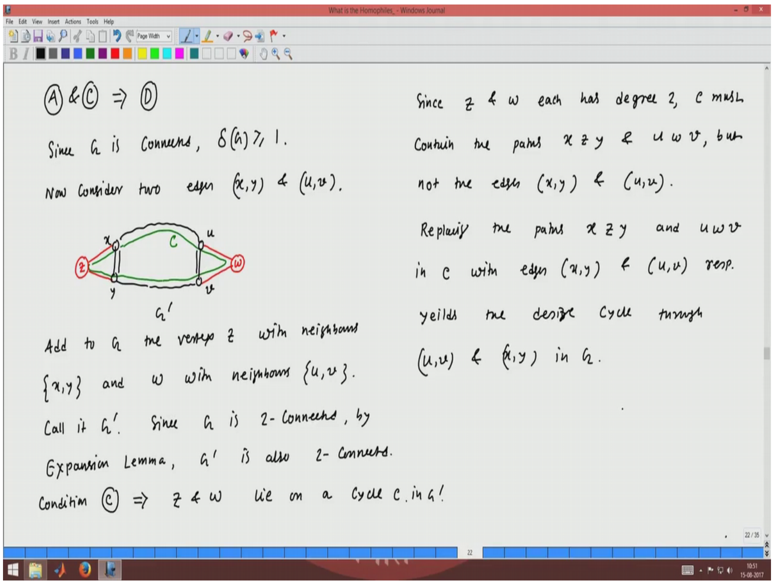

So, we proved that D implies C right. Next we prove that condition A and C implies D, which is nontrivial. So, condition A is that G is connected and has no cut vertex. Since G is connected delta G is greater than equal to 1. The minimum degree has to be at least 1. So, we have proved the first part of D, the first part of D says that the minimum degree is greater than equal to 1 and then we have to prove that every pair of every pair of edges in g lie on a common cycle right.

Now, consider 2 edges x y and u v and we have to prove that there exists the cycle containing this 2 edges. So, here is the edge x y and the other edge u v. Now we use the expansion lemma. So, we add to G the vertex a new vertex z with neighbour with neighbours x y. And w a new vertex with neighbours u v. So, what we do is that we add a new vertex? Z here with neighbour x and y, and we add another new vertex w with neighbour u and b. And call this new graph you call it G prime, it is like the expansion lemma not like it is exactly the expansion lemma.

Now, since G is 2 connected by expansion lemma G prime is also 2 connected right. Now so, this new graph G prime which is obtained by adding 2 new vertices z and w. See you can think in this way first you add z and by expansion lemma the graph with new vertex z is 2 connected. And then with that graph again you add w and again by the expansion lemma the graph G prime will be 2 connected. So, condition C says that for every pair there is cycle through those 2 vertices for every pair of vertices if you see condition C for all x y for every pair of vertices x and y there is a cycle through x and y. So, condition C implies z and w are the 2 new vertices in the graph 2 connected graph G prime. So, z and w lie on a cycle c. So, here is the cycle the cycle you can. So, this is the cycle basically this is the cycle c the green one is the cycle c. Now cycle c in G prime, since z and w each has degree 2 c must contain the paths x z y.

So, c must since z is up degree to the cycle must contain this part x z y and u w v, the see the cycle must contain this path, but not the edges x y and u v, because c c is a path through z w. So, it can not contain the edges x y and we u v. Now replacing the paths x and z y and u w v in c with edges x y and u v respectively yeild the desire cycle through u v and x y in G. So now in the cycle c if I if I remove this path x z y if I remove this path, and at this edge a similarly if I remove the path v u w v and at this edge in the cycle, then I get a cycle containing x y and v w. That is what the condition d says that is what we have to prove that every pair of edges. So, he took 2 pairs randomly x y and u v, and every pair of edges in cycle G lie on a common cycle and we have constructed that cycle c here.

So, we have proved this theorem this characterization theorem. Now we talk about 2 problems.

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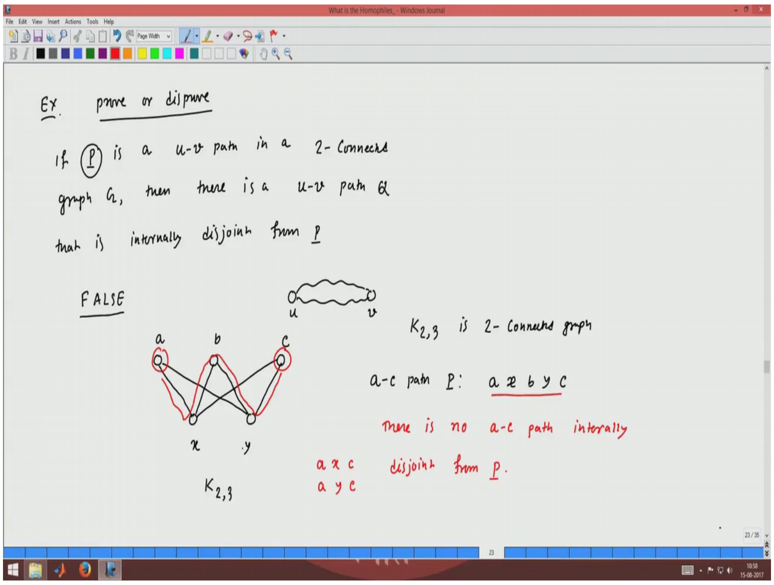

Say, the first problem is you have to prove or disprove if p is a u v path in a 2 connected graph G, then there is a u v path Q that is internally disjoined from P. So, what we now is that if g is a 2 connected graph then to every pair of vertices x y, there are internally disjoined paths that is true, but this exercise says that it is specifying one path say let p be a path between u and v in a 2 connected graph, then there exist another path Q which is internally disjoined form P.

So, is this true or false? Surprisingly not surprising this is false statement, this is false. Because of the fact I will explain is not contradicting first theorem that we proved the only difference here you can see that you are specifying one path P. And the first theorem says that if g is 2 connected then between every pair of vertices u and v there are internally disjoined paths, but you specify one path you may not find another path which is internally disjoined from P. Let me just give an example to prove that this is false. Let me consider this graph. I hope that you understand the difference between this statement here and the statement that we proved in the theorem.

So, I label this vertex a b c x and y. A y is also adjacent. Similarly x and c are adjacent. So, basically this is k 2 3 it is complete by partite graph with one part is having 2 vertices and the other part is having 3 vertices. Now I specify one a c path a c path. Let me call it P. So, one a c path is a x b y c. And is easy to see that this k 2 3 is 2 connected graph. Because you can not find single vertex whose removal can disconnect the graph. So, k 2 3 is a 2 connected graph and I specify one path between a and c since it is 2 connected between every pair of vertices say for example, a and c there are 2 internally disjoined paths that is true you can see that the 2 internally disjoined paths are one is a x, sorry a x c is one path, and the other path is a y c a y c. So, there are 2 internally disjoined path between x and y because the graph is 2 connected. Now if I specify my one path to be say a x and then b y and c, then I can not find another a c path which is internally disjoined from this path. So, there is no a c path internally disjoint from P.

So, the difference. So, I hope that now you understood that if you specify one path between a pair of vertices you might not get another path which is internally disjoint from that specified path, but since the graph is 2 connected since the graph is 2 connected here. You can always find 2 internally disjoined path, but you can not specify one path and then look for another path which is internally disjoint from the other path.

So, I hope that you understood this is an nice example to illustrate that you can not even a 2 connected graph you can not specify one path between a pair of vertices, and then you can not ask for another path which is internally disjoint from the specified path. But in 2 connected graph always you can find 2 internally disjoint path between every pair of vertices. That is all for today.

Thank you very much.