Graph Theory
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Lecture – 13
Part – 1
2 - Connected Graphs

Welcome to the first part of lecture 13 on Graph Theory. So, here is the content of this lecture. So, we will talk about 2 connected graphs.

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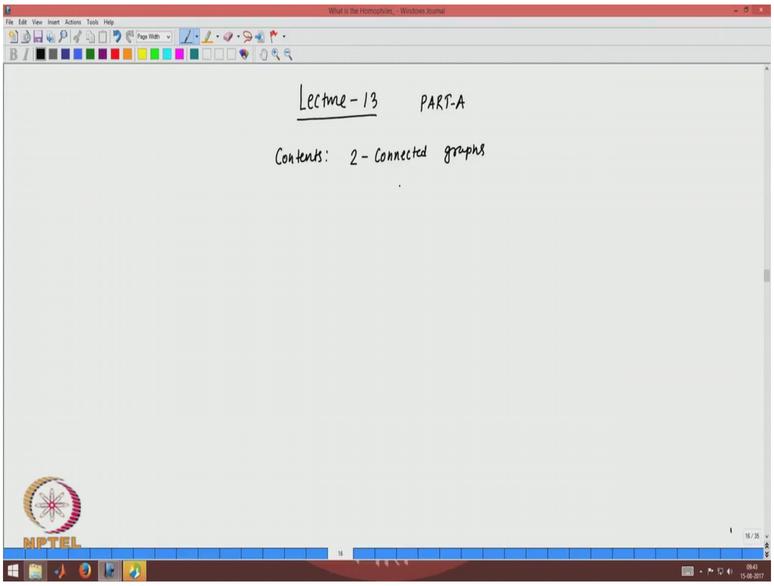

So, we know that a graph is k connected, if there does not exist a set of k minus 1 vertices, whose removal disconnects the graph. So, similarly 2 connected a graph is 2 connected if there does do not does not exist one vertex whose removal disconnects the graph.

So, in this lecture we will characterize 2 connected graphs. So, let me start with an example maybe.

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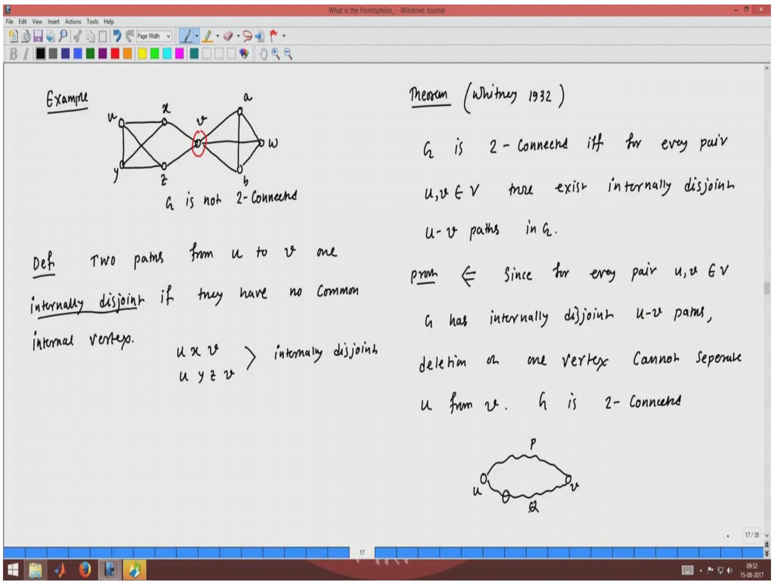

Example of 2 connected graph maybe, this is not 2 connected. So, you can see that if you remove this if you remove this vertex; this vertex from the graph the graph will become disconnected. So, this is So, G this graph G is obviously not 2 connected ok.

So, you want to characterize when a graph is to connected. First we give a definition 2 paths from u to v are internally disjoint if they have no common internal vertex. So, we are introducing internally disjoint paths. Let me give an example using this graph for example, maybe say this is my u and this is my v, and I call these vertices say x y z. Then between these 2 vertices I have I can see there are many paths, but u x v this is one path and the other part is u y z v. So, these 2 paths are internally disjoint right.

Now, if I say this vertex is w and this is a b. You can see that you can not find internally disjoint paths between u and w. So, that is why this graph is not 2 connected. And here is the theorem which sort of characterize 2 connected graphs, theorem 1932. So, it says that G is 2 connected, if and only if for every pair u v n v. There exists internally disjoint u v path in G. Well so, G is 2 connected if and only for every pair of vertices there exist internally disjoint u v paths.

Now, this condition is not true for this graph, because I can see that there does not exist internally disjoint paths between u and w. Because every path from u to w has go through this vertex v. So, you can not find internal disjoint paths between u and w. So, the graph G is not 2 connected. This is just an example to illustrate the meaning of this theorem.

Let us prove this theorem. So, this part that for every pair of vertices there exist internal disjoint paths then the graph is 2 connected. Since for every pair u v belongs to v G has internally disjoint u v paths. Deletion of one vertex cannot separate u from v. So, G is 2 connected.

Since between u and v for every pair it will take a random pair u and v, and there are internally disjoint paths. So, these are the 2 internal disjoint paths P and Q. Then obviously, just deleting one vertex you can not separate u and v right. So, deletion of one vertex is not enough to disconnect the graph g. So, the graph is 2 connected.

Well, next we will prove the other part.

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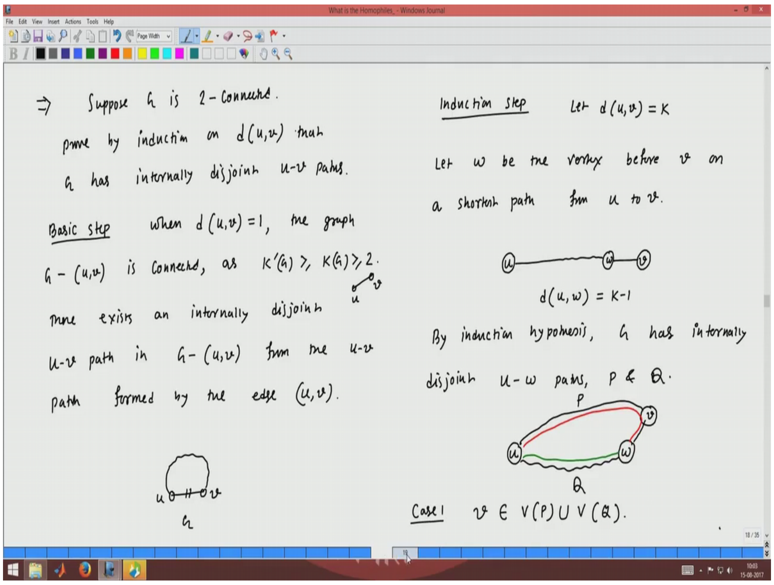

That suppose G is 2 connected. Then we prove by induction on d u v. So, d is the distance between u and v. So, you take a pair of vertices which are at distance d u v, and prove that they have internally disjoint paths. So, proof by induction on d u v that G has internally disjoint u v paths.

So, what the idea here is that we start with a pair of vertices which are at distance one. And then we will prove that there exist internally disjoint paths between u and v when u and v are at distance one. And then we will assume that this is true for any pair of vertices which are at distance k minus 1 and then finally, we will prove that this is also true for any pair of vertices which are at distance k.

So, first step is the basic step basic step, when distance of you take a pair of vertices which are at distance one; that means, u and v is an edge basically right. When this is one the graph G minus u v is connected. Why this is true? Because when the distance is one; that means, u and v is an edge in the graph. And now if you remove this edge from the graph then the graph becomes G minus u v this is connected. Because as k prime G is always greater than equal to k G which is greater than equal to 2. So, the graph is 2 connected. So, the connectivity is at least is greater than or equal to 2 the vertex connectivity. And we know that the edge connectivity is always greater than equal to the vertex connectivity. So, the edge connectivity is greater than equal to 2; that means, by removing one edge from the graph you can not make the graph disconnected.

So, that is why G minus u v is connected; that means, there exists and internally disjoint u v path in G minus u v from the from the u v path formed by the edge u v. So, what we proved is that. So, u v is an edge at this moment because the distance is equal to 1. And if I remove this edge from the graph G there is a path between u and v right. Because after removing the edge u v the graphs still remain connected; that means, there is a path between u and v. So, in the graph G there are 2 paths. One is this path and the other path is that the path obtained by this using the edge itself. So, we proved that when the distance between u and v is one there exist internally disjoint paths between u and v.

Now, the induction step induction step is that let distance between u and v be k. And we will prove that there exists internally disjoint paths between u and v when a when they are at distance k.

Let w be the vertex before v on a on a shortest path from u to v. So, u and v are 2 vertices, which are a distance k. And our final goal is to prove that there are 2 internally disjoint paths between u and v. So, at this moment what we are assuming is that suppose this is the shortest path between u and v which is; that means, the distance between u and v is k and w with the vertex before v on the shortest path from u to v.

So that means, the distance between u and w is equal to k minus 1 right. Now by induction hypothesis, induction hypothesis G has internally disjoint u w paths. Because we have assumed that this is a proof by induction. So, we will assume that there are internal disjoint paths between a pair of vertices which are at distance k minus 1, and then we use this fact to prove that there are internally disjoint paths between a pair of vertices which are at distance k ok.

So, by induction hypothesis G has internally disjoint u w paths, P and Q. So, u and w suppose this is. So, they are a distance k minus 1. So, this is one path say Q and the other path this is one case I am considering, that other path between u and w which is P, P contains v also. So, the path is the path P is this path basically the path P is this path from u to w, and path Q is this path. So, this is one case this is the case one when v when v is part of this path. So, this is the vertices of the path P union the vertices of path Q.

So, in this case we can see easily that. So, here you can see that v is a vertex in the path P. So, under case one we can say that there are internal disjoint paths between u and v. So, the one path is between u and v one path is this one and the other path is this one. So, in this case we can say that under case one if V is in vertex set of P union vertex set of Q then we can easily find internally disjoint u v paths in G.

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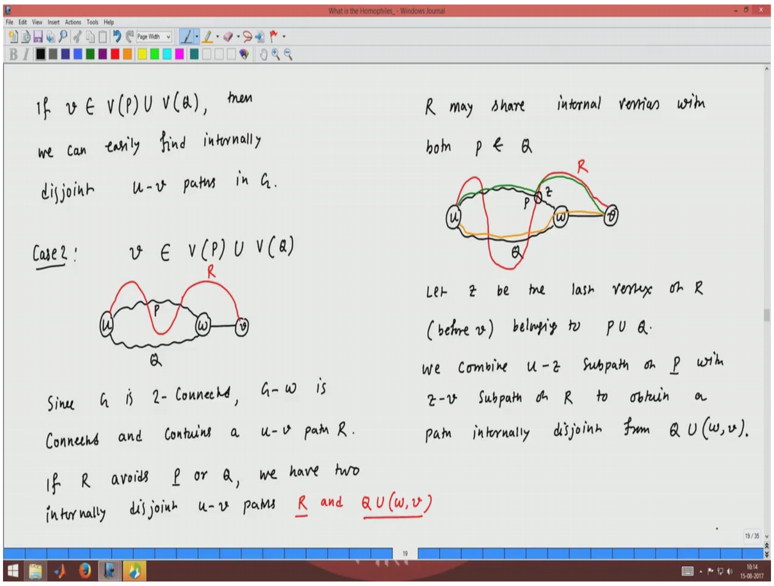

Now, the case 2 is that v is not a vertex on the path v P union v Q. So, this is the situation where u is here w is here, and u and w are at distance k minus 1. So, there are internally disjoint paths between u and w. This is P, this is Q and v is here. So, v is not on the path.

Now, since G is 2 connected by removing one vertex say vertex w you cannot make the graph disconnected. So, G minus w is connected from the definition of 2 connected graph 2 connected graph means, you can not you can not find a single vertex whose removal disconnect the graph. So, if you remove the vertex w from the graph the graph will still remain connected. So, G minus w is connected, and contains u v path R. So, again there are 2 situations if R avoids P or Q, we have 2 internally disjoint u v paths. Why this is true? So, G minus w is connected and contains a u v paths. And if R is this one then clearly we have internally disjoint paths between u and v. One is this R itself and the other one is Q union w v.

Now, suppose R avoids one of them not both of them. Suppose R is like this suppose R is like this. So, R intersects with P, then this is my R, and then also I can have 2 internally disjoint u v paths one is R this R and the other one is Q union w v. So, these are the 2 internally disjoint u v paths. So, this is the first case in terms of R of course, the other one is R may share internal vertices with both P and Q.

So, in this case let me draw the figure again. So, u w v the distance between u and w is k minus 1. So, by induction hypothesis there are 2 internally disjoint paths P and Q between u w. Now since the graph is 2 connected if you remove w from the graph the graph still remain connected; that means, there is a path R between u and v, and that path share internal vertices with both P and Q. So, this is my R know right.

Now, I have to in this situation I have to find a to internally disjoint paths between u and v. Let z be the last vertex of R before v belonging to P union Q. So that means, z is this vertex z be the last vertex of R before v before v belonging to P union Q. Now what we do is that we need to get 2 internally disjoint paths. So, we get the 2 internal disjoint paths in this way. So, we combine u z sub path of P with z v sub path of R to obtain a path internally disjoint from Q union w v. So, what is the meaning of this one? So, I need to find in under this situation I need to find 2 internally disjoint paths from between u and v, one path is this path Q, sorry u z sub path of P union z v sub path of R. So, this I should draw in this way. So, basically this is one path, this is one path. And the other path is the other path is this path. And you can see that these 2 paths are these 2 paths mean one this green and the orange these 2 paths are internally disjoint.

So, what we have proved here is that when we took 2 vertices at distance k. So, there are internally disjoint paths between those 2 vertices. Assuming the industry induction hypothesis that there exist internal disjoint paths between 2 vertices which are at distance k minus 1. So, what we have proved is that a graph is 2 connected if and only if there exists internally disjoint paths between every pair of vertices. That is all.

Thank you very much.