Graph Theory
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Lecture – 12
Part – 2
Graph Connectivity

Welcome to the second part of lecture 12. Today we will talk about expansion lemma. So, here is the content of today's lecture.

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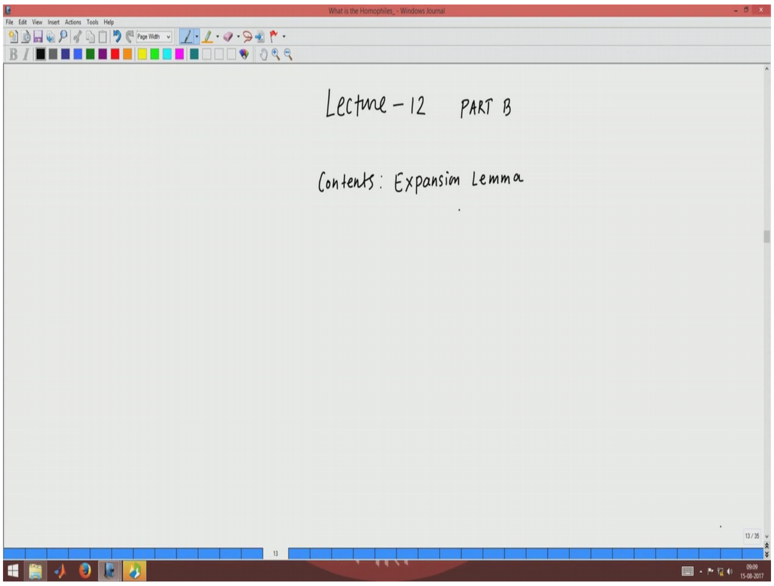

So, we will talk about expansion lemma, and we know that the vertex connectivity is the minimum number of vertices that need to be removed from the graph to make the graph disconnected. And edge connectivity is similarly the minimum number of edges that need to be removed from the graph to make the graph disconnected. And also we have proved in the first part of this lecture that the edge connectivity is always greater than or equal to the vertex connectivity for a simple graph.

So, today we will start with a result which says that the vertex connectivity is equal to the edge connectivity for a specific graph like graph 3 regular graph. A 3 regular graph means all the vertices are having the degree same degree 3 ok.

So, we will start with that result.

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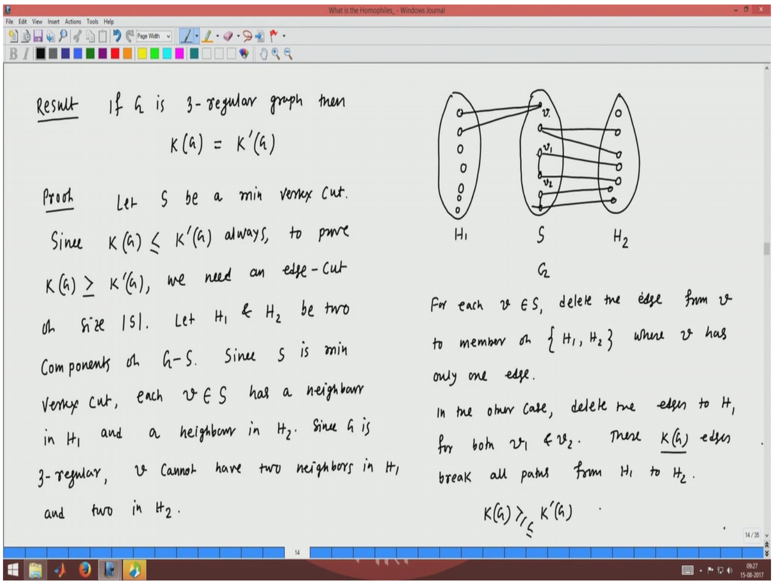

If G is 3 regular graph. Then vertex connectivity k G is equal to the edge connectivity k prime G. Well so, we will prove this theorem or result. Let s be a minimum vertex cut. We know that vertex cut means if the set of vertices such that if you remove that set from the graph then the graph will become disconnected. And the cardinality of the minimum vertex cut is the vertex connectivity of the graph.

Since k G the vertex connectivity is always less than or equal to k prime G, to prove k G the vertex connectivity greater than or equal to k prime G we need and edge cut of size cardinality of s. So, we started with a minimum vertex cut is and in order to prove that this is true we want to find the edge cut of size cardinality of s. Note that s is a minimum vertex cut. So, if you remove s from the graph G you will be left with 2 components. Let H 1 and H 2 be 2 components of G minus s.

So, s is a vertex cut. So, it is consists of some vertices. If you remove these vertices from the graph, then the graph will become disconnected. And it will have 2 components. So, one component is H 1 and the other component is H 2 and this is the whole graph G.

Now, since s is minimum vertex cut. Each vertex in the vertex cut s has a neighbor in H 1, and a neighbor in H 2. So, what you want to say is that you take a vertex v in s, note that this is a 3 regular G is a 3 regular graph. So, it has degree 3. So, this v will have a neighbor in H 1 and one neighbor in, at least one neighbor in H 2. It can have 2 neighbors in H 1 also or otherwise the third neighbor could be in s itself also. So, these are the possibilities. So, the other possibility is that v 1 vertex v here has 2 neighbors in H 2, and one neighbor in H 1. What we want to say is that all the 3 neighbor of v can not be in one component can not be in H 1, then this would not be in s and since s is a minimum vertex cut. That is why each vertex has a neighbor in H 1 and a neighbor in H 2. So, all the 3 neighbor can not be in one component. In that case the vertex v need not to be in the vertex cut, I hope that you understood this.

Now, since G is 3 regular we cannot have 2 neighbors in H 1 and 2 in H 2. So, this is quite clear because the graph G is a 3 regular graph. So, it can not have 2 neighbor in H 1 and 2 neighbor in H 2. So, it can have 2 neighbors in H 1, and then one neighbor in H 2 or it can have one neighbor in H 1 and 2 neighbors in H 2. The other possibilities are the vertex v has one neighbor here, and one neighbor in H 2 and one neighbor here. This is also possible. So, in that case this vertex is also will have one neighbor in H 2 and one neighbor in H 1. So, let me repeat this thing for these 2 vertices also.

Now, we want to construct an edge cut of cardinality s that is what our aim is now for each v belongs to s delete the edge from v to members to member of H 1 H 2 where v has only one edge. So, what you want to say is that we need to delete one edge corresponds to one vertex because we want to construct a edge cut of size v only.

So, in this case for this v will consider we could choose this edge we delete this edge. For v this v will delete this edge. Because it says that for every v delete the edge from v to member of H 1 H 2, where v has only one edge and the other case is this one, where the vertex v has one edge here one edge in H 2 and one edge within the vertex cut is. So, in such situation you delete the edges to H 1 for both v 1 and v 2. So, in the other case delete the edges to H 1 for both v 1 and v 2.

Now, similarly here also you remove these 2 edges. Now what we did is that you see that corresponds to every vertex we have selected one edge. And you delete this edges, and then you will see that the graph this, this edges disconnect this set of edges disconnect H 1 and H 2. So, these k G edges break all paths from H 1 to H 2. So, we constructed an edge cut of size cardinality of s or size k G. So, what we have proved is that you start from a vertex cut of size k G, and then you construct an edge cut of size k G again. So, the edge connectivity is less than or equal to the vertex connectivity right.

So, finally, we proved and the other case is always true that edge connectivity is greater than equal to the vertex connectivity. So, we proved that we proved that k G is equal to k prime G for 3 regular graph.

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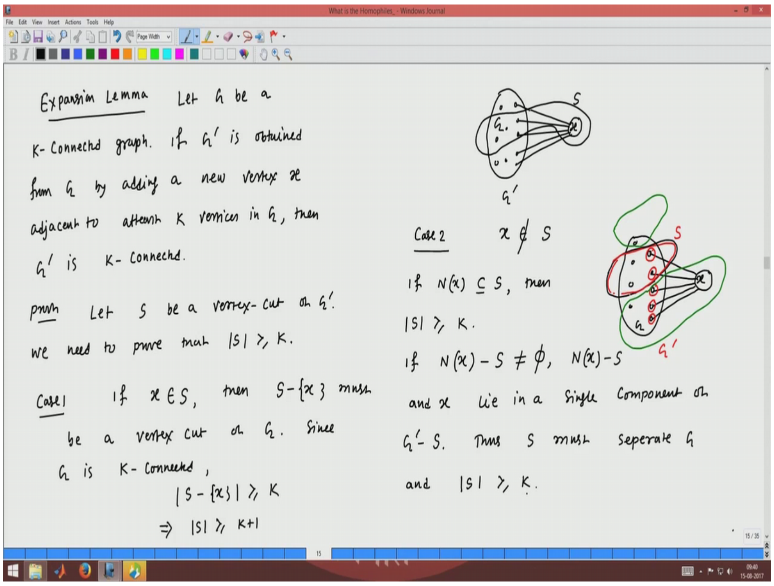

So next we move to the expansion lemma which says that suppose G is a k connected graph, and then how to expand this graph into G prime? So, that G prime is also k connected. So, this is expansion lemma. Let us G be a k connected graph. If G prime is obtained from G by adding a new edge, sorry by adding a new vertex x adjacent to at least k vertices in G, then G prime is k connected.

So, we will prove this lemma which is very important. Let s be a vertex cut of G prime. And we want to prove that G prime is k connected; that means, the cardinality of s must be greater than or equal to k. So, we need to prove that the cardinality of s is greater than equal to k. So, we know the definition of k connected graph right. So, graph is k connected if there does not exist a set of k minus 1 vertices whose removable disconnect the graph. So, since s is a vertex cut in G prime we have to prove that the cardinality of s is greater than or equal to k.

Now, here the graph is suppose this is the graph G, which is k connected. And it has many vertices, and I add a new vertex this is what the experiencing lemma is you I need a I add a new vertex x, which is adjacent to at least k vertices of G. Then the new graph G prime. So, this is G and this is G prime the new graph is also kth connected that is what we want to prove. And then we need to find the vertex we need to prove that a vertex cut of G prime is of cardinality minimum k. So, the case one is that if x is in the vertex cut s of G prime.

So, let me just a drew, draw s for example, suppose this is my s, this is my s. And this s is a vertex cut which include x. If x is in s then s minus x must be a vertex cut of G. I hope this is easy to understand. Now since G is k connected, the cardinality of this should be s minus x is a vertex cut for G. And G is k connected so, s minus x the cardinality of this one is greater than equal to k. Because s minus x is a vertex cut for the graph g. So, this implies that the cardinality of s is greater than equal to k plus 1.

So, we proved that when the new vertex x is in s, then in that case the cardinality of s is greater than equal to k plus 1. So, this is the case one and case 2 is x is not in is the new vertex is not in the vertex cut. So, s is the is a vertex cut of G prime. So, let me draw this again suppose this is my G it has many vertices. And this is G, and this is my new vertex x which is adjacent to at least k vertices of G. And I am considering the second case that x is not in s. So, s is something like this. So, this is my s, this is my s. And s this is a vertex cut for G prime this is G prime. And it does not include x.

Now, there could be 2 cases again if I didn't draw this one if all the neighbor of x is in s. See if you remove all the neighbors of s then this is a trivial vertex cut for G prime, because if you remove this vertices then you sort of disconnect x from the remaining vertex. So, this is also a vertex cut.

So, that is why if all the neighbors of x is in s then the cardinality of s must be greater than equal to k. Because the cardinality of the neighbor of x is greater than equal to k. The other case is that if all the neighbors of x is not in s; that means, n minus s is not equal to phi; that means,. So, these are the neighbor of x right. So, this is one neighbor of x this is another neighbor of x. So, these are the neighbor of x and all of them are not in s. So, this is the figure which represent this case then it is clear that N x minus s; that means, these vertices this vertex this vertex and this vertex. And x lie in a single component of G prime minus s. So, very clear from the figure that if all the neighbors are not in s. Then the neighbors which are not in s those neighbors and x they will be in one component. If you remove if you remove the vertices in s from the graph G prime then you can see that. So, these vertices are in one component, and these vertices are in the other component of G prime s. Thus s must separate G, and hence the cardinality of s must be greater than equal to k ok.

So, in all the cases in case one and in case 2 we prove that a vertex cut of G prime a vertex cut of G prime s, that has cardinality greater than or equal to k. So, G prime which is obtained from G is also k connected. So, the expansion lemma expansion lemma says that if G is a k connected graph. And you construct another graph G prime from G by adding a new vertex. And make that new vertex adjacent to at least k vertices of G, then the new graph G prime is also k connected. So, this is how you can sort of construct a bigger k connected graph from an existing k connected graph. That is all.

Thank you very much.