Graph Theory
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Lecture – 11
Part – 1
Stable Matching

Welcome to the 11th lecture on Graph Theory. This is the first part of course. So, today we talk about stable marriage problem or stable matching, and this problem has many applications. So, the input to this problem is that you are given a set of say n men and a set of a n women and their preference list and you have to find a stable matching. So, I will define um what is the meaning of this stable matching. We know what is matching, but now we talk about what is stable matching we will talk about an algorithm to find stable matching.

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So, let me start with the definition of a stable matching. As I said that the input to this problem is a list of men women and their preference list. And output to this algorithm is a stable matching. Let me start with an example. So, this is the set of men suppose I consider 4 men. And the preference of the first men the first preference is woman 2 the second preference is 4 1 3. So, one is the third preference and 3 is the 4th preference for man, first man and for the second man the first preference first preference is woman 3.

Second preference is woman 1 and third preference is woman 4 and 4th preference is woman 2.

Similarly, for the third man the preference list is here 2 3 1 4 4 1 3 2. So, this is the men preference list. And similarly we have a preference list for the women there are 4 women 1 2 3 4 and for the first woman her first preference is man 2, second preference is man 1, man 4, man 3 and for the second woman the preference list is 4 3 1 2. For the third woman it is 1 4 3 2. And for the 4th woman the preference list is 2 1 4 3.

So, this is this list is women's preference list. Now slowly we talk about what is the stable matching. We know what is matching right a matching is a one to one correspondence between, between the men and the women right. So, for example, 1 2 2 3 3 1 4 4. So, this is this is the man women pair right. So, this is the first component is the man and the second component is the woman. So, first man is matched with the second woman. So, this is you can see that one man is matched with only one woman.

So, this is this is a matching. So, this is the one to one correspondence right. So, this is a matching. So, some technical some notations now if m w; that means, it is a like 1 2 if m w belongs to the matching m let me call this as m. So, this is a matching; that means, this 2 are matched man m is matched with woman w. So, they are matched in the matching m if this belongs to m, then we say that m is equal to P M w; that means, the partner of w woman w in the matching m is man m. And similarly w is the partner of man m in the matching m.

So, this is the notation for partner in matching P M. Now we talk about the most important part. That is called the blocking pair. A pair say m w which is not in the matching a pair m w not in m is said to be a blocking pair a blocking pair for m if m prefers w to it is matched partner that is a P M m. So, m prefers w to it is matched partner in m and w prefer this is very important w prefers m to it is matched partner in m. Then this pair is called a blocking pair ok.

So, this is so, a pair a m w is a blocking pair if m prefers w more compared to it is matched partner in m. And similarly w prefers m more compared to it is matched partner. So, it is sort of analog of a sort of a extramarital affair type of a thing, because you know they prefer m and w prefer each other more compared to their matched partner in m. Then this pair is a blocking pair and if a matching has at least one blocking pair or if

there is a there exist a blocking pair with respect to a matching m then that matching is not a stable matching.

So, this is what the idea of a stable matching or stable marriage problem, let me give example to illustrate this blocking pair this a very important concept for this topic. So, let me give one example of blocking pair example. So, let me just prove that 3 2 is a blocking pair is a blocking pair, in m in m means this m I mean in the in the in example one is a blocking pair in m of say example 1. So, why 3 2 is a blocking pair. So, we will understand this first. So, 3 in the matching 3 is matched with the man 3 is man 3 is matched with matched with one right, and 3 prefers 2 more compared to it is matched partner 1 in m.

So, here 3 is matched with one. So, 3 prefers the man 3 prefers 2 to it is match partner 1 2 to 1. So, man 3 prefers 2 more compared to it is match partner 1. Because this is this is from this list you can see that for 3, 1 is third preference and 2 is the first preference. So, 3 prefers 2 more compared to one. Now look at the girl look at a woman 2. A woman 2 is a matched with 1, and for woman 2 she is matched with 1, she is matched with 1, but she prefer 3 more compared to 1.

So, 2 prefers or woman 2 this is man 3 woman 2 prefers 3 to it is match partner match partner 1 right. So, that is why 3 2 is a is a that is why 3 2 is a blocking pair.

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Let me give more example say m 2, m 2 is equal to I am talking about another matching. So, m 2 is this set 1 1 2 3 3 2 4 4. So, the question is this is another matching and question is, is there any blocking pair with respect to m 2? For the same instance let me just copy the instance once more because we will be discussing this in detail again.

So, the man preference list 2 4 1 3, second man 3 1 4 2 third man 2 3 1 4 the 4th man 4 1 3 2. This is men's preference list the same list and the other one is 1 2 3 4 2 1 4 3 4 3 1 2 1 4 3 2 2 1 4 3. This is women's preference list. Now the question is there any blocking pair with respect to m 2 for the same instance this is the instance given. See here you can see that 1 is matched with in this matching one is the man 1 is matched with a woman 1 which is his third preference, sorry which is his third preference.

Now second man is man 2 is matched with 3, which is his first preference right similarly if. So, sort of a man 2 is happy I mean he can not be in the blocking pair. So, 2 will not be involved in the blocking pair with respect to this matching this you have to understand. This is intuitively a man 3 is matched with 2 a man 3 is matched with 2 which is his first preference. So, man 3 also will not be involved in the blocking pair. A man 4 is matched with 4. So, for man 4 4 is the first preference. So, man 4 will not be involved in the blocking pair.

So, the only possibility when I ask the question is there any blocking pair with respect to m 2, then you need to understand that if there is a blocking pair that blocking pair will involve man 1 because man 1 is sort of unhappy he is he is matched with the third preference one. So, so you need to check 2 things is the first thing is that is 1 2 is a blocking pair. This is this blocking pair concept is very important in the stable matching problem. You need to understand this clearly. That is why this is another example to explain blocking pair a 1 2 is a 1 2 is a blocking pair this is a question, let us see. So, so 1 2 is first of all 1 2 is not in a matching. So, that is why these blocking pair is never in the matching right.

So, one is matched with a 1 and it prefers 2 it refers 2 more than 1. So, man 1 prefers 2 to it matched partner 1. So, it is 1 man 1 is matched with woman 1 in the matching, but see, but the man 1 prefer woman 2 to woman 1. Now what about so, this part is correct. What about a woman 2? So, woman 2 is matched with 3. And so, woman 2 is matched with 3 of, but one she does not prefer 1 to 3.

So, man woman 2 prefers her matched partner 3 to to one right. Because you see look at the woman 3 a list woman 3 is right now matched with sorry woman 2 is right now matched with 3. This is her matched partner. And she prefers 3 more to compare to one. So, so the other path is not clear not true that 2 prefers one to it is matched partner that is not true. That is why 1 2 is not a blocking pair right, I hope you understood.

Now the question second question is since 1 2 is not a blocking pair, because of the fact that let me write rewrite the same thing again. Woman 2 does not prefers 1 to 3 that is why 1 2 is not a blocking pair. The second question the other one I said that you know a only the man 1 is unhappy, because he is matched with his third preference. So, the other possibility is that 1 4 could be a blocking pair. So, the question is 1 4 is a blocking pair. So, for that first we will check whether one prefer 4 to it is match partner.

So, a man 1 prefers 4 man 1 man 1 it is match partner is one is this is the matched partner, but she, but he prefer 4, more compared to it is match partner. So, man 1 prefer 4. So, this is true for one to be a blocking pair one side is true. Man 1 prefer 4 to his matched partner 1 right. Similarly let us look at woman 4 now. Now women 4 is matched with 4 yeah. So, woman 4 this is woman 4 is matched with man 4, and she prefer one more compared to it is matched partner 4.

So, man 4 prefers sorry woman 4 prefers one to it is match partner I can write also in this way match partner in m 2, when match partner of 4 in m 2 is equal to 4 because of this one. And you can see that 4 of woman 4 prefer 1 to 4. So, that is why both side is true that you know one prefer of 4 to it is match partner 1, and similarly 4 also woman 4 also prefer 1 to it is match partner 4. So, that is why 1 4 is a is a blocking pair. I hope that these 3 examples of blocking pair will be enough to understand what is what is a blocking pair with respect to a matching.

Now we go to the definition of stable matching. A matching, a matching for which there is at least one blocking pair is called unstable, otherwise stable. Now you can check on your own that the matching the following matching the matching of 1 4 2 3 3 2 4 1 is stable. So, this is now probably it is easy to check that this matching that I wrote at the end is a stable matching. Because you can try to find a blocking pair with respect to this matching you will not be able to find a blocking pair with respect to the matching that I wrote at the end.

So, in the next part of this lecture we talk about an algorithm to find stable matching given an instance.

Thank you very much.